



Industrial Control Systems

Chapter Three: Laplace Transform

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العلاقة تاجبة مع الزمن $v = \frac{I}{R}$ ما بتغير سوتو مع الزمن

Linear Time-invariant Systems

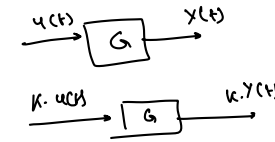
Relation between input + output.
علاقة انظركم (المدخل مع المخرج)
بتغير زانق النظام

Linearity: A system S is linear if it satisfies both

- **Homogeneity:** If $y = Sx$, and a is a constant then

$$ay = S(ax).$$

if the input is multiplied by a then the output is also multiplied by a (any change done on the input should be done on the output)



- **Superposition:** If $y_1 = Sx_1$ and $y_2 = Sx_2$, then

$$y_1 + y_2 = S(x_1 + x_2).$$



Combined Homogeneity and Superposition:

If $y_1 = Sx_1$ and $y_2 = Sx_2$, and a and b are constants,

$$ay_1 + by_2 = S(ax_1 + bx_2)$$

If H is time invariant, delaying the input and output both by a time τ should produce the same response

$$h_\tau(t) = h(t - \tau).$$

In terms of ODEs

- The syst is linear if and only if contain terms of the form:

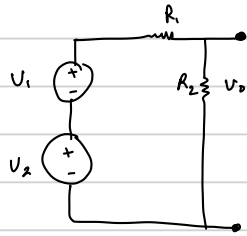
$$a \cdot x^{(k)}(t) \text{ and they must be added together.}$$

k^{th} = derivative of $x(t)$

e.g. of linear syst - $a \cdot x^{(k)}(t) + b \cdot x^{(k)}(t) + c = 0$
 $mL^2 \ddot{\theta} + mgL \theta = 0$

"non" - $mL^2 \ddot{\theta} + mgL \sin \theta = 0$

eg on SUPERPOSITION on linear systems

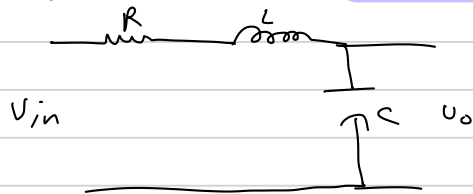


$$V_{01} = V_1 \frac{R_2}{R_1 + R_2}$$

$$V_{02} = V_2 \frac{R_2}{R_1 + R_2}$$

$$V_0 = V_{01} + V_{02} = (V_1 + V_2) \left[\frac{R_2}{R_1 + R_2} \right]$$

Dynamic linear system: (Linear time Invariant syst (LTI syst.))



$R, L, C \Rightarrow$ const with time

$V_{in}, V_{out} \Rightarrow$ are const with time for the same components

at any time V_{in} at any time (t)
 is always the same because the relation
 between (V_{in}) with the components is const.



Linear Time-invariance Systems

How they can be both !

- A system can have dynamics (i.e., it's a dynamic system) but still be time-invariant **if the laws that define its dynamics are constant.**
- For example, a simple RLC circuit with constant resistance, inductance, and capacitance is a dynamic system because its output is described by a differential equation involving past inputs. It is also time-invariant because the values of R , L , and C are constant, meaning its response to a specific input will be the same no matter when the input is applied.
- Conversely, a system that is time-variant has dynamics that change over time, such as a circuit where the resistance changes with temperature.



Laplace Transforms

- Important analytical method for solving ^{any syst consists of} *linear* ordinary differential equations.

Application to nonlinear ODEs? Must linearize first.

- Laplace transforms play a key role in important process control concepts and techniques.

