Weekly Notebook

OR

first semester
2021-2022

Chapter 3

Introduction to Linear Programming:-

3.1 The prototype example

Decision variables:-

x_1:- number of product 1 batches/week

x_2:- number of product 2 batches/week

Objective:-

maximize total profit

objective function :-

$$Z = 3000x_1 + 5000x_2$$

Constraints:-

$x_1 \leq 4$

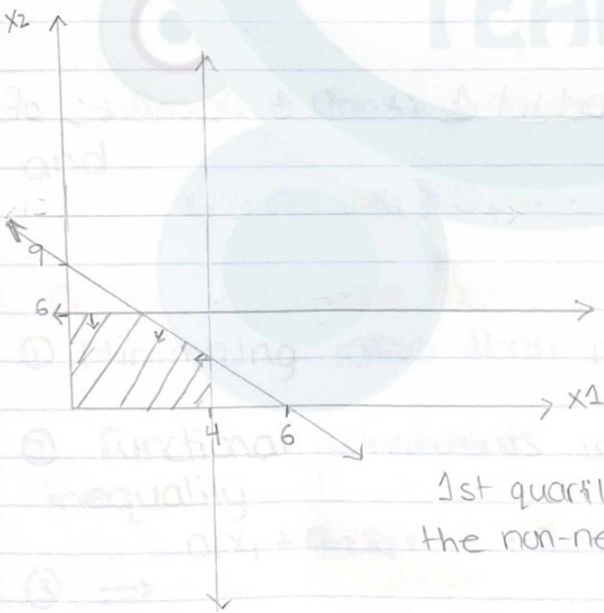
$2x_2 \leq 12$

$3x_1 + 2x_2 \leq 18$

$x_1 \geq 0$, $x_2 \geq 0$

nonnegativity constraints.

Graphical Solution:-



$x_1 = 4$ $x_2 = 6$

$x_2 = -1.5x_1 + 9$

$(6, 0)$, $(0, 9)$

1st quartile bcz of the non-negativity constraints.

* When the model is a linear continuous deterministic ~~mode~~, and the feasible area is of a convex nature, the optimal solution is at one of the corner points.

Corner points:-

(0,6)

(4,0)

$3x_1 + 2(6) = 18$

$3x_1 = 6$

$x_1 = 2$

$(2,6)$

$3(4) + 2x_2 = 18$

$2x_2 = 6$

$x_2 = 3$

$(4,3)$

* Substitute in z:-

$z = 3000(2) + 5000(6)$

$= \$36,000$

$z = 3000(4) + 5000(3)$

$= \$27,000$

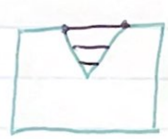
Optimal Solution:-

* 2 batches of product 1 and 6 batches of product 2.

By

* ~~Convex, we mean that the feasi~~

Examples of convex regions.



→ not convex

(bcz when drawing a line between two corners the line will be outside the wanted region

Alternative method of solving:-

→ plot the objective function where z is substituted as zero.

→ then ~~the~~ we keep on increasing the constant till the last point ~~it intersects~~ of intersection with the feasible area.

* The corner infeasible points are taken into consideration once the post-optimality is considered.

Standard form of the Model:-

Objective
maximize $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$,

subject to the restrictions

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \cdots a_{1n}x_n \leq b_1$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 \cdots a_{mn}x_n \leq b_n,$$

and

$$x_1 \geq 0, \; x_2 \geq 0, \; \cdots, \; x_n \geq 0$$

Other legitimate forms:-

① Minimizing rather than maximizing

② functional constraints with a greater-than-or-equal to inequality

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq b_1$$

③ → $=$

④ Deleting the non-negativity constraints for some decision ~~forms~~ variables.

Solution:- any specification of values for the decision variable.

Feasible solution:- is a solution for which all the constraints are satisfied.

Infeasible solution:- is a solution for which at least one constraint is violated.

Feasible Region:- is the collection of all feasible solutions.

Optimal Solution:- is a feasible solution that has the most favorable value of the objective function.

~~Most favorable Solution:-~~

CPF (corner-point feasible) solution

is a solution that lies at a corner of the feasible region.

often referred to as extreme points.

Assumptions of Linear Programming

① Proportionality:-
→ all the functions are linear

② Additivity:-
→ every function in a linear programming model is the sum of the individual contributions of the respective activities.

③ Divisibility:-
→ d.v.s are not restricted to just integer values;

④ Certainty:-
→ the value assigned to each parameter of a linear programming model is assumed to be a known constant.

Design of Radiation Therapy

Decision variables:

x_1 :- the dose at the entry point for beam 1

x_2 :- the dose at the entry point for beam 2

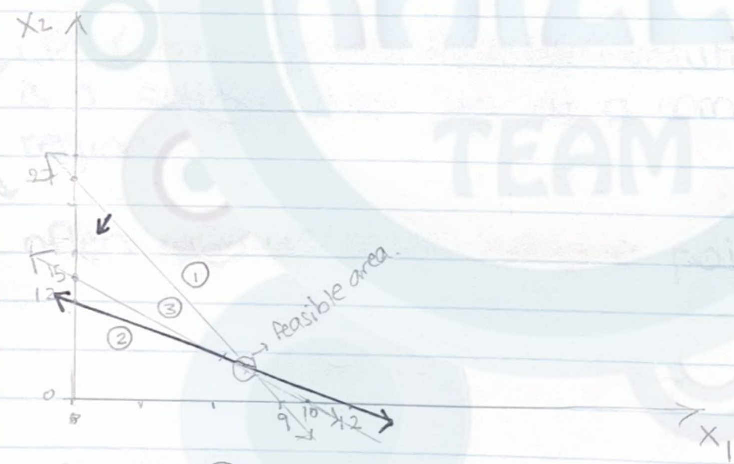
Objective :-

Minimize $z = 0.4x_1 + 0.5x_2$

subject to

$$0.3x_1 + 0.1x_2 \leq 2.7 \quad -- ①$$
$$0.5x_1 + 0.5x_2 = 6 \quad -- ②$$
$$0.6x_1 + 0.4x_2 \geq 6 \quad -- ③$$

$$x_1 \geq 0, \; x_2 \geq 0$$



① and ②

$$(0.3x_1 + 0.1x_2 = 2.7) * -5$$
$$0.5x_1 + 0.\cancel{5}x_2 = 6$$
$$-1.5x_1 - 0.\cancel{5}x_2 = -13.5$$

$$-x_1 = -7.5 \qquad x_2 = 4.5$$
$$x_1 = 7.5$$

$$(7.5, 4.5)$$

2 and 3

$0.5x_1 + 0.5x_2 = 6 \quad \text{--- } ②$

$x_1 = 12 - x_2$

plug ② in ③

$0.6(12 - x_2) + 0.4x_2 = 6$

$7.2 - 0.6x_2 + 0.4x_2 = 6$

$-0.2x_2 = -1.2$

$x_2 = 6$

$x_1 = 12 - 6$

$= 6$

$(6, 6)$

Try both solutions in $z \geq x$

$(7.5, 4.5)$ $\qquad$ $(6, 6)$

$z_1 = 5.25$ $\qquad$ $z_2 = 5.4$

$z_1 < z_2$

therefore it is optimal to use a total dose at the entry point of 7.5 Kilorads for beam 1 and 4.5 Kilorads for beam 2.

Controlling Air Pollution Page 51

- Specify which type of abatement methods will be used and at what fractions of their capacities for ① Blast furnaces or ② Open-hearth furnaces

this small paragraph is important by helping determine the decision variables.

Find a plan that satisfies the requirements with the smallest possible cost.
obj → minimization.

Decision variables:
x_{ij}: the fraction of abatement capacity i for furnace j.

i: 1 → taller smokestacks
2 → Filters
3 → Better Fuels

j: B → Blast furnace
H → Open-hearth ″

Objective
$$\min\ z = 8x_{1B} + 10x_{1H} + 7x_{2B} + 6x_{2H} + 11x_{3B} + 9x_{3H}$$

Subject to
$$12x_{1B} + 9x_{1H} + 25x_{2B} + 20x_{2H} + 17x_{3B} + 13x_{3H} \geq 60$$
$$35x_{1B} + 42x_{1H} + 18x_{2B} + 31x_{2H} + 56x_{3B} + 49x_{3H} \geq 150$$
$$37x_{1B} + 53x_{1H} + 28x_{2B} + 24x_{2H} + 29x_{3B} + 20x_{3H} \geq 125$$

$x_{ij} \leq 1$, $x_{ij} \geq 0$

since it is a fraction where you can either use the entire capacity or less.

Reclaiming Solid Wastes

Grade B → any percentage of material 3 could be used.
Grade C → " " " " 2, 3, 4 " " "

Decision variables:
x_{ij} = material i used in grade j
i: 1, 2, 3, 4
j: A, B, C

Obj maximization of net profit (sales income − Amalgamation cost)

max

$$z = (8.5-3)(X1A + X1B + X1C + X1D) \rightarrow \text{materials in grade A}$$
$$+ (7-2.5)(X2A + X2B + X2C + X2D) \rightarrow \text{" " " B}$$
$$+ (5.5-2)(X3A + X3B + X3C + X3D) \rightarrow \text{" " " C}$$

Subject to

$$X1A \leq 0.3(X1A + X2A + X3A + X4A)$$
$$X2A \geq 0.4(X1A + X2A + X3A + X4A)$$
$$X3A \leq 0.5(X1A + X2A + X3A + X4A)$$
$$X4A = 0.2(X1A + X2A + X3A + X4A)$$

$$\text{total } B = X1B + X2B + X3B + X4B$$
$$X1B \leq 0.5(\text{total } B)$$
$$X2B \geq 0.1(\text{total } B)$$
$$X4B = 0.1(\text{total } B)$$

No constraint on material 3.

~~total C = X1C + X2C + X3C + X4C~~

$$X1C \leq 0.7(X1C + X2C + X3C + X4C)$$

~~X1A + X2A + X~~

pounds/week available

$3000 \geq X1A + X1B + X1C \geq 1500 \quad (\frac{1}{2}(3000))$

$2000 \geq \cancel{X1B} + X2A + X2C \geq 1000$

$4000 \geq X3A + X3B + X3C \geq 2000$

$1000 \geq X4A + X4B + X4C \geq 500.$

$$3(X1A + X1B + X1C) + 6(X2A + X2B + X2C) + 4(X3A + X3B + X3C) + 5(X4A + X4B + X4C) = 30{,}000$$

$X_{ij} \geq 0 \rightarrow$ Non-negativity constraint

3-1-9

Decision Variables:

x_1 : Special Risk Insurance

y : Mortgages

objective

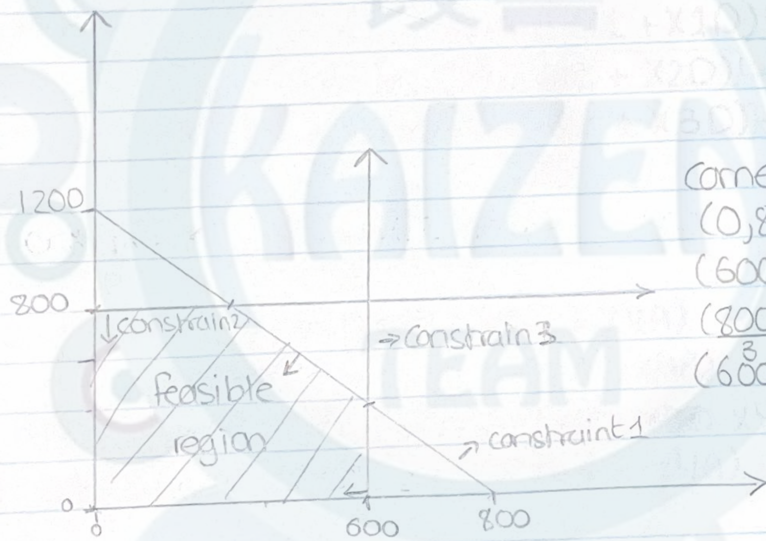
Max $z = 5x + 2y$

Subject to

$3x + 2y \le 2400$ $\quad x \ge 0, y \ge 0$

$0x + y \le 800$

$2x + 0y \le 1200$



Corner points :-

(0,800) (0,0)

(600,0)

($\frac{800}{3}$, 800)

(600, 300).

$y = 800$

$2x = 1200$

$x = 600$

$3x + 2y = 2400$

$(, 0) \Rightarrow 3x + 2(0) = 2400$

$x = 800$

$(0,) \Rightarrow 3(0) + 2y = 2400$

$y = 1200$

intersection of constraints 2 & 3 :-

$3x + 2y = 2400$ $\quad y = 800$

$3x + 2(800) = 2400$

$3x = 2400 - 1600$

$3x = 800$

$x = \frac{800}{3}$

inter. of const. 1 & 3 :-

$3(600) + 2y = 2400$ $\quad 2x = 1200$

$x = 600$

$y = 300$

z at $(800/3, 800)$
$= \$2933$

z at $(600, 300)$
$= \$3600$

Optimal Solution $\Rightarrow (600, 300)$

3-2-3

Decision variables:

x_1 = fraction of full partnership in venture 1
x_2 = fraction of full partnership in venture 2

Objective:

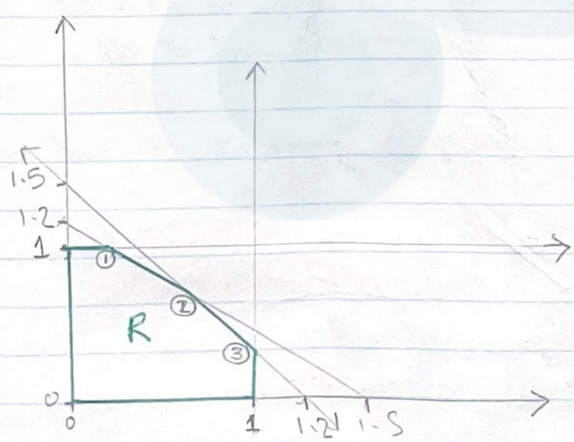
$$\text{Max } z = 9000x_1 + 9000x_2$$

Subject to

$$10000x_1 + 8000x_2 \leq 12000 \qquad x_1, x_2 \geq 0.$$
$$400x_1 + 500x_2 \leq 600$$

$0 \leq x_1 \leq 1, \quad x_2 \leq 1$

because it's a fraction where it's a part of the full partnership.



$(0, \ ) \rightarrow 8000x_2 = 12000$
$x_2 = 1.5$
$(\ , 0) \rightarrow 10000x_1 = 12000$
$x_1 = 1.2$
$(0, \ ) \rightarrow 500x_2 = 600$
$x_2 = 1.2$
$(\ , 0) \rightarrow 400x_1 = 600$
$x_1 = 1.5$

Corner points

(0,0)

(1,0)

(0,1)

① $400x_1 + 500x_2 \leq 600$ and $x_2 \leq 1$

$x_2 = 1 \rightarrow 400x_1 + 500(1) = 600$

$400x_1 = 100$

$x_1 = 0.25$

$(0.25, 1) \rightarrow z = 11250$

② $400x_1 + 500x_2 \leq 600$ and $10000x_1 + 8000x_2 \leq 12000$

$400x_1 + 500x_2 = 600$

$10000x_1 + 8000x_2 = 12000$

$x_1 = 2/3$; $x_2 = 2/3 \rightarrow z = 12{,}000$

③ $10{,}000x_1 + 8000x_2 \leq 12000$ and $x_1 \leq 1$

$10000(1) + 8000x_2 = 12000$ $(1, 0.25)$

$x_2 = 0.25$

Optimal point:

$(2/3, 2/3)$

3-4-10

Decision variables:-

x_m: the number of full-time consultants working in the morning shift (8 am - 4pm)

x_a = " " " " " " " " " " afternoon " (noon - 8 pm)

x_e = " " " " " " " " " " evening shift (4 pm - midnight)

P_1 = no. of parttime workers in the first shift

P_2 = no. of parttime workers in the second shift

P_3 = no. of parttime workers in the third shift

P_4 = no. of parttime workers in the fourth shift

Obj

maximize $z = 40(8)(fm + fa + fe) + 30(4)(P_1 + P_2 + P_3 + P_4)$

↑no. of hours ↑no. of hours

Subjected to

$x_m + P_1 \geq 4$

$x_m + x_a + P_2 \geq 8$

$x_a + x_e + P_3 \geq 10$

$x_e + P_3 \geq 6$

Morning: 8am-noon — Morning only; Noon-4pm — Afternoon

evening: 4pm - 8pm — Afternoon; 8pm-midnight — evening only

2 ft foreach 1pt.

1pt → 2ft

$x_m \geq 2P_1$

$x_m + x_a \geq 2P_2$

$x_a + x_e \geq 2P_3$

$x_e \geq 2P_4$

$x_m, x_a, x_e, P_1, P_2, P_3, P_4 \geq 0$

3-4-11

X_{ij}: the number of units sent from Factory i to customer j

i: 1,2

j: 1,2,3

Objective

minimize

$Z = 600X_{11} + 800X_{12} + 700X_{13} + 400X_{21} + 900X_{22} + 600X_{23}$

total shipping cost

subjected to

$X_{11} + X_{21} = 300$

$X_{12} + X_{22} = 200$

$X_{13} + X_{23} = 400$

$X_{11} + X_{12} + X_{13} \leq 400$

$X_{21} + X_{22} + X_{23} \leq 500$

$X_{ij} \geq 0$

3-4-14

Decision Variables:-

X_{ij}: proportion of cargo i in compartment j

i: 1,2,3,4

j: F, C, B

Objective:-

maximize

$Z = 320*20\,(X_{1F} + X_{1C} + X_{1B}) + 400*16\,(X_{2F} + X_{2C} + X_{2B})$
$+ 360*25\,(X_{3F} + X_{3C} + X_{3B}) + 290*13\,(X_{4F} + X_{4C} + X_{4B})$

$20X_{1F} + 16X_{2F} + 25X_{3F} + 13X_{4F} \leq 12$
$20X_{1C} + 16X_{2C} + 25X_{3C} + 13X_{4C} \leq 18$
$20X_{1B} + 16X_{2B} + 25X_{3B} + 13X_{4B} \leq 10$

$500(20)(X_{1F}) + 700(16)(X_{2F}) + 600(25)(X_{3F}) + 400(13)(X_{4F}) \leq 7000$

$500(20X_{1B}) + 700(16X_{2B}) + 600(25X_{3B}) + 400(13X_{4F}) \leq 9000$

$500(20X_{1C}) + 700(16X_{2C}) + 600(25X_{3C}) + 400(13X_{4C}) \leq 5000$

X_{ij} ← نسبة عشان هيك لازم نضربها بالوزن الكلي عشان يطلع معنا الوزن

مثلاً : $X_{1F} = 0.4 \rightarrow 0.4 * 20 \text{ tons} = 8 \text{ tons}$ → يوجد 8 طن من الشحنة 1 في الحجرة الأمامية.

$X_{ij} \geq 0$ → non-negativity constraint.

$$\frac{(X_{1F} + X_{2F} + X_{3F} + X_{4F})}{12} = \frac{(X_{1C} + X_{2C} + X_{3C} + X_{4C})}{18}$$

$$\frac{(X_{1F} + X_{2F} + X_{2F} + X_{4F})}{12} = \frac{(X_{1B} + X_{2B} + X_{3B} + X_{4B})}{10}$$

$$\left.\begin{aligned} X_{1F} + X_{1C} + X_{1B} &\leq 1 \\ X_{2F} + X_{2C} + X_{2B} &\leq 1 \\ X_{3F} + X_{3C} + X_{3B} &\leq 1 \\ X_{4F} + X_{4C} + X_{4B} &\leq 1 \end{aligned}\right\} \text{ cz } X_{ij} \text{ is a proportion}$$

Chapter 4
Simplex Method

* The mathematical model should be written in the standard form.

* the method will be explained on the windor example :-

objective maximize $z = 3x_1 + 5x_2 \rightarrow z - 3x_1 - 5x_2 = 0$

Add slack variables

$x_1 \leq 4 \rightarrow x_1 + x_3 = 4$

$2x_2 \leq 12 \rightarrow 2x_2 + x_4 = 12$

$3x_1 + 2x_2 \leq 18 \rightarrow 3x_1 + 2x_2 + x_5 = 18$

$x_1 \geq 0 \quad x_2 \geq 0 \qquad x_3, x_4, x_5 \geq 0$

slack variables represent the unutilized capacity which is the available capacity.

Start with the feasible solution (0,0).

Simplex tableau

	Z	x_1	x_2	x_3	x_4	x_5	RHS
R_0	1	-3	-5	0	0	0	0
R_1	0	1	0	1	0	0	4
R_2	0	0	2	0	1	0	12
R_3	0	3	2	0	0	1	18

Basic variables → x_3, x_4, x_5

* there is an identity column below them

no. of basic variables = no. of constraints (equations)

Nonbasic → x_1, x_2

Optimality step (related to the objective row)
→ If Ro contains any negative values, the solution isn't optimal

⇓

to improve the solution, look for the most ~~op~~ negative value in Ro.

⇓

x_2 → most negative, so it will be the entering value to the basic variables set

Note:-

basic var.

they take the value of RHS of the same row that contains the leader!

ex:-

$x_3 = 4$

$x_4 = 12$

$x_5 = 18$

Nonbasic var:

→ is always equal to zero.

Feasibility step (related to the constraints only)

→ Divide RHS by the coefficient of the entering variable.

→ then look for the minimum result.
→ the leader in the same row of the minimum will be the leaving variable of the basic variables set.

x_2 → entering

↙ pivot column

z	x_1	x_2	x_3	x_4	x_5	RHS	RHS / coeff. of ent. var
1	-3	-5	0	0	0	0	
0	1	0	1	0	0	4	$4/0 = \infty$
0	0	2	0	1	0	12	$12/2 = 6$
0	3	2	0	0	1	18	$18/2 = 9$

pivot row

pivot element

Analysis of ∞, 6, 9 :-

∞ is related to the 1st constraint
→ if x_2 is increased to ∞, x_1 won't be affected as it is ind. of x_2, so this constraint will always be achieved.
* coeff. of $x_2 = 0$, so no contribution.

6 → 2nd constraint.
→ 6 is the highest value x_2 can be increased to and still achieve the constraint.

9 → 3rd constraint
→ in this step $x_1 = 0$, so the highest x_2 can be increased to and still achieve the constraint is 9.

⇓

we choose the minimum to achieve all constraints.

* The pivot element should become a leader, so it should become 1 and the elements above and below it become 0.

⇓

X_4 is the leaving variable.

$R_2/2$

		z	x_1	x_2	x_3	x_4	x_5	RHS	RHS/(coeff)
$5R_2+R_0$	R_0	1	-3	0	0	2.5	0	30	
-	R_1	0	1	0	1	0	0	4	4/1 = 4
	R_2	0	0	1	0	0.5	0	6	6/0 = ∞ ig
$-2R_2+R_3$	R_3	0	3	0	0	-1	1	6	6/3 = 2

X_2, X_3, X_5 → basic $\quad X_1, X_4$ → nonbasic.

$X_2 = 6 \quad X_3 = 4 \quad X_5 = 6.$

→ solution still not optimal.

X_1 → most negative

X_1 → entering variable

X_5 → leaving variable.

$R_3/3$

		z	x_1	x_2	x_3	x_4	x_5	RHS
$3R_3+R_0$	R_0	1	0	0	0	1.5	1	36
$-R_3+R_1$	R_1	0	0	0	1	1/3	-1/3	2
	R_2	0	0	1	0	0.5	0	6
	R_3	0	1	0	0	-1/3	1/3	2

optimal solution as there is no negative in the objective row.

X_1, X_2, X_3 → basic variables

$X_1 = 2$

$X_2 = 6$

$X_3 = 2$ → unutilized capacity is 2 hours.

$X_4, X_5 = 0.$ → 0 unutilized capacity

4.1.6

~~$-3x_1+x_2$~~

Maximize $z = 2x_1 + 3x_2 \longrightarrow z - 2x_1 - 3x_2 = 0$

$-3x_1 + x_2 \le 1 \longrightarrow -3x_1 + x_2 + x_3 = 1$

$4x_1 + 2x_2 \le 20 \longrightarrow 4x_1 + 2x_2 + x_4 = 20$

$4x_1 - x_2 \le 10 \longrightarrow 4x_1 - x_2 + x_5 = 10$

$-x_1 + 2x_2 \le 5 \longrightarrow -x_1 + 2x_2 + x_6 = 5$

and $x_1 \ge 0$ $x_2 \ge 0$

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS	RHS/coeff of x_2
R_0	1	-2	-3	0	0	0	0	0	
R_1	0	-3	1	1	0	0	0	1	$1/1 = 1 \leftarrow$ min.
R_2	0	4	2	0	1	0	0	20	$20/2 = 10$
R_3	0	4	-1	0	0	1	0	10	$10/-1 = -10$
R_4	0	-1	2	0	0	0	1	5	$5/2 = 2.5$

ignore cz $x_2 \ge 0$

Basic variables:-
x_3, x_4, x_5, x_6
$= 1, 20, 10, 5$

Nonbasic variables:-
$x_1, x_2 = 0$

Most negative $\rightarrow x_2$
so x_2 is entering variable.
x_3 is leaving variable.

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS	RHS/coeff of x_1
$3R_1+R_0$	1	-11	0	3	0	0	0	3	
	0	-3	1	1	0	0	0	1	$1/-3 = -1/3$
$-2R_1+R_2$	0	10	0	-2	1	0	0	18	$18/10 = 1.8$
R_1+R_3	0	1	0	1	0	1	0	11	$11/1 = 11$
$-2R_1+R_4$	0	5	0	-2	0	0	1	3	$3/5 = 0.6$

basic: $x_2 = 1$, $x_4 = 18$, $x_5 = 11$, $x_6 = 3$
non-basic: $x_1, x_3 = 0$

entering variable:- x_1
leaving variable:- x_6

$R_4/5$

		Z	X_1	X_2	X_3	X_4	X_5	X_6	RHS	RHS/coeff
$11R_4+R_0$	R_0	1	0	0	-1.4	0	0	2.2	9.6	9
$3R_4+R_1$	R_1	0	0	1	-0.2	0	0	0.6	2.8	2.8/-0.2
$-10R_4+R_2$	R_2	0	0	0	2	1	0	-2	12	12/2=6
$-R_4+R_3$	R_3	0	0	0	1.4	0	1	-0.2	10.4	10.4/1.4
	R_4	0	1	0	-0.4	0	0	0.2	0.6	0.6/(-0.

entering variable = X_3
Leaving variable = X_4

basic => $X_1 = 0.6$
$X_2 = 2.8$
$X_4 = 12$
$X_5 = 10.4$

nonbasic => $X_3 = 0$
$X_6 = 0$

$R_2/2$

	Z	X_1	X_2	X_3	X_4	X_5	X_6	RHS
$1.4R_2+R_0$	1	0	0	0	0.7	0	0.8	18
$0.2+R_1$	0	0	1	0	0.1	0	0.4	4
	0	0	0	1	0.5	0	-1	6
$-1.4R_2+R_3$	0	0	0	0	-0.7	1	1.2	2
$0.4R_2+R_4$	0	1	0	0	0.2	0	-0.2	3

optimal solution

basic variables:
$X_1 = 3$
$X_2 = 4$
$X_3 = 6$
$X_5 = 2$

Nonbasic variables:
$X_4 = 0$
$X_6 = 0$

$Z = 18$

* Ignore RHS/coeff of entering value when it is equal to either infinity or negative.

Tie Breaking in the Simplex Method

Case 1
Tie for the entering basic variable

	z	x_1	x_2	x_3	x_4	x_5	RHS	
R_0	1	$\boxed{-3}$	$\boxed{-3}$	0	0	0	0	
R_1	0	1	2	1	0	0	4	$4/1 = 4$
R_2	0	2	1	0	1	0	6	$6/2 = 3$
R_3	0	0	2	0	0	1	8	$8/0 = \infty$ ignore

~~x_1 and x_2 are the~~

The coefficients of both x_1 and x_2 are the most negative. (There is a tie).

You can choose any of them as the entering variable.

$R_2/2$

$x_1 \rightarrow$ entering $\quad x_4 \rightarrow$ leaving

z	x_1	x_2	x_3	x_4	x_5	RHS		
1	0	-1.5	0	1.5	0	9		$3R_2 + R_0$
0	0	1	1	-0.5	0	1	$1/1 = 1$	$-R_2 + R_1$
0	1	0.5	0	0.5	0	3	$3/0.5 = 2$	
0	0	2	0	0	1	8	$8/2 = 4$	

$x_2 \rightarrow$ entering $\quad x_3 \rightarrow$ leaving

z	x_1	x_2	x_3	x_4	x_5	RHS	
1	0	0	1	1	0	10	$1.5R_1 + R_0$
0	0	1	2/3	-1/3	0	2/3	
0	1	0	-1/3	2/3	0	8/3	$-0.5R_1 + R_2$
0	0	0	-5/3	2/3	1	20/3	$-2R_1 + R_3$

$x_1 = 8/3$

$x_2 = 2/3 \qquad x_3 = 0$

$x_5 = 20/3 \qquad x_4 = 0$

Case
Tie for the leaving basic variable

z	x_1	x_2	x_3	x_4	x_5	RHS	
1	-3	-2	0	0	0	0	
0	1	2	1	0	0	2	$2/1 = 2$] tie
0	3	1	0	1	0	6	$6/3 = 2$] tie
0	0	2	0	0	1	8	$8/0 = \infty$ ignore

entering $\rightarrow x_1$ leaving $\rightarrow x_3$ (I chose it)

Choose any of them it doesn't matter.

$R_1 =$

z	x_1	x_2	x_3	x_4	x_5	RHS	
1	0	4	3	0	0	6	$3R_1 + R_0$
0	1	2	1	0	0	2	
0	0	-5	-3	1	0	0	$-3R_1 + R_2$
0	0	2	0	0	1	8	

$x_1 = 2$ $\boxed{x_4 = 0}$ $x_5 = 8$

$x_2, x_3 = 0$

we chose x_3 to leave yet the value of x_4 became 0 as well. This phenomena is called degenerating.