- Examples:

- Transfer functions
- Frequency response
- Control system design
- Stability analysis



Definition of Laplace Transform

$$\mathbf{L}[f(t)] = F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

time domain (under $f(t)$) *frequency domain* (under $F(s)$)

$s = \sigma + j\omega$ (in a purple box)

کے لیے اہم ہے۔ اس سے ہمیں بہتر طریقہ پر مسئلہ حل کرنے میں مدد ملتی ہے۔

- Convert **time-domain** functions and operations into **Complex frequency-domain**

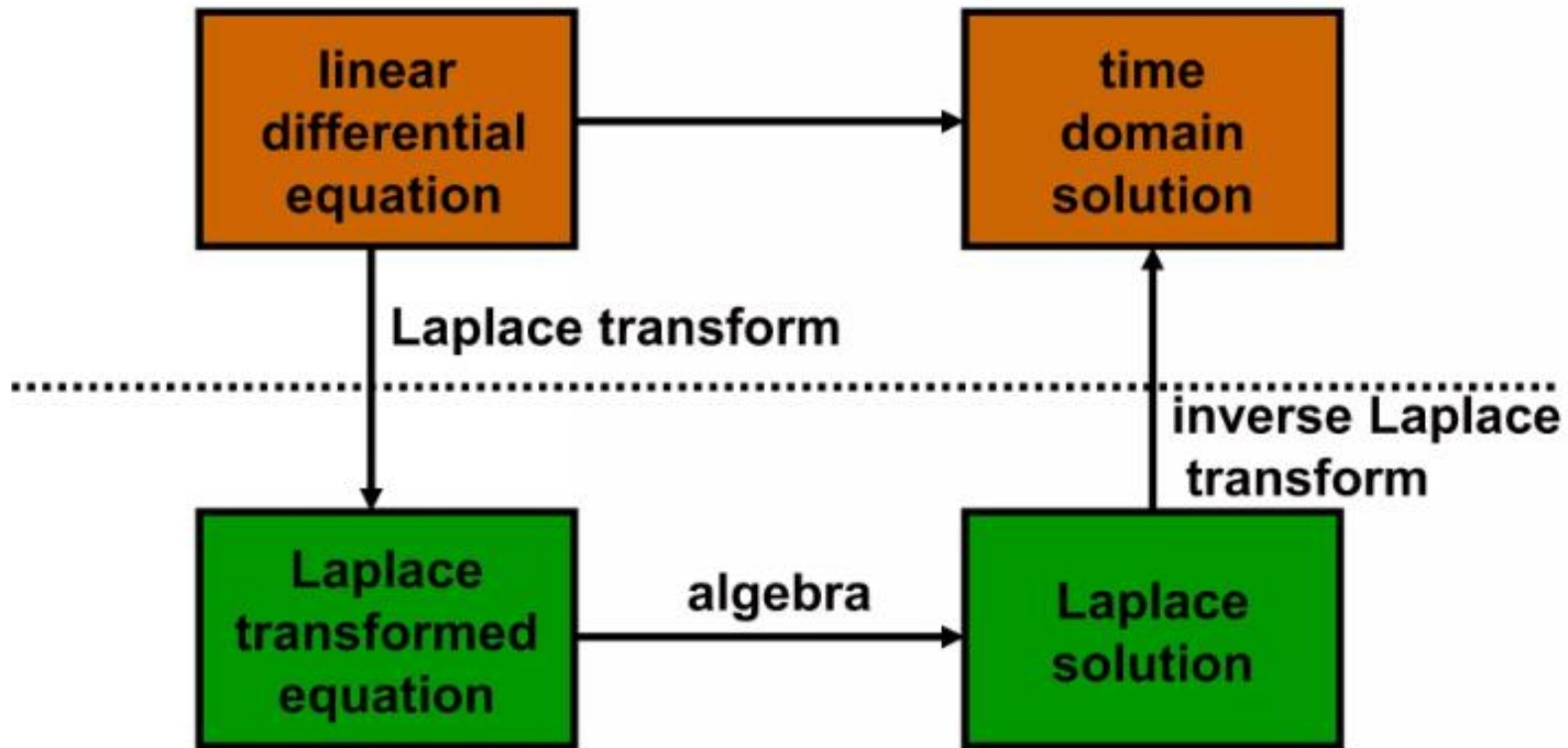
- $f(t) \rightarrow F(s) \quad (t \in R, s \in C)$
- Linear differential equations (LDE) \rightarrow algebraic expression in Complex plane