* You should always look for the minimum RHS/coeff of entering variable. If the tie isn't in the most minimum values, ignore it and choose the minimum.

case 3
No leaving basic variable

z	x_1	x_2	x_3	x_4	x_5	RHS
1	-3	-2	0	0	0	0
0	-1	2	1	0	0	4
0	-2	1	0	1	0	6
0	0	2	0	0	1	8

$4/-1 = -4$ ignore
$6/-2 = -3$ ignore
$8/0 = \infty$ ignore

$x_1 \rightarrow$ entering variable

the constraints in this tableau.

$-x_1 + 2x_2 \leq 4$
$-2x_1 + x_2 \leq 6$
$0x_1 + 2x_2 \leq 8$

for the 3 constraints, we can increase x_1 to infinity without breaking any constraint.

therefore this model is unbounded (بلا حدود)

* There would be a problem in the mathematical model if it is unbounded.

Case 4
Multiple Optimal Solutions.

* Whenever a problem has more than one optimal basic feasible solution, at least one of the nonbasic variables has a coefficient of 0 in the objective row in the final simplex tableau. Increasing any such variable will not change the value of Z.

Z	x_1	x_2	x_3	x_4	x_5	RHS
1	0	0	0	0	1	18
0	1	0	1	0	0	4
0	0	0	1.5	1	-0.5	3
0	0	1	-1.5	0	0.5	3

This an optimal final simplex tableau.

basic variables → x_1, x_2, x_4
Nonbasic → x_3, x_5

* The coefficient of x_3 ⇒ 0

entering variable → x_3 leaving variable → x_4

Z	x_1	x_2	x_3	x_4	x_5	RHS	
1	0	0	0	0	1	18	
0	1	0	0	-2/3	1/3	2	$-R_2 + R_1$
0	0	0	1	2/3	-1/3	2	
0	0	1	0	1	0	6	$1.5R_2 + R_3$

$P_1 = (4,3)$ $P_2 = (2,6)$
Both are optimal

All the optimal solutions are a weighted average of these 2 optimal CPF solutions:

$$(x_1, x_2) = w_1(2,6) + (1-w_1)(4,3)$$

Post optimality Analysis:

Windor glass company example

z	x_1	x_2	x_3	x_4	x_5	RHS
1	0	0	0	1.5	1	36
0	0	0	1	1/3	-0.33	2
0	0	1	0	0.5	0	6
0	1	0	0	-0.33	0.33	2

$x_1 = 2$
$x_2 = 6$
$x_3 = 2$
$x_4 = 0$
$x_5 = 0$

The slack variables represent the unutilized capacity.

x_3 represents the unutilized capacity in the first factory and $x_3 = 2$.

If the ~~merge~~ management wants to increase the profit (more than the optimal), they should increase the RHS, however to make the right choice they should ~~too~~ take into consideration the values of the slack variables.
(By the way, the changes would be small).

For instance, it won't make sense to increase the capacity of the first factory by 1 unit since $x_3 = 2$ hours, which means 2 hours of the capacity are already not used.

However, it would be logical to increase the RHS of the 2nd and 3rd constraints since the entire ~~constrai~~ capacities are utilized. ($x_4, x_5 = 0$)

Therefore, the 2nd and the 3rd constraints are called the functional constraints or tight constraints.

⇓

the constraints which determined the ~~VO~~ optimal point.

⇓

the point of intersection between them is the optimal point.

Shadow price :- (Sensitivity Analysis)

. the amount of increase in the profit optimal when the RHS is increased by one unit.

. It can only be determined through the final simplex method.

. It is represented by the coefficients of the slack variables in the objective row.

For example,
if RHS of the first constraint from 4 to 5, the profit will increase by 0
(the coefficient of x_3 (slack variable related to the 1st constraint) is 0).

Increasing RHS of the 2nd ~~from~~ by 1 unit, will increase z by 1.5.
x_4 is related to the 2nd constraint and its coefficient is 1.5.

$Max\ z = c_1 x_1 + 5x_2$

Sub to

$x_1 \leq 4$ --- 1

$2x_2 \leq 12$ --- 2

$3x_1 + 2x_2 \leq 18$ --- 3

$x_1 \geq 0$; $x_2 \geq 0$.

Optimal point → (2,6).

Find the value of c_1 such that (2,6) remains ~~an~~ as the optimal solutions.

~~the~~ (2,6) is the intersection of the 2 functional constraints, and the last point of inter. btwn the objective & feasible region.
The slope of the objective must be confined between the slopes of the functional constraint to get the same optimal point.

$$\frac{0}{2} \leq \frac{c_1}{5} \leq \frac{3}{2}$$

$$0 \leq \frac{c_1}{5} \leq \frac{3}{2}$$

$$0 \leq c_1 \leq \frac{15}{2}$$

→ optimal point won't change however the objective will change.

Big M method

→ used when the model is not in the standard form.

Radiation therapy example

min
$z = 0.4x_1 + 0.5x_2 + Ma_1 + Ma_2 \Rightarrow$ max $-z = -0.4x_1 - 0.5x_2 - Ma_1 - Ma_2$

$-z + 0.4x_1 + 0.5x_2 + Ma_1 + Ma_2 = 0$

Subject to

$0.3x_1 + 0.1x_2 \leq 2.7 \rightarrow 0.3x_1 + 0.1x_2 + x_3 = 2.7$

$0.5x_1 + 0.5x_2 = 6 \rightarrow 0.5x_1 + 0.5x_2 + xa_1 = 6$

$0.6x_1 + 0.4x_2 \geq 6 \rightarrow 0.6x_1 + 0.4x_2 + xa_2 - x_4 = 6.$

$x_1, x_2 \geq 0$ $\quad x_3, x_4, xa_1, xa_2 \geq 0$

The artificial variables should take a value of 0 in the final simplex method.
↓
It will only take a value when the model is infeasible.

z	x_1	x_2	x_3	x_4	xa_1	xa_2	RHS
-1	0.4	0.5	0	0	M	M	0
0	0.3	0.1	1	0	0	0	2.7
0	0.5	0.5	0	0	1	0	6
0	0.6	0.4	0	-1	0	1	6

Step 1
Restoring the Gaussian Form.

$-MR_2 + R_0$

$-MR_3 + R_0$

z	x_1	x_2	x_3	x_4	xa_1	xa_2	RHS	
-1	$0.4-1.1M$	$0.5-0.9M$	0	M	0	0	$-12M$	
0	0.3	0.1	1	0	0	0	2.7	$2.7/0.3 = 9$
0	0.5	0.5	0	0	1	0	6	$6/0.5 = 12$
0	0.6	0.4	0	-1	0	1	6	$6/0.6 = 10$

entering → x_1

leaving → x_3

z	x_1	x_2	x_3	x_4	xa_1	xa_2	RHS	
-1	0	$(11/30 - 0.533M)$	$(-4/3 + 11/3M)$	M	0	0	$-3.6 - 2.1M$	$(-0.4 + 1.1M)R_1 + R_0$
0	1	1/3	10/3	0	0	0	9	
0	0	1/3	-5/3	0	1	0	1.5	$-0.5R_1 + R_2$
0	0	0.2	-2	-1	0	1	0.6	$-0.6R_1 + R_3$

x_2 → entering variable

xa_2 → leaving variable

z	x_1	x_2	x_3	x_4	xa_1	xa_2	RHS
-1	0	0	$7/3 - 1.66M$	$11/6 - 1.66M$	0	$-1.83 + 2.66M$	$-4.7 - 0.5M$
0	1	0	20/3	5/3	0	-5/3	8
0	0	0	5/3	5/3	1	-5/3	0.5
0	0			-5	0	5	3

x_4 → entering

xa_1 → leaving

$(-11/30 + 0.533M)R_3 + R_0$

$-\frac{R_3}{3} + R_1$

$-\frac{R_3}{3} + R_2$

	Z	x_1	x_2	x_3	x_4	x_{a_1}	x_{a_2}	RHS
R_0	-1	0	0	0.5	0	$-1.1+9.96M$	$-1.83+2.66M$	-5.25
R_1	0	1	0	5	0	-1	0	7.5
R_2	0	0	0	1	1	0.6	-1	0.3
R_3	0	0	1	-5	0	3	0	4.5

$R_2/(5/3)$ $\quad (\frac{-11}{6}+1.66M)R_2+R_0$ $\quad \frac{-5}{3}R_2+R_1$ $\quad 5R_2+R_2$

$Z = 5.25$ → optimal value

$x_1 = 7.5 \quad x_2 = 4.5$ → optimal solution

Two-phase method:-

min

$z = 0.4x_1 + 0.5x_2 + Mxa_1 + Mxa_2$

Sub to

$0.3x_1 + 0.1x_2 \leq 2.7$

$0.5x_1 + 0.5x_2 = 6$

$0.6x_1 + 0.4x_2 \geq 6$

max

$z + 0.4x_1 + 0.5x_2 + Mxa_1 + Mxa_2 = 0$

$0.3x_1 + 0.1x_2 + x_3 = 2.7$

$0.5x_1 + 0.5x_2 + xa_1 = 6$

$0.6x_1 + 0.4x_2 - x_4 + xa_2 = 6$

Phase 1 :- (surplus, slack, and artificial values)

Max $-z + xa_1 + xa_2 = 0$

Phase 2 :-

Max $-z + 0.4x_1 + 0.5x_2 = 0$

Phase 1

z	x_1	x_2	x_3	x_4	xa_1	xa_2	RHS
-1	0	0	0	0	1	1	0
0	0.3	0.1	1	0	0	0	2.7
0	0.5	0.5	0	0	1	0	6
0	0.6	0.4	0	-1	0	1	6

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Step 1:-
Restoring Gaussian Form:-

$-MR_2 + R_0$
$-MR_3 + R_0$

z	x_1	x_2	x_3	x_4	xa_1	xa_2	RHS	
-1	-1.1	-0.9	0	1	0	0	-12	
0	0.3	0.1	1	0	0	0	2.7	9
0	0.5	0.5	0	0	1	0	6	12
0	0.6	0.4	0	-1	0	1	6	10

entering → x_1
leaving → x_3

z	x_1	x_2	x_3	x_4	xa_1	xa_2	RHS		
-1	0	-0.53	11/3	1	0	0	-2.1		$1.1R_1 + R_0$
0	1	1/3	10/3	0	0	0	9	27	
0	0	1/3	-5/3	0	1	0	1.5	4.5	$-0.5R_1 + R_1$
0	0	0.2	-2	-1	0	1	0.6	3	$-0.6R_1 + R_3$

entering → x_2
leaving → xa_2

z	x_1	x_2	x_3	x_4	xa_1	xa_2	RHS	
-1	0	0	-5/3	-5/3	0	8/3	-0.5	$+0.53R_3 + R_0$
0	1	0	20/3	5/3	0	-5/3	8	$-\frac{R_3}{3} + R_1$
0	0	0	5/3	5/3	1	-5/3	0.5	$-\frac{R_3}{3} + R_2$
0	0	1	-10	-5	0	5	3	

entering → x_4 leaving → xa_1 xa_1 & $xa_2 = 0$ → used them to find feasible solution now delete them

z	x_1	x_2	x_3	x_4	xa_1	xa_2	RHS	
-1	0	0	0	0	1.1	0	0	$5/3R_2 + R_0$
0	1	0	5	0	-1	0	7.5	$-5/3R_2 + R_1$
0	0	0	1	1	3/5	-1	0.3	
0	0	1	-5	0	3	0	4.5	$5R_2 + R_3$

↑ feasible and optimal with respect to phase 1.

Phase 2

Z	X_1	X_2	X_3	X_4	RHS
-1	0.4	0.5	0	0	0
0	1	0	5	0	7.5
0	0	0	-1	1	0.3
0	0	1	5	0	4.5

$-0.4R_1 + R_0$

$-0.5R_3 + R_0$

Z	X_1	X_2	X_3	X_4	RHS
-1	0	0	0.5	0	-5.25
0	1	0	5	0	7.5
0	0	0	1	1	0.3
0	0	1	-5	0	4.5

$Z = 5.25$

$X_1 = 7.5 \quad X_2 = 4.5 \quad X_4 = 0.3$

Simplex Method → we use it when the model is written in the standard form
⇓
otherwise we use 2 phase or Big M.

→ when the obj. is min. → multiply by negative
→ constraint ≥ +ve RHS → surplus & artificial value.
→ constraint = +ve RHS → artificial value.

*if the RHS of a constraint ($\leq$ or $\geq$) is negative, multiply it by ~~a~~ negative 1.

→ To solve using the simplex method, the variables should always be greater than 0.

Therefore ~~However~~, if a variable is allowed to take negative values, it should be modified.

Example:-

$x_1 \geq -50$

$x_1 + 50 \geq 0 \longrightarrow x_1' = x_1 + 50 \longrightarrow x_1' \geq 0$

$x_1 = x_1' - 50$

Suppose the objective is

Max $z = 3x_1 + 2x_2$

$z = 3(x_1' - 50) + 2x_2 \rightarrow z - 3x_1' - 2x_2 = -150$

Subject to

$x_1 + x_2 \leq 100 \longrightarrow (x_1' - 50) + x_2 \leq 100$

$2x_1 - 4x_2 \geq 40 \longrightarrow 2(x_1' - 50) - 4x_2 \geq 40$

$x_1' + x_2 \leq 150$

$2x_1' - 4x_2 \leq 140$

$x_1' \geq 0 \quad x_2 \geq 0$

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Another case

Max $z = 3x_1 + 2x_2$

Sub to:-

$x_1 + x_2 \leq 100$

$2x_1 - 4x_2 \geq 40$

$x_1 \geq 20$ → it implicitly achieves the non-negativity constraint (already greater than 0) ; so just deal with it as a normal constraint.

$x_1 \geq 0$, $x_2 \geq 0$.

Max $z = 3x_1 + 2x_2$

Sub to:-

$x_1 + x_2 \leq 100$

$2x_1 - 4x_2 \geq 40$

$x_1 \leq 20$ → ~~it is~~ treat it as a normal constraint.

$x_1, x_2 \geq 0$ — It doesn't impose any problems.

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Max $z = 3x_1 + 2x_2 \rightarrow z = 3(x_1^+ - x_1^-) + 2x_2$

Sub to

$x_1 + x_2 \leq 100 \longrightarrow (x_1^+ - x_1^-) + x_2 \leq 100$

$2x_1 - 4x_2 \geq 40 \longrightarrow 2(x_1^+ - x_1^-) + 4x_2 \geq 40$

$x_1 \; (-\infty, \infty) \longrightarrow$ unconstrained.

$x_2 \geq 0$

$x_1 = x_1^+ + x_1^- \qquad x_1^+$ and $x_1^- \geq 0$

$x_1 = 20$

$x_1^+ = 20, \quad x_1^- = 0$

$x_1 = -30$

$x_1^+ = 0, \quad x_1^- = 30$

Chapter 6
Duality Theory

* Every linear programming ~~model~~ problem has ~~ass~~ an associated problem called the dual.