- Graphical solution for key LDE characteristics
- Discrete systems use the analogous z-transform



Definition of Laplace Transform

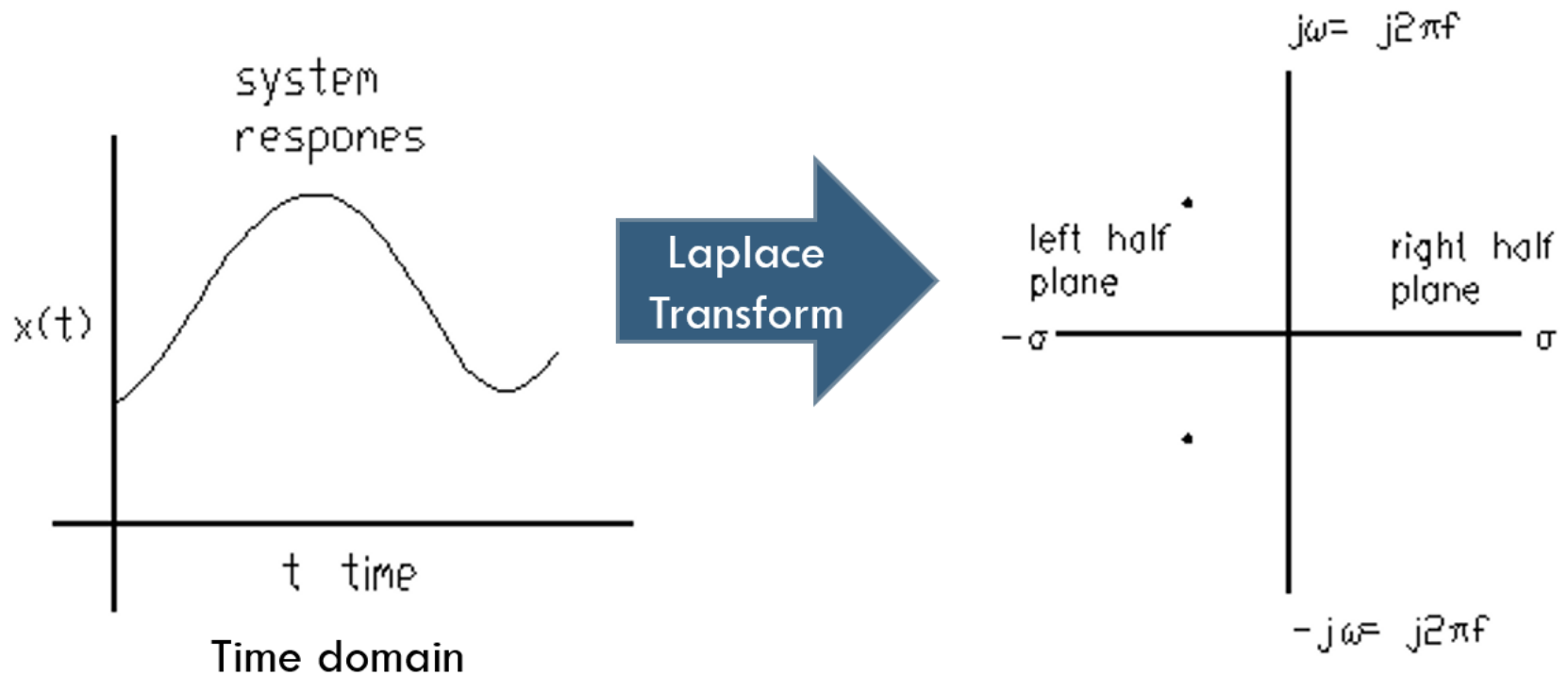
time domain



Laplace domain or
complex frequency domain

Laplace Transform

Laplace transform converts time domain problems into functions of a complex variable, s , that is related to the frequency response of the system