Original → primal

Example:

↓ written in standard form.

$Max = 4x_1 + 2x_2$

~~coeff~~ $\boxed{\text{RHS of constraints in dual}}$ — Assign each constraint a dual decision variable

Subject to :-

coeff. of x_1 will be coeff of dual D.V in same order.

$3x_1 + 2x_2 \leq 8$ --- y_1

$x_1 + x_2 \leq 5$ --- y_2

$x_2 \leq 6$ --- y_3

coefficients of dual d.v in dual objective.

$x_1 ; x_2 \geq 0$

Min

$W = 8y_1 + 5y_2 + 6y_3$

Subject to :-

$3y_1 + y_2 + 0y_3 \geq 4$ → constraint related to x_1

$2y_1 + y_2 + y_3 \geq 2$ → constraint related to x_2

$y_1, y_2, y_3 \geq 0$

$$\text{Max } z = 3x_1 + 5x_2$$

Subject to

$x_1 \leq 4$ --- y_1

$2x_2 \leq 12$ --- y_2

$3x_1 + 2x_2 \leq 18$ --- y_3

$x_1, x_2 \geq 0$

$$\text{Min } w = 4y_1 + 12y_2 + 18y_3$$

Subject to

$y_1 + 0y_2 + 3y_3 \geq 3$

$0y_1 + 2y_2 + 2y_3 \geq 5$

$y_1, y_2, y_3 \geq 0$

$$\text{Max } z = 3x_1$$

Subject to

$3x_1 + 2x_2 \leq 10$ --- y_1

$2x_1 \leq 6$ --- y_2

$x_1, x_2 \geq 0$

$$\text{min } w = 10y_1 + 6y_2$$

sub. to

$3y_1 + 2y_2 \geq 3$ → only one constraint since the second is redundant

$2y_1 \geq 0$

$y_1 \geq 0$

$y_1, y_2 \geq 0$

* The values of y_1, y_2, y_3 in the simplex tablue in each iteration will be equal to the coefficients of the slack variables in the objective row, where each ~~slack variable belongs~~ takes the value of the slack belong to the same constraint

General Relationships between primal and dual problems:-

① Parameters for a functional constraint in either problem are the coefficients of a variable in the other problem.

② Coefficients in the objective function of either problem are the RHS for the other problem

Duality theorem:-

① weak Duality Property:-
→ if x is a feasible solution for the primal and y is a feasible solution for the dual

$$\underbrace{cx}_{} \leq \underbrace{yb}_{}$$

value of objective

② Strong Duality Property:-
→ if x* is an optimal solution for the primal & y* is an optimal solution for the dual:

$$cx^* = by^*$$

③ Complementary Solutions Property:
→ For each iteration in the simplex method, you can determine the values of x_t and at the same time the values of y.

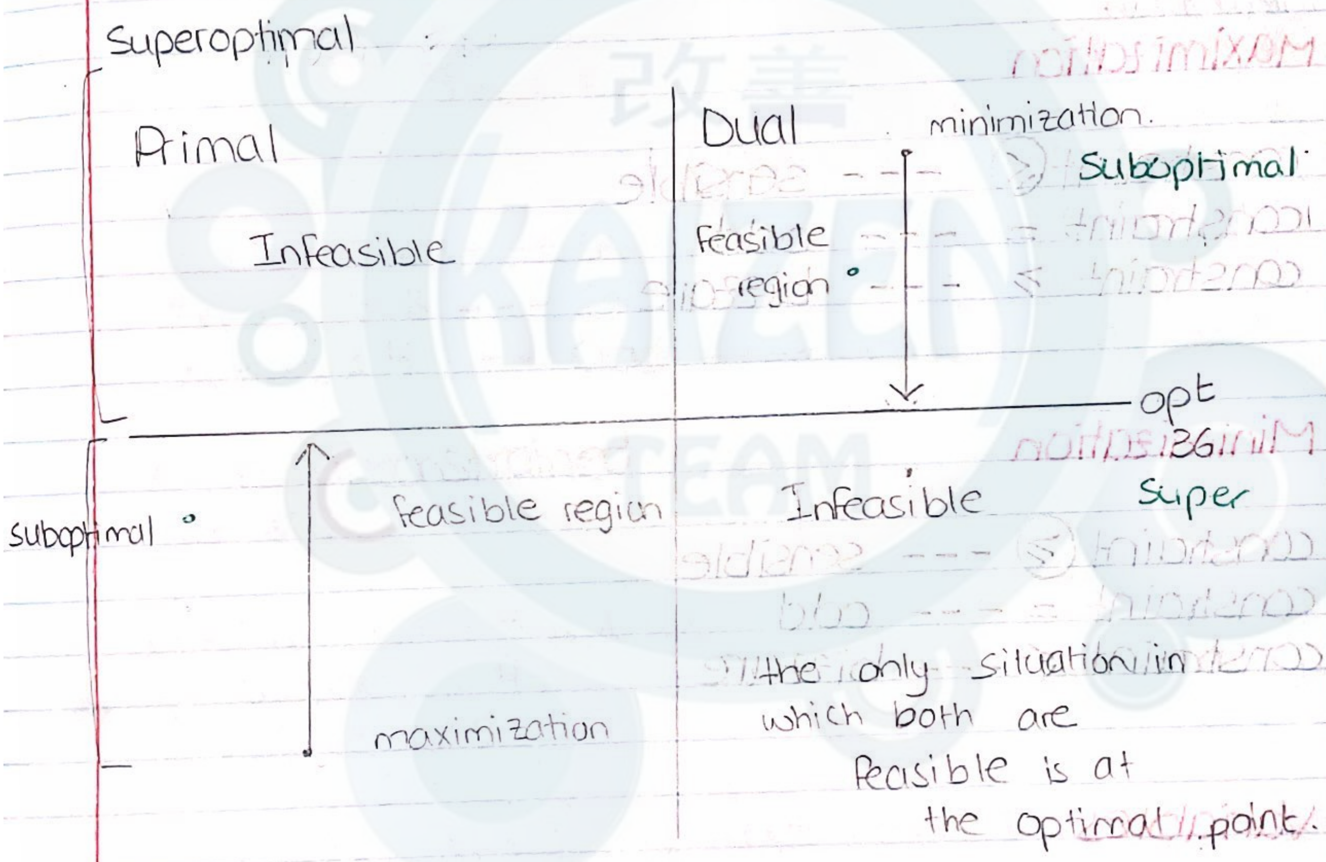
→ If x is not optimal for the primal problem then y is infeasible for the dual problem.

④ Complementary optimal solutions property:-

→ in the final iteration, there will be an optimal solution for the primal and a complementary optimal solution for the dual.

⑤ Symmetry property:-

All relationships between them must be symmetric



Economic Interpretation of the dual:-

The values of y in the dual represent the shadow prices.

→ Feasible and not optimal
Sub optimal
→ infeasible and optimal
superoptimal
→ feasible and ~~not~~ optimal
optimal
→ infeasible and not optimal
neither feasible nor superoptimal.

Adapting to other primal forms.

SOB method
Sensible-Odd-Bizzare

Maximization

constraint ≤ --- sensible
constraint = --- odd
constraint ≥ --- bizzare

Minimization

constraint ≥ --- sensible
constraint = --- odd
constraint ≤ --- bizzare

Variables

var. ≥ 0 --- sensible
var. unconstrained --- odd
var ≤ 0 --- bizarre

	Primal		Dual
	Max		Min
	constraint:-		Variable:-
S	$\leq$	----	$y \geq 0$
O	$=$	----	y unconstrained
B	$\geq$	----	$y \leq 0$
	Variable ~~constraint~~		constraint ~~Variable~~
S	$x \geq 0$	----	$\geq$
O	x unconstrained	---	$=$
B	$x \leq 0$	-----	$\leq$

Example:

Max $z = 4x_1 + 5x_2$

Subject to:

$x_1 + x_2 \leq 10$ --- Sensible --- y_1 --- $y_1 \geq 0$

$2x_1 + x_2 \geq 8$ --- Bizzare --- $y_2 \leq 0$

$x_1 - x_2 = 4$ --- Odd ---- y_3 unconstrained

x_1 is unconstrained

$x_2 \leq 0$

Min $w = 10y_1 + 8y_2 + 4y_3$

$y_1 + 2y_2 + y_3 = 4$

$y_1 + y_2 - y_3 \leq 5$

$y_1 \geq 0$, $y_2 \leq 0$, y_3 unconstrained

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Example

$\text{Min } z = 3x_1 + 2x_2 + 4x_3$

Subject to:-

$x_1 + 2x_2 + 3x_3 \geq 20$ --- Sensible --- $y_1 \geq 0$

$x_1 + x_2 + x_3 = 10$ --- Odd --- y_2 unconstrained

$-x_1 + x_2 - x_3 \leq 15$ --- bizzare --- $y_3 \leq 0$

$x_1 \geq 0$ -- Sensible

x_2 unconstrained -- odd

$x_3 \leq 0$ -- bizzare

$\text{Max } w = 20y_1 + 10y_2 + 15y_3$

sub to:-

$y_1 + y_2 - y_3 \leq 3$

$2y_1 + y_2 + y_3 = 2$

$3y_1 + y_2 - y_3 \geq 4$

$y_1 \geq 0$, y_2 unconstrained , $y_3 \leq 0$

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Min $z = 5x_1 + 4x_2$

Sub to :-

$2x_1 + x_2 \geq 10$ --- Sensible --- $y_1 \geq 0$

$-2x_1 + 2x_2 = 8$ --- Odd --- y_2 unconstrained

$x_1 + x_2 \geq 2$ --- Sensible --- $y_3 \geq 0$

$x_1 \geq 0, \; x_2 \geq 0.$

$\quad$ S $\qquad$ S

Max $w = 10y_1 + 8y_2 + 2y_3$

Sub to

$2y_1 - 2y_2 + y_3 \leq 5$

$y_1 + 2y_2 + y_3 \leq 4$

$y_1 \geq 0$, y_2 unconstrained, $y_3 \geq 0$

Max $z = 3x_1 + 5x_2 + x_3$

sub to

$x_3 + x_1 \leq 4$

$2x_2 \leq 12$

$3x_1 + 2x_2 \leq 18$

$x_3 + x_1, x_2 \geq 0$

$\longrightarrow$

Min $w = 4y_1 + 12y_2 + 18y_3$

sub to

$4y_1 + 3y_3 \geq 3$

$2y_2 + 2y_3 \geq 5$

$y_1 + y_3 \geq 1 \rightarrow$ is redundant

$y_1, y_2, y_3 \geq 0$

it will not affect the optimal solution.

$(0, 1.5, 1)$

if max $z = 3x_1 + 5x_2 + 5x_3$

third constraint $\rightarrow$ $y_1 + y_3 \geq 5$

not satisfied, so it will affect the optimal solution, solve using revised simplex method.

Chapter 5 The Theory of the Simplex Method

The constraint boundary equation for any constraint is obtained by replacing $\leq, =,$ or $\geq$ sign with an $=$ sign.

CPF solution (Corner-point feasible)
is a feasible solution that does not lie on any line segment connecting two other feasible solutions.

* Each CPF solution is simultaneously the solution of ~~two~~ n constraint boundary equations.
$\Downarrow$
where n is the number of D.V.

Example : Wyndor Glass Company

CPF solution	Defining Equations	
(0,0)	$x_1 = 0$, $x_2 = 0$	$\rightarrow$ intersection of the non-negativity cst.
(0,6)	$x_1 = 0$, $2x_2 = 12$	
(2,6)	$2x_2 = 12$, $3x_1 + 2x_2 = 18$	
(4,3)	$3x_1 + 2x_2 = 18$, $x_1 = 4$	
(4,0)	$x_1 = 4$, $x_2 = 0$	

Max
$z = 2x_1 + 5x_2$ → In matrix form
$z = CX'$

Subject to
$2x_1 + x_2 \leq 20$
$x_1 + x_2 \leq 10$ $\quad Ax' = b$
$x_1 - x_2 \leq 5$

$x_1, x_2 \geq 0$

C = matrix of the coefficients of the decision variables in the objective.

$C = [2, 5]$

A = matrix of the coefficients of the decision variables in the constraints

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$$

b = matrix of the RHS

$$b = \begin{bmatrix} 20 \\ 10 \\ 5 \end{bmatrix}$$

↑ those are constant, they don't change.

For each iteration:-

B = is the matrix of the coefficients of the basic variables in the constraint.
C_B = is the matrix of the coefficients of the basic variables in the objective.

take them from the original equations.

Initial Simplex Tableau.

Z	X_1 X_2	X_3 X_4 X_5	RHS
1	$-C$	0	0
0 0 0	A	I	b

For each iteration

Z	X_1 X_2	X_3 X_4 X_5	RHS
1	$(C_B * B^{-1} * A) - C$	$C_B * B^{-1}$	$C_B * B^{-1} * b$
0 0 0	$B^{-1} * A$	B^{-1}	$B^{-1} * b$

Max $z = 2x_1 + 5x_2$
Subject to

$2x_1 + x_2 + x_3 = 20$

$x_1 + x_2 + x_4 = 10$

$x_1 - x_2 + x_5 = 5$

$$A = \begin{array}{cc} x_1 & x_2 \\ \downarrow & \downarrow \end{array}$$
$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 5 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} 20 \\ 10 \\ 5 \end{bmatrix} \rightarrow 3 \times 1$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad C_B = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \quad X_B = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$x_2 \rightarrow$ entering variable

divide b by 2nd row of A element by element to determine leaving variable.

This step will differ in the following iterations.

$20/1 = 20$
$10/1 = 10 \rightarrow$ minimum $\rightarrow$ 2nd row $\rightarrow$ the leader in 2nd row belongs to x_4
$5/-1 = -5$ ignore

$x_4 \rightarrow$ leaving variable

new B and XB & CB بدّل x_4 ب x_2 بنفس الترتيب

$$X_B = \begin{bmatrix} x_3 \\ x_2 \\ x_5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad C_B = \begin{bmatrix} 0 & 5 & 0 \end{bmatrix}$$

coeff. of x_1 and $x_2 \Rightarrow C_B * B^{-1} * A - C$

$$= \underset{1\times 3}{\begin{bmatrix} 0 & 5 & 0 \end{bmatrix}} * \underset{3\times 3}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}^{-1}} * \underset{3\times 2}{\begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}} - \begin{bmatrix} 2 & 5 \end{bmatrix}$$

Use the calculator to solve this

$$= \begin{bmatrix} 5 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 0 \end{bmatrix}$$

coeff of slack variables in objective row:-

$C_B * B^{-1}$

$= [0 \quad 5 \quad 0]$

coeff of D.V in constraints:-

$B^{-1} * A$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$$

coeff of slack variables in constraints:-

B^{-1}

$$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

RHS of objective:-

$C_B * B^{-1} * b$

$= 50$

RHS of constraints:-

$B^{-1} * b$

$$= \begin{bmatrix} 10 \\ 10 \\ 15 \end{bmatrix}$$

Z	X_1	X_2	X_3	X_4	X_5	RHS
1	3	0	0	5	0	50
0	1	0	1	-1	0	10
0	1	1	0	1	0	10
0	2	0	0	1	1	15

↑ optimal tableau.