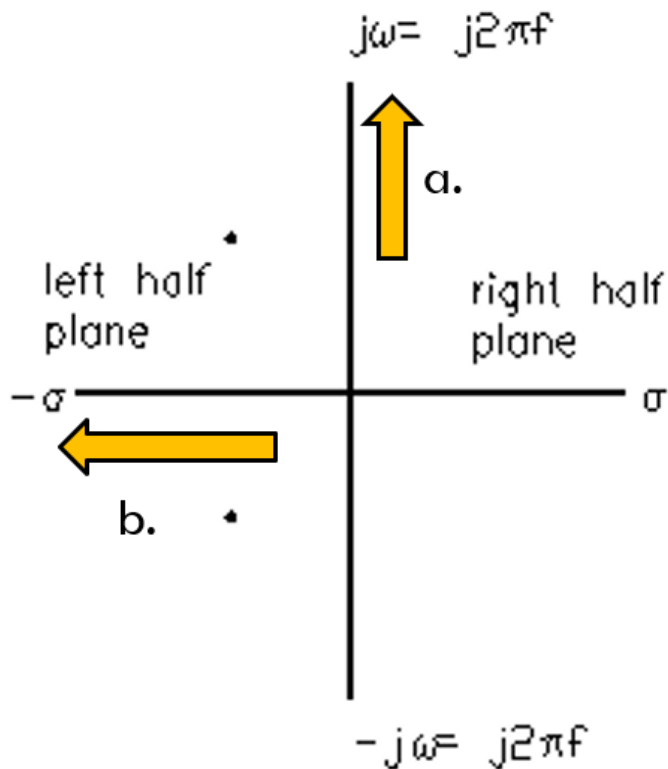


lesson10et438a.pptx **Complex frequency**

Handwritten notes: $s = \sigma + j\omega$

Laplace Transform

Complex frequency combines transient response with sinusoidal steady-state response to get total response of system to input



$$s = \sigma + j\omega$$

Complex Frequency

σ = exponential decay/increase constant that is related to time constants of systems transient response. $RC = L/R = \sigma$ in circuit analysis

$e^{\sigma t}$ Exponentially increasing function over time

$e^{-\sigma t}$ Exponentially decreasing function over time

- a.) Higher frequency
- b.) Faster time constants

Laplace Transform

The radian frequency $\omega = j2\pi f$ same frequency used in phasor analysis

Laplace related to sine response through Euler's Identity.
Euler's relates complex exponentials to sine and cosine time functions

$$e^{j\omega t} = \cos(\omega t) + j \cdot \sin(\omega t)$$

$$e^{-j\omega t} = \cos(\omega t) - j \cdot \sin(\omega t)$$

Adding and subtracting the above relationships gives the exponential forms of sine and cosine

Laplace Transform

Add the identities

$$\begin{aligned} e^{j\omega t} &= \cos(\omega t) + j \cdot \sin(\omega t) \\ e^{-j\omega t} &= \cos(\omega t) - j \cdot \sin(\omega t) \end{aligned} \quad \left. \vphantom{\begin{aligned} e^{j\omega t} \\ e^{-j\omega t} \end{aligned}} \right\} +$$

$$e^{j\omega t} + e^{-j\omega t} = 2 \cdot \cos(\omega t)$$

$$\frac{e^{j\omega t} + e^{-j\omega t}}{2} = \cos(\omega t)$$

حقیقی حصہ
میں سے سینا اجتناب

Exponential form of Cosine

Since $e^{st} = e^{(\sigma + j\omega)t} = e^{\sigma t} \cdot e^{j\omega t}$

Subtract the identities

$$\begin{aligned} e^{j\omega t} &= \cos(\omega t) + j \cdot \sin(\omega t) \\ e^{-j\omega t} &= \cos(\omega t) - j \cdot \sin(\omega t) \end{aligned} \quad \left. \vphantom{\begin{aligned} e^{j\omega t} \\ e^{-j\omega t} \end{aligned}} \right\} -$$

$$e^{j\omega t} - e^{-j\omega t} = 2j \cdot \sin(\omega t)$$

$$\frac{e^{j\omega t} - e^{-j\omega t}}{2j} = \sin(\omega t)$$

Exponential form of Sine

Laplace can give complete response: dc transient and steady-state sinusoidal

Laplace Transform

Time Domain Function

Laplace Domain Function

$\delta(t)$ Impulse



1

Value at zero

$u_s(t)$ Unit Step



$\frac{1}{s}$

e^{-at}



$\frac{1}{s+a}$

e^{at}



$\frac{1}{s-a}$

$\sin(\omega t)$



$\frac{\omega}{s^2 + \omega^2}$

$\cos(\omega t)$



$\frac{s}{s^2 + \omega^2}$

t Linear ramp (slope 1)



$\frac{1}{s^2}$



Note: time functions multiplied by constants give Laplace function multiplied by constant

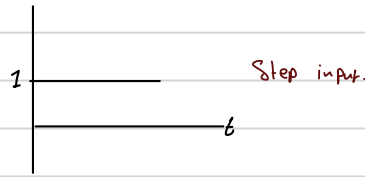
Examples:

$$5 \cdot u_s(t) \rightarrow \frac{5}{s}$$

$$3 \cdot \sin(4t) \rightarrow 3 \cdot \left(\frac{4}{s^2 + 16} \right) \quad \omega = 4$$

$$f(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} 1 \cdot e^{-st} dt = \left[-\frac{1}{s} e^{-st} \right]_0^{\infty} \\ = \frac{1}{s}$$



easy way to check a dynamic syst
is by doing a step response (consider a
standard step input)

Laplace for the first derivative of $f(t)$

$$\mathcal{L}[\dot{f}(t)] = \int_0^{\infty} f'(t) e^{-st} dt$$

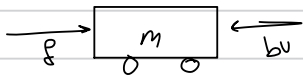
first derivative

Integration by Part

$$\begin{aligned} \int u dv &= uv - \int v du \\ &= f(t) e^{-st} - \int_0^{\infty} f(t) (-s e^{-st}) dt \\ &= -f(0) + s \int_0^{\infty} f(t) e^{-st} dt \\ &= s F(s) - f(0) \end{aligned}$$

Laplace for 2nd derivative

$$\mathcal{L}[\ddot{f}(t)] = s^2 F(s) - s f(0) - \dot{f}(0)$$



$$F = m\dot{v} + bv$$

$$\begin{aligned} \dot{v} &= \frac{F}{m} - \frac{bv}{m} \\ &= \frac{1}{m} (F - bv) \end{aligned}$$

block diagram



لپلاس انورس
(2) تبدیل به جریب

Laplace transform

$$\begin{aligned} \mathcal{L}[m\dot{v} + bv] &= \mathcal{L}[F] \\ m \mathcal{L}[\dot{v}] + b \mathcal{L}[v] &= \mathcal{L}[F] \\ m(sV(s) - v(0)) + bV(s) &= F(s) \\ (ms + b)V(s) &= F(s) + m v(0) \\ V(s) &= \frac{1}{ms + b} F(s) + \frac{m v(0)}{ms + b} \end{aligned}$$

$$V(s) = \frac{F(s)}{m} \left[\frac{1}{s(s + \frac{b}{m})} \right] + \frac{v(0)}{s + \frac{b}{m}}$$