5-1-4

Max $z = 2x_1 - x_2 + x_3$

Sub to

$3x_1 + x_2 + x_3 \leq 60$

$x_1 - x_2 + 2x_3 \leq 10$ ←

$x_1 + x_2 - x_3 \leq 20$

and $x_1, x_2, x_3 \geq 0$ $x_2 \geq 0$, $x_3 \geq 0$.

~~Initial~~ Simplex Tableau after one iteration.

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	0	-1	3	0	2	0	20
0	0	4	-5	1	-3	0	30
→ 0	1	-1	2	0	1	0	10
0	0	2	-3	0	-1	1	10

$c = [2 \quad -1 \quad 1]$

$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}$ $b = \begin{bmatrix} 60 \\ 10 \\ 20 \end{bmatrix}$

$X_B = \begin{bmatrix} x_4 \\ x_1 \\ x_6 \end{bmatrix}$ → leader in 1st row
→ leader in 2nd row,
→ leader in 3rd row

should be written in this order.

a) CPF in iteration 1

$x_1 = 10 \quad x_2 = 0 \quad x_3 = 0$

b) The constraint boundary equations defining this CPF:-

↓ the constraint whose slack variable became a 0

so 2nd constraint since $x_5 = 0$ & belongs to 2nd cons.

$x_1 - x_2 + 2x_3 \leq 10$

The other 2 constraints → $x_2 \geq 0$ and $x_3 \geq 0$

since x_2 and x_3 are still equal to 0.

5-1-1 Max $z = 3x_1 + 2x_2$

subject to

$2x_1 + x_2 \leq 6 \rightarrow 2x_1 + x_2 + x_3 = 6$

$x_1 + 2x_2 \leq 6 \rightarrow x_1 + 2x_2 + x_4 = 6$

and $x_1 \geq 0,\ x_2 \geq 0$

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad C = [3 \ \ 2] \quad b = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$

$X_B = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad C_B = [0 \ \ 0]$

entering $\rightarrow x_1$
leaving $\rightarrow x_3$

$X_B = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \quad C_B = [3 \ \ 0]$

coeff of d.v in obj. row

$= C_B * B^{-1} * A - C$

$= [3 \ \ 1.5] - [3 \ \ 2]$

$= [0 \ \ -0.5]$

coeff. of slack variables in obj. row

$= C_B * B^{-1}$

$= [1.5 \ \ 0]$

coeff of d.v in constraints:-

$= B^{-1} * A$

$= \begin{bmatrix} 1 & 0.5 \\ 0 & 1.5 \end{bmatrix}$

coeff. of slack variables in constraints:-

$= B^{-1}$

$= \begin{bmatrix} 0.5 & 0 \\ -0.5 & 1 \end{bmatrix}$

RHS of objective

$= C_B * B^{-1} * b$

$= [9]$

RHS of constraints

$= B^{-1} * b$

$= \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

constraints in the simplex tableau

$$\begin{array}{ccccc|c} \rightarrow x_1 & x_2 & x_3 & x_4 & RHS & \\ \hline \end{array}$$

$$\begin{bmatrix} 1 & -0.5 & 0.5 & 0 & 3 \\ 0 & 1.5 & -0.5 & 1 & 3 \end{bmatrix} \begin{matrix} 3/-0.5 \\ 3/1.5 \end{matrix}$$

entering $\rightarrow x_2$ leaving $\rightarrow x_4$

$X_B = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ $C_B = [3 \; 2]$ $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

coeff. of O.V in objective row:-

$= C_B * B^{-1} * A - C$

$= [3 \; 2] - [3 \; 2]$

$= [0 \; 0]$

coeff of slack variables in objective row:-

$= C_B * B^{-1}$

$= [1.333 \quad 0.333]$

RHS of objective row:-

$= C_B * B^{-1} * b$

$= [10]$

RHS of constraints.

$= B^{-1} * b$

$= \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

* The tableau is optimal now as there no negative coefficients in the objective row.

* Optimal objective:
  $z = 10$

* Optimal solution:
  $x_1 = 2$
  $x_2 = 2$

5-1-4

$\text{Max } z = 2x_1 - x_2 + x_3$

subject to

$3x_1 + x_2 + x_3 \leq 60$

$x_1 - x_2 + 2x_3 \leq 10$

$x_1 + x_2 - x_3 \leq 20$

$x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0$

First iteration of the simplex tableau:-

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS	
1	0	-1	3	0	2	0	20	
0	0	4	-5	1	-3	0	30	30/4 = 7.5
0	1	-1	2	0	1	0	10	10/-1 ignore
0	0	2	-3	0	-1	1	10	10/2 = 5

$C = [2 \ -1 \ 1]$

$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}$ $\quad b = \begin{bmatrix} 60 \\ 10 \\ 20 \end{bmatrix}$

$X_B = \begin{bmatrix} x_4 \\ x_1 \\ x_6 \end{bmatrix}$

$x_2 \rightarrow$ entering $\qquad x_6 \rightarrow$ leaving

$X_B = \begin{bmatrix} x_4 \\ x_1 \\ x_2 \end{bmatrix}$ $\quad C_B = [0 \ 2 \ -1]$

$B = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

coeff of d.v in R_0 :-

$= C_B * B^{-1} * A - C$

$= [2 \quad -1 \quad 2.5] - [2 \quad -1 \quad 1]$

$= [0 \quad 0 \quad 1.5]$

coeff of slack variables in R_0 :

$= C_B * B^{-1}$

$= [0 \quad 1.5 \quad 0.5]$

optimal since no negatives.

RHS of obj :-

$= C_B * B^{-1} * b$

$= 25$

RHS of constraints :-

$= B^{-1} * b$

$$= \begin{bmatrix} 10 \\ 15 \\ 5 \end{bmatrix} \begin{array}{l} \rightarrow x_4 \\ \rightarrow x_1 \\ \rightarrow x_2 \end{array}$$

optimal value $\Rightarrow$ 25

optimal solution $\Rightarrow x_1 = 15$, $x_2 = 5$, $x_3 = 0$

Chapter 7
Linear Programming under Uncertainty

→ In all the previous examples of linear programming, we assumed that all the parameters are known constants.

↓

they are actually just estimates based on a prediction of future conditions.

↓

therefore, the associated uncertainty must be taken into consideration.

↓

thus, it is important to perform sensitivity analysis to investigate the effect on the optimal solution provided by the simplex method if the parameters take on other possible values.

↓

first thing is to do is to know which parameters are sensitive and which are not.

(postoptimality)

Performing Sensitivity analysis on the Wyndor Glass company example

Max $z = 3x_1 + 5x_2$

subject to

$x_1 \le 4$

$2x_2 \le 12$

$3x_1 + 2x_2 \le 18$

$x_1 \ge 0,\ x_2 \ge 0$

After we changed

Max $z = \cancel{3}\ 7x_1 + 5x_2$

subject to

$x_1 \le 4$

$x_2 \le 20$

$4x_1 + 2x_2 \le 18$

$x_1 \ge 0,\ x_2 \ge 0.$

Optimal Simplex Tableau.

z	x_1	x_2	x_3	x_4	x_5	RHS
1	0	0	0	1.5	1	36
0	0	0	1	1/3	1/3	2
0	0	1	0	1/2	0	6
0	1	0	0	-1/3	1/3	2

① Identify y, ~~which~~

y = the matrix of the coefficients of the slack variables in the objective ~~funct~~ row

$$y = [\,0 \quad 1.5 \quad 1\,]$$

② Identify S

S = the matrix of the coefficients of the slack variables in the constraints

$$S = \begin{bmatrix} 1 & 1/3 & 1/3 \\ 0 & 1/2 & 0 \\ 0 & -1/3 & 1/3 \end{bmatrix}$$

③ Identify the basic variables in the exact same order.

$$\text{Basic Variables} = \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix}$$

The changes to be made:-

Changing parameters

$$C[3\ 5] \longrightarrow C_N[7\ \ 5]$$

$$A\begin{bmatrix}1 & 0\\ 0 & 2\\ 3 & 2\end{bmatrix} \rightarrow A_N\begin{bmatrix}1 & 0\\ 0 & 2\\ 4 & 2\end{bmatrix}$$

$$b\begin{bmatrix}4\\ 12\\ 18\end{bmatrix} \longrightarrow b_N\begin{bmatrix}4\\ 20\\ 18\end{bmatrix}$$

	Z	X_1	X_2	X_3	X_4	X_5	RHS
R_0	1	-3	0	0	1.5	1	48
R_1	0	-1/3	0	1	1/3	-1/3	14/3
R_2	0	0	1	0	0.5	0	10
R_3	0	$\boxed{4/3}$	0	0	-1/3	1/3	-2/3

should be a leader

* No changes will be made on the Z, X_3, X_4, X_5 columns.

↓

Generally, ~~the~~ only the columns under the original variables and the RHS undergo changes

Coefficients of the original variables in the ~~objective~~ ~~new~~ objective row in the new simplex tableau:-

$$\overset{y^*}{[0\ 1.5\ 1]} * \overset{A_n}{\begin{bmatrix}1 & 0\\ 0 & 2\\ 4 & 2\end{bmatrix}} - \overset{C_N}{[7\ \ 5]} = [-3\ \ 0]$$

Coefficients of the original variables in the constraints:

$S^* A_n$

$$\begin{bmatrix} 1 & 1/3 & -1/3 \\ 0 & 0.5 & 0 \\ 0 & -1/3 & 1/3 \end{bmatrix} * \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} -1/3 & 0 \\ 0 & 1 \\ 4/3 & 0 \end{bmatrix}$$

RHS of the objective:

$Y^* b_n$

$$\begin{bmatrix} 0 & 1.5 & 1 \end{bmatrix} * \begin{bmatrix} 4 \\ 18 \\ 20 \end{bmatrix} = [48]$$

RHS of the constraints:

$S^* b_n$

$$\begin{bmatrix} 0 & 1/3 & -1/3 \\ 1 & 0.5 & 0 \\ 0 & -1/3 & 1/3 \end{bmatrix} * \begin{bmatrix} 4 \\ 18 \\ 20 \end{bmatrix} = \begin{bmatrix} 14/3 \\ 10 \\ -2/3 \end{bmatrix}$$

* X_1 should be a basic variable, however there is no identity below it.
Therefore the identity should be restored.
The leader was in R_3

$R_3 / 4/3$

~~This~~ ↓ Revised Simplex Tableau.

Z	X_1	X_2	X_3	X_4	X_5	RHS	
1	0	0	0	0.75	1.75	46.5	$3R_3 + R_0$
0	0	0	1	0.25	-0.25	4.5	$R_3/3 + R_1$
0	0	1	0	0.5	0	10	
0	1	0	0	-0.25	0.25	-0.5	

* The Revised Simplex Tableau is infeasible as ~~one of~~ the last constraint has a negative RHS.

↓

The solution is superoptimal.

	Z	X_1	X_2	X_3	X_4	X_5	RHS
R_0	1	0	0	0	0.75	1.75	46.5
R_1	0	0	0	1	0.25	-0.25	4.5
R_2	0	0	1	0	0.5	0	10
R_3	0	1	0	0	-0.25	0.25	-0.5

pivot element ↑ to fix this solve using dual simplex

$1/0=\infty$ $0/0=0$ $0/0=0$ $\frac{-0.25}{0.75}=\frac{-1}{3}$ $\frac{0.25}{1.75}=\frac{1}{7}$

→ Choose the most negative in RHS column

↓

then divide ~~the~~ each element in the associated row by each corresponding element in the objective row.

↓

Ignore ∞, and positive values
Choose the smallest negative value.

leaving → x_1
entering → x_4

Z	X_1	X_2	X_3	X_4	X_5	RHS	
1	3	0	0	0	2.5	45	$-0.75R_3+R_0$
0	1	2	1	0	0	4	$-0.25R_3+R_1$
0	2	1	0	0	0.5	9	$-0.5R_3+R_2$
0	-4	0	0	1	-1	2	$R_3/-0.25$

⇓

This tableau is feasible and optimal.

The allowable range for a RHS:- for wyndor glass

$b_1^* = 2 + \frac{1}{3}\Delta b_2$ - the change in b.

$b_2^* = 6 + \frac{1}{2}\Delta b_2$

$b_3^* = 2 - \frac{1}{3}\Delta b_2$

2nd column of S

$2 + \frac{1}{3}\Delta b_2 \geq 0$ & $6 + \frac{1}{2}\Delta b_2 \geq 0$ & $2 - \frac{1}{3}\Delta b_2 \geq 0$

$\Delta b_2 \geq -6$ $\quad \Delta b_2 \geq -12$ $\quad \Delta b_2 \geq 6$

$\Downarrow$

$-6 \leq \Delta b_2 \leq 6$

*For each b_i, the allowable range gives this range if none of the other bi are changing at the same time.

Linear Programming Under UnCertainty

→ Sensitivity Analysis:-

① Changing parameters
② Adding a constraint
③ Adding a decision variable

investigate their effect on the optimal solution.

Adding a constraint

Original:-

$\text{Max } z = 3x_1 + 5x_2$

Subject to:-

$\cancel{2x_2}\ x_1 \leq 4$ $\qquad\qquad x_1, x_2 \geq 0$

$2x_2 \leq 12$

$3x_1 + 2x_2 \leq 18$

new constraint: $7x_1 + 3x_2 \leq 40$

* whenever a new constraint is added, you should check whether it's redundant or not.

redundant → it won't affect the solution.

optimal solution → $(2, 6)$

$7(2) + 3(6) \overset{?}{\leq} 40$

$32 \leq 40$ → satisfied, so this constraint is redundant.

new constraint: $7x_1 + 3x_2 \leq 28$ → $7x_1 + 3x_2 + x_6 = 28$

$32 \leq 28$ × not satisfied, so the simplex tableau should be changed.