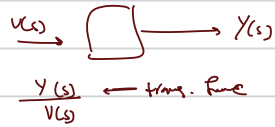
$$\mathcal{L}^{-1} \left[V(s) = \frac{F(s)}{m} \left[\frac{1}{s(s + \frac{b}{m})} \right] + \frac{v(0)}{s + \frac{b}{m}} \right]$$

Table 3.1 (13)

$$\frac{F(s)}{m} (1 - e^{-bt/m}) + v(0) e^{-(b/m)t}$$

$$F(s) = \frac{F_0}{s} \Rightarrow \text{step function}$$

to connect output + input \Rightarrow transfer function



always assume initial conditions = 0

$$V(s) = F(s) \frac{1}{ms+b}$$

$$\frac{V(s)}{F(s)} = \frac{1}{ms+b}$$

block diagram



Slide
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Laplace Transform for Elementary Functions

1) $x(t) = \delta(t)$ **Impulse signal**

Solution

$$\begin{aligned} \mathcal{L}[\delta(t)] &= \int_0^{\infty} \delta(t)e^{-st} dt \\ &= e^{-s(0)} = 1 \end{aligned} \quad \because \delta(t) = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{for } t \neq 0 \end{cases}$$

$\mathcal{L}^{\infty} = \text{Zer}$

2) $x(t) = u(t)$ **Step signal**

Solution

$$\mathcal{L}[u(t)] = \int_0^{\infty} u(t)e^{-st} dt = \int_0^{\infty} 1e^{-st} dt = \left[\frac{e^{-st}}{-s} \right]_0^{\infty} = \frac{1}{s}$$

$$\because u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

Laplace Transform for Elementary Functions

3) $L(1) = \frac{1}{s}$ **Constant**

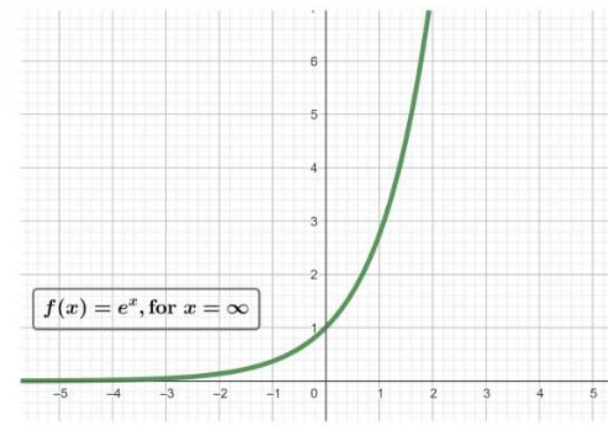
Solution

$$L(1) = \int_0^{\infty} e^{-st} \cdot 1 dt = \left[\frac{e^{-st}}{-s} \right]_0^{\infty} = \frac{1}{s}, (s > 0)$$

4) $L(e^{at}) = \frac{1}{s - a}$ **Exponential signal**

Solution

$$L(e^{at}) = \int_0^{\infty} e^{-st} \cdot e^{at} dt = \int_0^{\infty} e^{-(s-a)t} dt = \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^{\infty} = \frac{1}{s - a} \text{ if } s > a$$



Laplace Transform for Elementary Functions

5) $L[e^{-at}] = \frac{1}{s+a}$, $s > -a$ **Exponential signal**

6) $L[\sinh at] = \frac{a}{s^2 - a^2}$

Solution

W.k.t $\sinh at = \frac{e^{at} - e^{-at}}{2}$

$$\begin{aligned} L(\sinh at) &= L\left[\frac{e^{at} - e^{-at}}{2}\right] = \frac{1}{2}[L(e^{at}) - L(e^{-at})] \\ &= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{a}{s^2 - a^2}, s > |a| \end{aligned}$$

Laplace Transform for Elementary Functions

7) $\mathcal{L}[\cosh at] = \frac{s}{s^2 - a^2}, s > |a|$ **Hint** $\cosh at = \frac{e^{at} + e^{-at}}{2}$

8) $x(t) = \cos \omega_0 t u(t)$

$$\begin{aligned}\mathcal{L}[\cos \omega_0 t u(t)] &= \frac{1}{2} \mathcal{L}[e^{j\omega_0 t} u(t) + e^{-j\omega_0 t} u(t)] \\ &= \frac{1}{2} \left[\frac{1}{s - j\omega_0} + \frac{1}{s + j\omega_0} \right] = \frac{s}{s^2 + \omega_0^2}\end{aligned}$$

9) $x(t) = \sin \omega_0 t u(t)$

Laplace Transform for Elementary Functions

$$8) \quad L[\sin at] = \frac{a}{s^2 + a^2} \text{ and } L[\cos at] = \frac{s}{s^2 + a^2}, s > 0$$

Solution

Using Euler's Formula

$$e^{iat} = \cos at + i \sin at \quad \text{----> (1)}$$

$$\begin{aligned} L[\cos at + i \sin at] &= L[e^{iat}] = \frac{1}{s - ia} * \frac{(s + ia)}{(s + ia)} \\ &= \frac{s + ia}{s^2 + a^2} = \frac{s}{s^2 + a^2} + i \frac{a}{s^2 + a^2} \quad \text{----> (2)} \end{aligned}$$

Compare (1) and (2)

$$\text{Real part } L[\cos at] = \frac{s}{s^2 + a^2} \quad \text{Imaginary part } L[\sin at] = \frac{a}{s^2 + a^2}$$

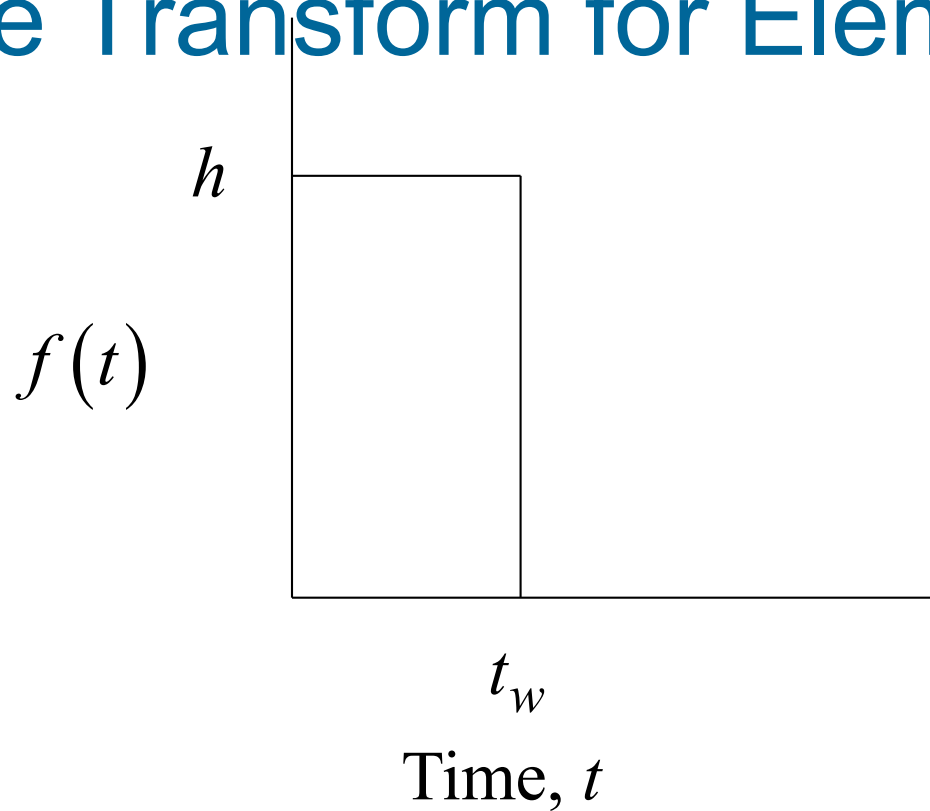
Laplace Transform for Elementary Functions