	Z	X_1	X_2	X_3	X_4	X_5	X_6	RHS
	1	0	0	0	1.5	1	0	36
X_3	0	0	0	1	1/3	-1/3	0	2
X_2	0	0	1	0	0.5	0	0	6
X_1	0	1	0	0	-1/3	1/3	0	2
X_6	0	7	3	0	0	0	1	28

should be zeros so restore them

	Z	X_1	X_2	X_3	X_4	X_5	X_6	RHS	
R_0	1	0	0	0	1.5	1	0	36	
R_1	0	0	0	1	1/3	-1/3	0	2	
R_2	0	0	1	0	0.5	0	0	6	
R_3	0	1	0	0	-1/3	1/3	0	2	
R_4	0	0	0	0	5/6	-7/3	1	-4	$-7R_3+R_4$ $-3R_2+R_4$

infeasible → superoptimal

X_6 → leaving
X_5 → entering

B.V	Z	X_1	X_2	X_3	X_4	X_5	X_6	RHS	
	1	0	0	0	13/7	0	3/7	34.3	$-R_4+R_0$
X3	0	0	0	1	3/14	0	1/7	18/7	$\frac{R_4}{3}+R_1$
X2	0	0	1	0	0.5	0	0	6	
X1	0	1	0	0	-3/14	0	1/7	10/7	$-\frac{R_4}{3}+R_3$
X5	0	0	0	0	$-\frac{5}{14}$	1	$-\frac{3}{7}$	$\frac{12}{7}$	

Optimal and feasible

$X_1 = 10/7$ $X_2 = 6$ $X_3 = 18/7$ $X_5 = 12/7$

* Adding a constraint to a mathematical linear programming model ~~causes the~~ worsens the ~~optimal solution~~ objective
  - → Maximization problem → decrease objective
  - → Minimization problem → increase objective

  ↳ worse.

or no effect if it's redundant

Adding a decision variable

$$\text{Max} \quad z = 3x_1 + 5x_2 + 4x_n$$

Subject to:-

$2y_1 + 2y_2 + y_3 \geq 4$

$$x_1 + 2x_n \leq 4$$
$$2x_2 + 2x_n \leq 12$$
$$3x_1 + 2x_2 + x_n \leq 18$$

$$x_1, x_2, x_n \geq 0$$

Adding a new decision variable means a new constraint is added in the dual problem.

The constraint related to x_n in the dual problem:-

$$2y_1 + 2y_2 + y_3 \geq 4$$

→ values of y_1, y_2, y_3 are equal to the coefficients of the slack variables in the objective row.

$y_1 = 0$, $y_2 = 1.5$, $y_3 = 1$

Substitute to check.

$$2(0) + 2(1.5) + 1 \overset{?}{\geq} 4$$

$4 \geq 4$ ✓ → no effect.

If

max

$$z = 3x_1 + 5x_2 + 6x_n$$

$4 \overset{x}{\geq} 6$ → not satisfied, so use the revised simplex method ~~tableau~~ to determine the effect.

Assume that x_n was present in the original problem however its coefficient in the objective and the constraints was 0.

$$\text{Max } z = 3x_1 + 5x_2 + 0x_n$$

Sub. to :

$$x_1 + 0x_n \geq 4$$
$$2x_2 + 0x_n \geq 12$$
$$3x_1 + 2x_2 + 0x_n \geq 18$$

$$\text{Max } z = 3x_1 + 5x_2 + 6x_n$$

Sub. to :

$$x_1 + 2x_n \geq 4$$
$$2x_2 + 2x_n \geq 12$$
$$3x_1 + 2x_2 + x_n \geq 18$$

Deal with it using the same method used when changing the parameters.

z	x_1	x_2	x_n	x_3	x_4	x_5	RHS
1	0	0	-2	0	1.5	1	36
0	0	0	7/3	1	1/3	-1/3	2
0	0	1	1	0	0.5	0	6
0	1	0	-1/3	0	-1/3	1/3	2

feasible

not optimal

$$C = [3\ \ 5\ \ 0] \longrightarrow C_N = [3\ \ 5\ \ 6]$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 2 & 0 \end{bmatrix} \rightarrow A_N = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

$$Y = [0\ \ 1.5\ \ 1] \qquad S = \begin{bmatrix} 1 & 1/3 & -1/3 \\ 0 & 0.5 & 0 \\ 0 & -1/3 & 1/3 \end{bmatrix}$$

① Coeff. of original D.V in obj row :-

$$= Y * A_N - C_N$$
$$= [0\ \ 0\ \ -2]$$

$x_3 \rightarrow$ leaving

$x_n \rightarrow$ entering

z	x_1	x_2	x_n	x_3	x_4	x_5	RHS	
1	0	0	0	6/7	25/14	5/7	37.7	$2R_1 + R_0$
0	0	0	1	3/7	1/7	-1/7	6/7	
0	0	1	0	-3/7	5/14	1/7	36/7	$-R_1 + R_2$
0	1	0	0	3/7	-2/7	2/7	16/7	$R_1/3 + R_3$

7-1-1

a) Max $z = 3x_1 + x_2 + 4x_3$

sub. to

$6x_1 + 3x_2 + 5x_3 \leq 25$ & $x_1, x_2, x_3 \geq 0$

$3x_1 + 4x_2 + 5x_3 \leq 20$

z	x_1	x_2	x_3	x_4	x_5	RHS
1	0	2	0	1/5	3/5	17
0	1	-1/3	0	1/3	-1/3	5/3
0	0	1	1	-1/5	2/5	3

optimal solution

$x_1 = 5/3$, $x_2 = 0$, $x_3 = 3$

optimal value.

Dual problem:-

b) Min $W = 25y_1 + 20y_2$

sub. to :

$6y_1 + 3y_2 \geq 3$

$3y_1 + 4y_2 \geq 1$ $\quad y_1, y_2 \geq 0$

$5y_1 + 5y_2 \geq 4$

c) $y_1 = 0.2$ $\quad y_2 = 0.6$

d) coeffs of x_2 changed so the 2nd constraint of the dual problem will change

$3y_1 + 4y_2 \geq 1 \longrightarrow 2y_1 + 3y_2 \geq 3$

$2(0.2) + 3(0.6) \overset{?}{\geq} 3$

$2.2 \not\geq 3$

not satisfied so the optimal solution will change.

(c) $y = [\ 1/5 \quad 3/5\ ]$ $S = \begin{bmatrix} 1/3 & -1/3 \\ -1/5 & 2/5 \end{bmatrix}$ $C_N = [3 \ 3 \ 4]$

$A_N = \begin{bmatrix} 6 & 2 & 5 \\ 3 & 3 & 5 \end{bmatrix}$

coeff. of x_2 in obj. row:

$y * A_{N12} - C_{N12}$

$= [0.2 \quad 0.6] * \begin{bmatrix} 2 \\ 3 \end{bmatrix} - [3]$

$= 2.2 - 3$

$= -0.8$

coeff. of x_2 in constraints

$= S * A_{N12}$

$= \begin{bmatrix} 1/3 & -1/3 \\ -0.2 & 0.4 \end{bmatrix} * \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} -1/3 \\ 0.8 \end{bmatrix}$

z	x_1	x_2	x_3	x_4	x_5	RHS
1	0	-0.8	0	1/5	3/5	17
0	1	-1/3	0	1/3	-1/3	5/3
0	0	0.8	1	-1/5	2/5	3

$x_3 \rightarrow$ leaving $x_2 \rightarrow$ entering

z	x_1	x_2	x_3	x_4	x_5	RHS	
1	0	0	1	0	1	20	$0.8R_2 + R_0$
0	1	0	5/12	0.25	-1/6	35/12	$\frac{R_2}{3} + R_1$
0	0	1	1.25	-0.25	0.5	3.75	

f) Max $z = 3x_1 + x_2 + 4x_3 + 2x_{new}$

sub. to

$6x_1 + 3x_2 + 5x_3 + 3x_{new} \leq 25$

$3x_1 + 4x_2 + 5x_3 + 2x_{new} \leq 20$

$3y_1 + 2y_2 \geq 2$

$3(0.2) + 2(0.6) \overset{?}{\geq} 2$

$1.8 \overset{\times}{\geq} 2$

Adding this variable will affect the solution.

z	x_1	x_2	x_3	x_n	x_4	x_5	RHS
1	0	2	0	-0.2	0.2	0.6	17
0	1	-1/3	0	1/3	1/3	-1/3	5/3
0	0	1	1	0.2	-0.2	0.4	3

$= [0.2 \quad 0.6] * \begin{bmatrix} 3 \\ 2 \end{bmatrix} - [2]$

$= -0.2$

$= \begin{bmatrix} 1/3 & -1/3 \\ -0.2 & 0.4 \end{bmatrix} * \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 1/3 \\ 0.2 \end{bmatrix}$

Adding a constraint $4x_1 + 7x_2 + 8x_3 \leq 20$

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	0	2	0	0.2	0.6	0	17
0	1	-1/3	0	1/3	-1/3	0	5/3
0	0	1	1	-0.2	0.4	0	3
0	4	7	8	0	0	1	20

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	0	2	0	0.2	0.6	0	17
0	1	-1/3	0	1/3	-1/3	0	5/3
0	0	1	1	-0.2	0.4	0	3
0	0	1/3	0	4/15	-28/15	1	-32/3

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	0	2.11	0	0.286	0	0.321	13.6
0	1	-0.39	0	0.286	0	-0.179	3.6
0	0	1.07	1	-1/7	0	0.214	0.71
0	0	-5/28	0	-1/7	1	-15/28	40/7

Chapter 9
The Transportation and Assignment Problems

→ Special cases of the integer linear programming model.

Prototype Example

Destinations

Shipping cost ($) per truckload

Sources	Warehouse 1	2	3	4	Output
Cannery 1	464	513	654	867	75
2	352	416	690	791	125
3	995	682	388	685	100
Allocation	80	65	70	85	

* When solving a transportation model, you should always check whether the sum of the allocations and the sum of outputs are equal

~~Decis~~

Σoutputs $= 75 + 125 + 100$
$= 300$

ΣAllocations $= 80 + 65 + 70 + 85$
$= 300$

They're equal ✓.

Decision Variable:-

x_{ij} = number of trucks that should be shipped from cannery i to warehouse j.

$i = 1, 2, 3$

$j = 1, 2, 3, 4$

Objective:-

Minimize

$$z = \sum_{j=1}^{4}\sum_{i=1}^{3} S_{ij} x_{ij}$$

S_{ij} = shipping cost from cannery i to warehouse j.

e.g. $S_{11} = \$464$

Subject to:-

$$\sum_{i=1}^{3} x_{ij} = A_j \quad \forall j$$

A_j = Allocation for warehouse j

e.g $A_1 = 80$

$$\sum_{j=1}^{4} x_{ij} = O_i \quad \forall i$$

→ equality because the total output is equal to the total allocated amount.

O_i = Output of cannery i.

$$x_{ij} \geq 0 \quad \forall i, \forall j$$

9-1-4

		Unit manufacturing cost / $				
		Plant				
	1	2	3	4	5	Required Quantity
Product 1	31	29	32	28	29	600
2	45	41	46	42	43	1000
3	38	35	40	M	M	800
4(D)	0	0	0	0	0	600
	400	600	400	600	1000	

Total required quantity
= 600 + 1000 + 800
= 2400

Total output
= 400 + 600 + 400 + 600 + 1000
= 3000

Total required quantity < Total output

2400 < 3000.

However, they should be equal bcz this is a transportation model

↓ So, increase the lower side by either adding a dummy product or dummy plant.

In this case, we need to increase the total required quantity to 3000 by adding a dummy product, whose required quantity will be ~~equal to~~ 600, which is the difference.

(3000 - 2400 = 600)

* Plants 4 and 5 can't produce ~~produce~~ product 3, therefore we assign a very large cost (M) to force the model to avoid assigning any amount of product 3 to plants 4 and 5.

* ~~The dummy product is imaginary, and thus it doesn't have a real cost. Hence~~, the cost assigned to ~~it~~ the dummy is \$0. as this way ~~in this case,~~ the dummy product will have a very low cost, therefore ~~it~~ the dummy product is chosen when a plant produces very expensive products

Decision variables:-

x_{ij} = number of units of product i to be produced in plant j.

$i = 1, 2, 3, 4$

$j = 1, 2, 3, 4, 5$

Objective:-

minimize

$$z = \sum_{j=1}^{5} \sum_{i=1}^{4} C_{ij} \, x_{ij}$$

C_{ij} = cost of product i to be produced in plant j

Subject to:-

$$\sum_{i=1}^{4} x_{ij} = O_j \qquad O_j = \text{output of plant } j \quad \forall j$$

$$\sum_{j=1}^{5} x_{ij} = Q_i \qquad Q_i = \text{quantity of product } i \quad \forall i$$

$$x_{ij} \geq 0 \quad \forall i, \forall j$$

9-1-6

		Customer				
		1	2	3	4	Output
Plant	1	$800	$700	$500	$200	60
	2	$500	$200	$100	$300	80
	3	$600	$400	$300	$500	40
Requirement		40	60	at least 20	as much as possible	

Profit

	1	2	3	4
Min	40	60	20	0
Max	40	60	80	60

Total output = 60 + 80 + 40
= 180

Total requirements =
40 + 60 + 20 → مبيع بالمية حيانوموا الزبائن
= 120

Difference = 180 - 120
= $\boxed{60}$

60 units aren't promised to any customer, and customers 3 and 4 are willing to buy as much as possible of the remaining products.

Therefore, the maximum amount that customer 3 can receive is (20 + 60) = 80 and the maximum amount that customer 4 can receive is (0 + 60) = 60.

We divided customer 3 into 3 and 3' because customer 3 must receive 20 units however the customer is able to receive an extra 60, which is not obligatory. So, it's divided to ensure this.

		Customer					
		1	2	3	3'(5)	4	
Plant	1	800	700	500	500	200	60
	2	500	200	100	100	300	80
	3	600	400	300	300	500	40
	4(D)	-M	-M	-M	0	0	
Requirement		40	60	20	60	60	

Total requirements = 40 + 60 + 20 + 60 + 60
= 240

Totol output = 180

Add a dummy plant with output (240 - 180 = 60) to equate the requirements and output.

P_{ij} = profit of product produced in plant i sent to customer j

P_{41}, P_{42}, and P_{43} = -M because customers 1, 2, & 3 must receive a certain amount, which will be produced in an actual plant.

If we allow the model to give a value to x_{41}, x_{42}, x_{43}, this means that the customers won't receive the full amount, which is unacceptable. Therefore, we should make this impossible by assigning a very large negative number. (since this objective is to maximize the profit).

$P_{43'}$ and P_{44} = 0, because they won't receive products for sure, so in case they didn't, it won't create a problem

Decision variables:-

x_{ij} = number of units sent from plant i to customer j.

$i = 1, 2, 3, 4$

$j = 1, 2, 3, 3', 4$

5

Objective:-

Maximize

$$Z = \sum_{i=1}^{4} \sum_{j=1}^{5} P_{ij} x_{ij}$$

P_{ij} = profit / unit sent from plant i to customer j.

Subject to:-

$$\sum_{i=1}^{4} x_{ij} = Q_j \quad \forall j$$

$$\sum_{j=1}^{5} x_{ij} = O_i \quad \forall i$$

$$x_{ij} \geq 0 \;, \quad \forall i \,\&\, \forall j$$

Northern Airplane Company
(Example with Dummy Destination) Page 327

		Month of installation					Max
		1	2	3	4	5 (D)	Prod.
Month of Production	1	1.08	1.095	1.110	1.125	0	25
	2	M	1.110	1.125	1.140	0	35
	3	M	M	1.100	1.115	0	30
	4	M	M	M	1.130	0	10
Scheduled Inst.		10	15	25	20	30	

Total Prod. = 25 + 35 + 30 + 10
= 100

Total Inst. = 70

100 - 70 = 30

Decision Variables:-
x_{ij} = number of engines produced in month i for installation in month j.

$x_{21}, x_{31}, x_{41}, x_{32}, x_{42}, x_{43} = M$
e.g. bcz it's impossible to install an engine in month 1 when it's supposed to be produced in month 2.

Production must precede installation

→ If the engine is produced in a certain month and not installed in the same month, $\$0.015 \times 10^6 = \$15,000$ will be paid for storage for each extra month.

~~Decision variables:~~
~~Subject to:~~ Objective:
Minimize

$$z = \sum_{i=1}^{4} \sum_{j=1}^{5} c_{ij} x_{ij}$$

c_{ij} = cost of engine produced in month i & to be installed in month j

Subject to:-

$$\sum_{i=1}^{4} x_{ij} = I_j \qquad \forall j$$

I_j = Scheduled Installations for month j.

$$\sum_{j=1}^{5} x_{ij} = \cancel{P_j}\ P_i \qquad \forall i$$

P_i = Maximum Production for month i.

9-1-9

i	j	1	2 (P_1)	3	4	5 (P_2)	6	7 (P_3)	Max Prod.
		M_1	M_2	M3	M_1	M_2	M3	(D)	
1 M1	RT	15	(15+1)=16	(16+2)=18	16	(16+2)=18	19	0	10
2	OT	18	(18+1)=19	(19+2)=21	20	22	(22+1)=23	0	3
3 M2	RT	M	17	20	M	15	16	0	8
4	OT	M	20	22	M	18	19	0	2
5 M3	RT	M	M	19	M	M	17	0	10
6	OT	M	M	22	M	M	22	0	3
Contracted Sales		5	3	4	3	5	4	12	

Total Prod = 10 + 3 + 8 + 2 + 10 + 3 = 36

Total Sales = 24

36 - 24 = 12 → Dummy product.

Decision Variable:-

x_{ij} = number of units to be produced in time period i to be used in time period j.

j = 1, 2, 3 → first product
4, 5, 6 → Second product
7 → Dummy

i = 1 → M_1, RT
2 → M_2, OT
3 → M_2, R_T
4 → M_2, OT
5 → M_3, RT
6 → M_3, OT

9.2 A streamlined Simplex Method for the Transportation Problem

· Initial Basic Feasible (BF) solution:-

① North-West Corner Rule:-

9-2-1

		Destination 1	2	3	Supply
Source	1	$\boxed{6}$ 4	3	5	4
	2	0 ↓ 4 →	M 2 →	7 1 ↓	3
	3	3	4	3 2	2
Demand		4 0	$\boxed{2}$ 0	2 $\boxed{3}$	

Demand < Supply

Solve using Northwest corner rule:-

① Choose x_{11} (Northwest corner of the table) and allocate the minimum between supply and demand.

⇓

② · If the supply is less than the demand, move ~~to~~ ~~vertically~~ downwards one unit. (to x_{21})

· If the demand is less the supply, move ~~to~~ one unit to the right. (to x_{12})

· ~~If the demand is equal to the supply, move to x_{22} (along the diagonal) as either~~

③ Continue the process until an allocation is made in the southeast corner.

Note:
once an allocation is made, the allocated amount should be subtracted from the ~~either~~ ~~minimum~~ maximum between the supply and demand.

Allocations

Starting at x_{11} (Northwest corner)

Allocation = 4 Subtract 4 from the demand

4 - 4 = 0

Moving to x_{21}

min [Demand = 0 , supply = 3]

Allocated amount = 0.

Moving to x_{22}

min [D = 2 , S = 3]

D = 2 → Allocated amount → 2

Subtract 2 from S (3 - 2 = 1)

Moving to x_{23}

min [D = 3 , S = 1]

Allocation = 1.

subtract 1 from D (3 - 1 = 2)

Moving to x_{33} (Southeast corner)

min (D = 2 , S = 2)

Doesn't matter.

Allocation = 2.

* In conclusion, the allocations in the same column should be equal to the demand under that column. and allocations in the same row should be equal to the supply of that row.

Row 1 → x_{11} → 4 (equal to the supply)

Row 2 → $x_{22} + x_{23}$ → 2 + 1 = 3 (equal to the supply)

Column 3 → $x_{23} + x_{33}$ → 1 + 2 = 3 (equal to the demand)

Initial Total cost = (6 x 4) + (M x 2) + (7 x 1) + (3 x 2)

= 37 + 2M

So, according to the North-west corner rule.

The basic variables are :-

$X_{11} = 4$ $X_{21} = 0$ $X_{22} = 2$ $X_{23} = 1$

$X_{33} = 2$

The rest are nonbasic variables and their value is equal to 0.

② Vogel's Approximation method (VAM)

Source \ Destination	1	2	3	4	5	Supply	Row diff
1	16	16	13	22	17	50	16-13=3
2	14	14	13	19	15	60	14-13=1
3	19	19	20	23	M	50	19-19=0
4(D)	M	0	M	0	0	~~50~~ 20	0-0=0
Demand	30	20	70	~~30~~ 0	60		
Column diff.	16-14=2	14-0 =14	13-13=0	19-0=0 19	15-0 =15		

① For each row and column, find the difference btwn the smallest two costs (or values).

② Choose the largest difference which is 19 here.

this is related to column 4

choose the cell with the smallest value in col 4.

$X_{44} = 0$

Allocate the minimum between supply & demand.

Allocation at $X_{44} = 30$

Since the demand is completely used up for column 4, eliminate column 4 and repeat the steps.

	1	2	3	5	Supply	Row diff
1	16	16	13	17	50	16-13 = 3
2	14	14	13	15	60	14-13 = 1
3	19	19	20	M	50	19-19 = 0
4(D)	M	0	M	$\boxed{0}$	~~20~~ 0	0-0 = 0
	30	20	70	~~40~~ ~~60~~		
Col. diff.	2	14	0	15		

Min. cost

largest difference

$X_{45} \rightarrow \min(60, 20)$

$X_{45} = 20$

	1	2	3	5	Supply	Row diff.
~~1~~	~~16~~	~~16~~	~~13~~	~~17~~	~~50~~ 0	3
2	14	14	13	15	60	1
3	19	19	20	M	50	0
Demand	30	20	~~70~~ 20	40		
Col. diff.	2	2	0	2		

$X_{13} \rightarrow \min[50, 70]$

$X_{13} \rightarrow 50$ Eliminate row 1 as its supply is exhausted.

	1	2	3	5	Supply	Row diff
2	14	14	13	$\boxed{15}$	~~60~~ 20	1
3	19	19	20	M	50	0
Demand	30	20	20	~~40~~ 0		
Col. diff.	5	5	7	$\boxed{M-15}$		

$X_{25} \rightarrow \min[40, 60]$

$X_{25} \rightarrow 40$

	1	2	3	Supply	Row diff.
~~2~~	~~14~~	~~14~~	~~13~~	~~20~~ 0	1
3	19	19	20	60	0
Demand	30	20	~~20~~ 0		
Col diff	5	5	7		

$X_{23} \rightarrow 20$

	1	2	3
3	19	19	20
	30	20	0

$X_{31} = 30$

$X_{32} = 20$

$X_{33} = 0$

Basic variables:-

$X_{44} = 30$ $\quad$ $X_{25} = 40$

$X_{45} = 20$ $\quad$ $X_{23} = 20$

$X_{13} = 50$ $\quad$ $X_{31} = 30$

$X_{32} = 20$ $\quad$ $X_{33} = 0$.

$Z = 30(0) + 20(0) + 50(13) + 20(19)$
$+ 40(15) + 20(13) + 30(19) + 0(20)$

$= 2{,}460$

③ Russell's Approximation Method:-

Source \ Destination	1	2	3	4	5	Supply
1	16	16	13	22	17	~~50~~ ~~40~~ 0
2	14	14	13	19	15	~~60~~ ~~30~~ 0
3	19	19	20	23	M	~~50~~ 30
4 (D~~emand~~)	M	0	M	0	0	~~50~~
	~~30~~	~~20~~ 0	~~70~~ ~~30~~ 0	30	~~60~~ ~~10~~	

① find max. cost for each row and column

Iteration	$\bar{u}_1$	$\bar{u}_2$	$\bar{u}_3$	$\bar{u}_4$	$\bar{v}_1$	$\bar{v}_2$	$\bar{v}_3$	$\bar{v}_4$	$\bar{v}_5$	Largest -ve Δ_{ij}	Allocation
1	22	19	M	M	M	19	M	23	M	$\Delta_{45} = -2M$	$X_{45} = 50$
2	22	19	M		19	19	20	23	M	$\Delta_{15} = -5 - M$	$X_{15} = 10$
3	22	19	23		19	19	20	23		$\Delta_{13} = -29$	$X_{13} = 40$
4		19	23		19	19	20	23		$\Delta_{23} = -26$	$X_{23} = 30$
5		19	23		19	19		23		$\Delta_{21} = -24$	$X_{21} = 30$
6											$X_{31} = 0$
											$X_{32} = 20$
											$X_{34} = 30$
											$Z = 2{,}570$

$\bar{u} \rightarrow$ max in row

$\bar{v} \rightarrow$ max in column

* After each iteration, cancel out the exhausted row or column.

1st iteration → Row 4 was exhausted so it was cancelled out.

Iteration 1 :-

$\Delta_{11} = C_{11} - \bar{v}_1 - \bar{u}_1$
$= 16 - 22 - M \quad = -6 - M$
$\Delta_{21} = C_{21} - \bar{u}_2 - \bar{v}_1 \quad = -5 - M$
$\Delta_{31} = 19 - 2M$
$\Delta_{41} = -M$

$\Delta_{12} = -25$
$\Delta_{22} = -24$
$\Delta_{32} = -M$
$\Delta_{42} = -19 - M$

$\Delta_{13} = -9 - M$
$\Delta_{23} = -6 - M$
$\Delta_{33} = 20 - 2M$
$\Delta_{43} = -M$

$\Delta_{14} = -23$
$\Delta_{24} = -23$
$\Delta_{34} = -M$
$\Delta_{44} = -23 - M$

$\Delta_{15} = -5 - M$
$\Delta_{25} = -4 - M$
$\Delta_{35} = -M$
$\boxed{\Delta_{45} = -2M}$ largest.

Choose the largest Difference.

$\underbrace{\Delta_{15}, \Delta_{25}, \Delta_{35}, \Delta_{14}, \Delta_{24}, \Delta_{34}, \Delta_{12}, \Delta_{22}, \Delta_{32}}_{\text{won't change}}$

In general, if only one row or column is left for consideration, select the remaining variables ~~to be basic~~ associated with that row or column to be basic variables with the only feasible allocation.

* Note :-

If the ~~dem~~ row and column have the same remaining supply and demand, then arbitrarily select the row as the one to be eliminated.

Chapter 10
Network optimization models

A network consists of a set of points and a set of lines connecting certain pairs of the points.

points ⟶ Nodes
lines ⟶ Arcs

→ The arcs are identified by the nodes found at both ends.

e.g AB → arc
node node

~~Directed Arc~~:
The arcs of a network may have a flow of some type through them.

Directed Arc:-
→ flow is allowed in only one direction.

Undirected Arc:
→ flow is allowed in either (both) direction.

will be referred to it as links.

When making a decision on the flow through an undirected arc, it is allowed to assign flow simultaneously in ~~diffe~~ opposite directions, however in the end find the net flow.

Directed Network:
↓
a network consisting of directed arcs only.
(negative Flow is unacceptable → infeasible)

Undirected network:
↓
a network consisting of undirected arcs only.

* A network consisting of a mixture of directed and undirected arcs can be converted to a directed network by replacing each undirected arc by a pair ~~a~~ of directed arcs in opposite directions.
↓
then you can choose whether to interpret the flows simultaneously in opposite directions or to find the net flow.

Path:- between two nodes is a sequence of distinct arcs connecting these nodes.

Directed Path from node i to node j is a sequence of connecting arcs whose direction is toward node j.

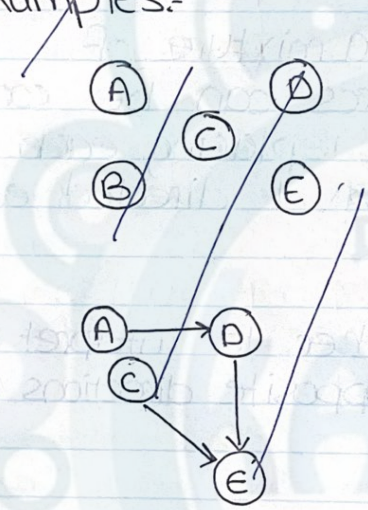
Undirected Path from node i to node j. is a sequence of connecting arcs whose direction can be either toward or away from node j.

* For undirected ~~arcs~~ paths, ~~it is~~ sometimes an assignment can be made in the wrong direction to reduce the flow in the right direction.

Connected Network:-
Every pair of nodes in the network has at least one undirected path between them.

Cycle:-
A path that begins and ends at the same node.

Examples:-



~~Un~~connected

~~Un~~connected

Arc Capacity:-
→ Maximum amount of flow that can be carried on a directed node.

Supply Node:-
→ Flow out exceeds flow in
e.g. factories

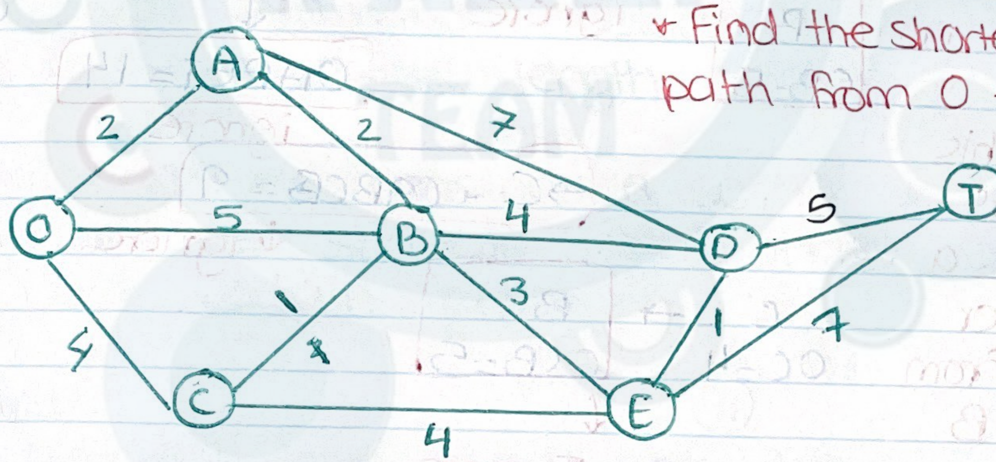
Demand Node:-
→ Flow in exceeds flow out
e.g. customers

Transshipment Node:-
→ Flow in equals flow out
e.g warehouses.

Find the shortest path

The Shortest-path problem

→ The objective of this problem is to find the shortest path (the path with the minimum total distance) from the origin to the destination.



* Find the shortest path from O to T.

Starting from O

A → B → D → OABDT

OA=2 OAB=4 OABD=8 = 13

$\boxed{\text{D OAD=9}}$ ignore

→ OABE=7 → OABED=8 ↓ OABEDT=13

↓ $\boxed{\text{OABET=14}}$ ignore

O $\boxed{\text{B OB=5}}$

Ignore this path as there is a shorter path from O to B (OAB).

$\boxed{\text{→C → OABC=9}}$ ↓ ignore

C → $\boxed{\text{B OCB=5}}$

OC=4

↓ Ignore.

cz OAB < OCB

The shortest path ~~can~~ ~~eit~~ is given by both OABDT and <u>OABEDT</u>.

↓ if the objective was to maximize the number of nodes, this would be the optimal distance.

The Minimum Spanning Tree

Similarities between the shortest path and the minimum spanning tree:-

① Undirected and connected network is being considered.

② The given information includes some measure of the positive length (distance, cost, time...)

③ Both involve choosing a set of links that have the shortest total length that satisfy a certain property.

Difference:-

* The shortest path problem ~~the~~ must provide a path between the origin and the destination, whereas the minimum spanning tree's objective is to find a path between each pair of nodes. (include all nodes)

↓

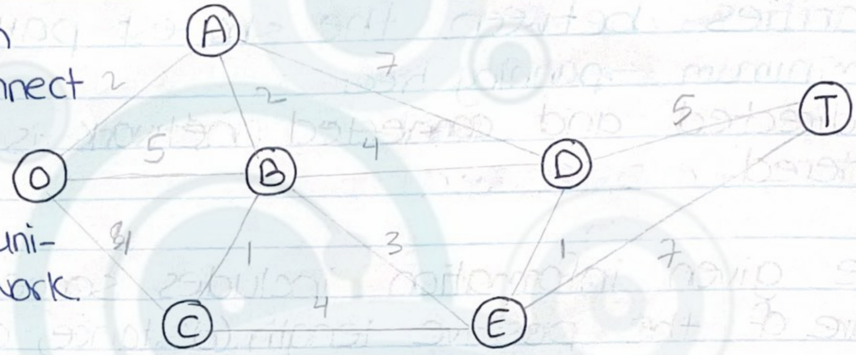
in such a way that minimizes the total length of the links inserted into the network.

* A network with n nodes requires (n-1) links to provide a path between each pair of nodes.

Seervade Park

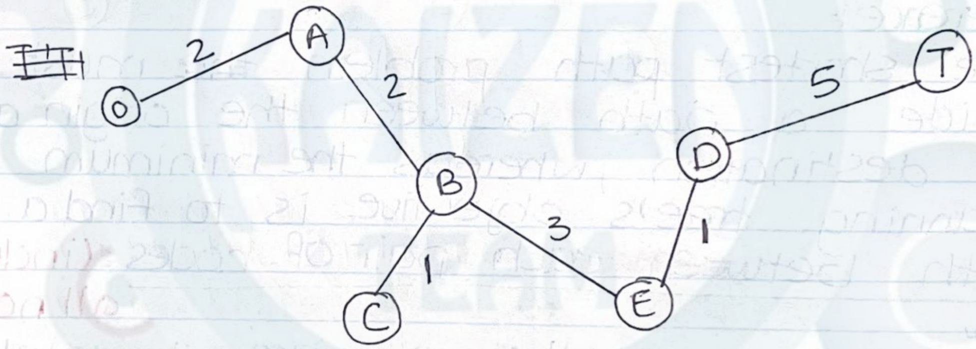
Find the minimum spanning tree:-

* The aim was to connect all nodes to build a telecommunication network.



↑ links drawn in pencil represent all the potential links.

Start from any desired point:-



* look for the closest unconnected node and form a link between them.

- The unconnected nodes to O are A, B, C.
  The closest is A.

- The unconnected nodes to O and A are D, B, C.
  The closest is B

* The closest unconnected nodes to ~~B~~ O, A, B are C, E, D.

The closest is C

* The closest unconnected nodes to O, A, B, C are E and D

The closest is E.

* The closest unconnected nodes to O, A, B, C, E are D and T.

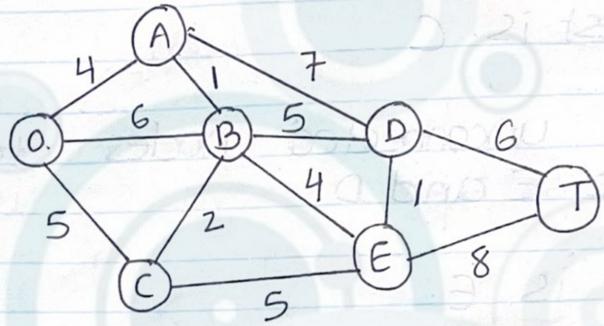
The closest is D.

Total distance = 2 + 2 + 1 + 3 + 1 + 5
= 14

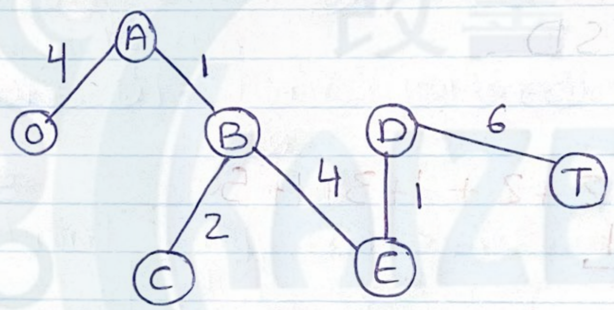
usually larger than the shortest path problem.

10-3-4

a)

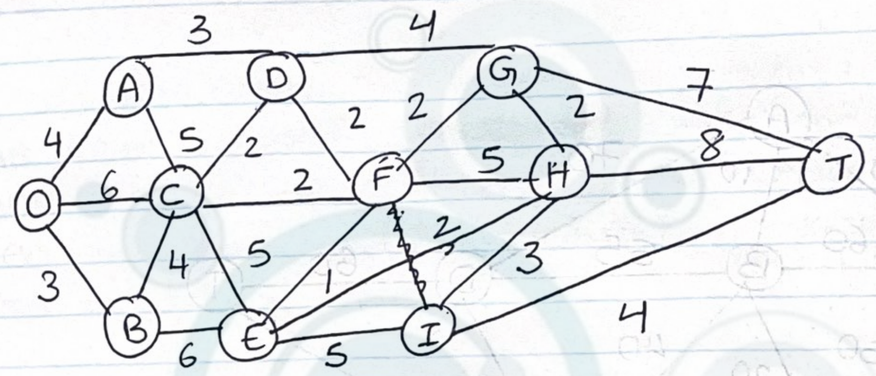


Minimum Spanning Tree.

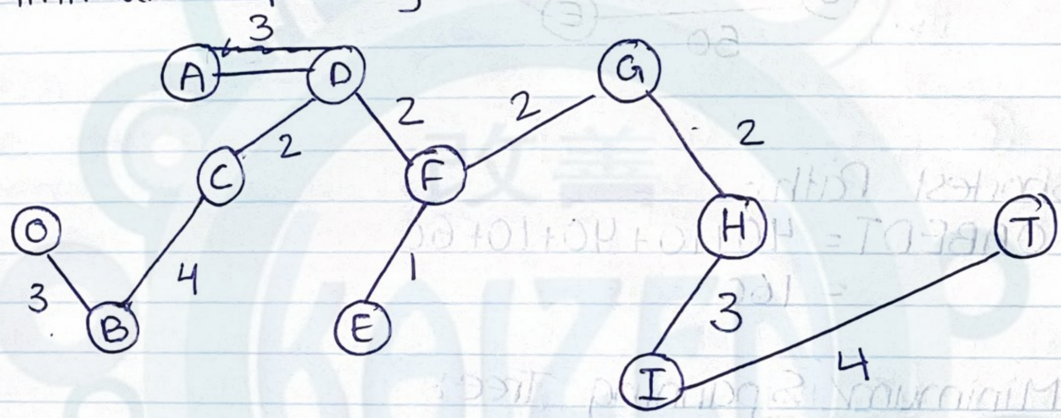


Total = 4+1 +2+4+1+6
= 18

b)

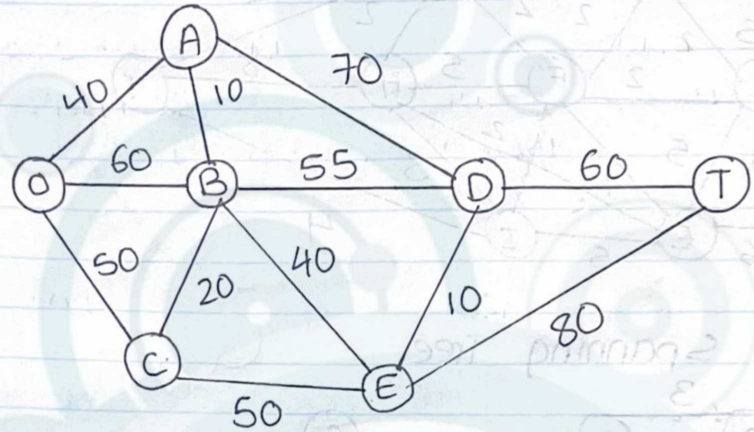


Minimum Spanning Tree



Total distance = 26

10-3-2

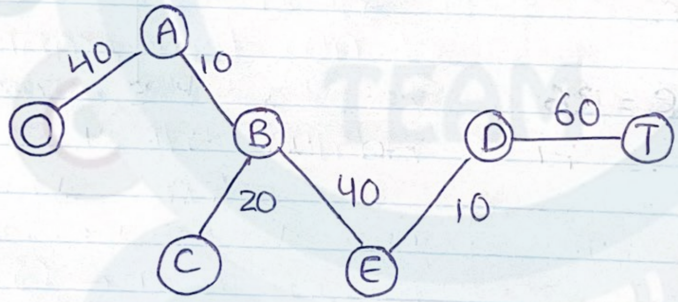


Shortest Path:-

OABEDT = 40+10+40+10+60

= 160

Minimum Spanning Tree:-



Total Distance

= 40 + 10 + 20 + 40 + 10 + 60

= 180

The Maximum Flow Problem

In Summary:-

1. All flow through a directed and ~~directed~~ connected network originates at one node, called the source, and terminates at one other node, called the sink.

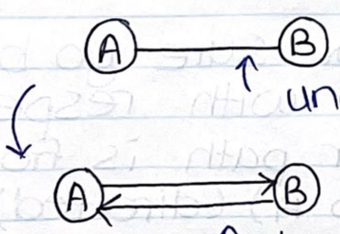
2. All the remaining nodes are transshipment nodes. (flow in equals flow out)

3. Flow through an arc is allowed only in the direction indicated by the arrowhead, where the maximum amount of flow is given by the capacity of that arc.
   ↓
   - At the source, all arcs point away from the node
   - At the sink, all arcs point ~~away~~ ~~towards~~ ~~into~~ into the node.

4. The objective is to maximize the total amount of flow from the source to the sink. (measured by ~~th~~ either the amount leaving the source or the amount entering the sink).

→ All arcs should be directed. In case there was an undirected arc, it should be replaced by ~~to~~ two opposite directed arcs.

e.g

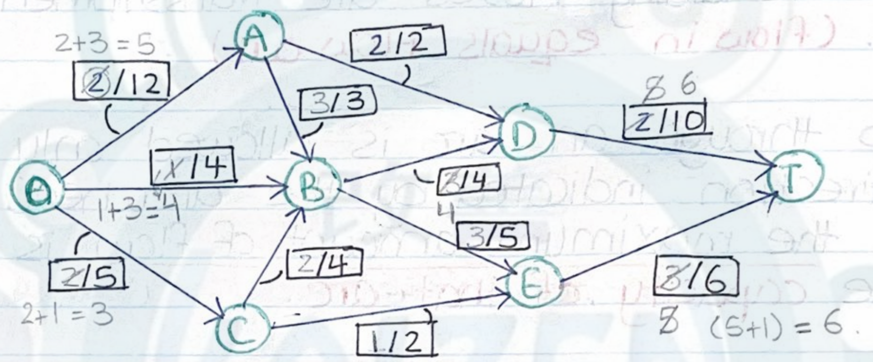


↑ undirected arc

↑ two opposite directed arcs.

Seervada Park.

· The objective is to find the maximum number of trams that can be sent from the park entrance (the source) to the scenic wonder. (the sink).



$\boxed{0/12}$ O → A

12 is the maximum number of trams that can be sent from O to A.

0 is the number of units that will be sent.

O → is a supply node. (source)
(flow in < flow out)

T → is a demand node. (sink)
(flow in > flow out)

The trams will for sure go back to O, however this is with respect to the sightseers. (Their path is from O to A) (directed)

Paths | Max flow of ~~this~~ each path

OADT

$\min(OA, AD, DT)$
$\min(12, 2, 10) = 2$

OABDT

*A is full, cz flow in reached max. cap.

$\min(OA, AB, BD, DT)$
$\min(10, 3, 4, 8) = 3$

* Sending a tram from B to A (opposite direction) actually means reducing the number of trams to be sent from A to B.
⇓
This is allowed as long as the number remains greater than or equal to zero.
⇓
this can be done because as the objective is to optimize the whole network, not a single path only.
(This step is usually applied when needed for the transhipment nodes).

OBDT

(D is full now cz flow in reached max. cap.)

$\min(OB, BD, DT)$
$\min(4, 1, 5) = 1$

OBET

$\min(3, 5, 6) = 3$

OCBET

(B is full now, as flow out reached the maximum capacity)

$\min(OC, CB, BE, ET)$
$= \min(5, 4, 2, 3) = 2$

OCET

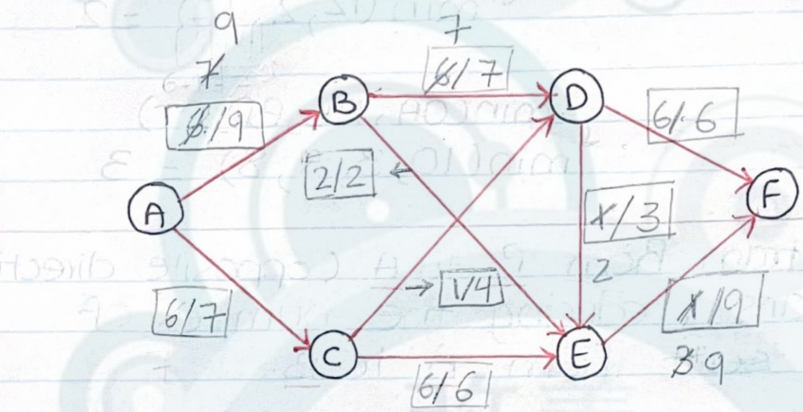
$\min(OC, CE, ET)$
$\min(3, 2, 1) = 1$

↓
Now, maximum flow is reached.
↓
bcz flow in to D is achieved, so no more flow out is possible & Flow out of E is achieved. E and D are connected to T.

Maximum Flow = 6 + 6

= 12.

10-5-6



ABDF

min(9, 7, 6) = 6 (DF is full now)

ABDEF

min(3, 1, 3, 9) = 1 (BD is full now)

ABEF

min(2, 2, 8) = 2 (AB is full now) & BE

ACEF

min(7, 6, 6) = 6 (CE is full now) and EF

We reached the maximum flow possible as the flow out of the supply node has reached the maximum capacity.

Maximum flow = 9 + 6

= 15

r transportation and assignment mode

no. of basic variables =

$n + m - 1$

umber of olumns

number of rows.