9) Rectangular Pulse Function

It is defined by:

$$f(t) = \begin{cases} 0 & \text{for } t < 0 \\ h & \text{for } 0 \leq t < t_w \\ 0 & \text{for } t \geq t_w \end{cases} \quad (3-20)$$

Laplace Transform for Elementary Functions



$$\begin{aligned} F(s) &= \int_0^{t_w} h e^{-st} dt \\ F(s) &= h \int_0^{t_w} e^{-st} dt \\ &= h \left[-\frac{1}{s} e^{-st} \right]_0^{t_w} \\ &= h \left[-\frac{1}{s} e^{-st_w} + \frac{1}{s} \right] \\ &= \frac{h}{s} (1 - e^{-st_w}) \end{aligned}$$

The Laplace transform of the rectangular pulse **function** is given by

$$F(s) = \frac{h}{s} (1 - e^{-t_w s}) \quad (3-22)$$

Laplace Transform for Elementary Functions

Match the following time functions to correct Laplace domain function using the transform pairs.

a.) $10t$ (4) $t \cos = \frac{1}{s^2}$

$\frac{1}{s+2}$ (1)

(3) $\frac{1}{s-5}$

b.) $t \cdot e^{-at}$ (6) $t \cdot e^{-at} = \frac{n!}{s^{n+1}}$

$3 \cdot \left(\frac{s}{(s^2+1)} \right)$ (2)

(4) $\frac{10}{s^2}$

c.) e^{5t} (3)

(5) $\frac{1}{(s+a)^2}$

d.) e^{-2t} (1)

e.) $3 \cdot \cos(t)$ (2)

Laplace Transform for Elementary Functions

Match the following time functions to correct Laplace domain function using the transform pairs.

a.) $10t$ (4)

$$\frac{1}{s+2} \text{ (1)}$$

$$\text{(3)} \frac{1}{s-5}$$

b.) $t \cdot e^{-at}$ (5) Laplace table 3.2 textbook

$$3 \cdot \left(\frac{s}{(s^2+1)} \right) \text{ (2)}$$

$$\text{(4)} \frac{10}{s^2}$$

c.) e^{5t} (3)

$$\text{(5)} \frac{1}{(s+a)^2}$$

d.) e^{-2t} (1)

e.) $3 \cdot \cos(t)$ (2)

Laplace Transform for Elementary Functions

	$f(t)$	$F(s)$
1	$\delta(t)$	1
2	$\delta(t - a)$	e^{-as}
3	$u_0(t)$	$\frac{1}{s}$
4	$tu_0(t)$	$\frac{1}{s^2}$
5	$t^n u_0(t)$	$\frac{n!}{s^{n+1}}$
6	$e^{-at} u_0(t)$	$\frac{1}{s + a}$
7	$t^n e^{-at} u_0(t)$	$\frac{n!}{(s + a)^{n+1}}$
8	$\sin(\omega t) u_0(t)$	$\frac{\omega}{s^2 + \omega^2}$
9	$\cos(\omega t) u_0(t)$	$\frac{s}{s^2 + \omega^2}$
10	$e^{\ominus at} \sin(\omega t) u_0(t)$	$\frac{\omega}{(s \oplus \bar{a})^2 + \omega^2}$
11	$e^{\ominus at} \cos(\omega t) u_0(t)$	$\frac{s + a}{(s \oplus \bar{a})^2 + \omega^2}$

Laplace Transform Theorems

Laplace of an unknown function

$$\mathcal{L}(f_1(t)) = F_1(s)$$

Capitalize unknown function name
Replace t with s

Laplace
Operator
Symbol

Examples

$$\mathcal{L}(i_1(t)) = I_1(s)$$

$$\mathcal{L}(v_1(t)) = V_1(s)$$

transform function

$$G(s) = \frac{\text{output Laplace transform}}{\text{input Laplace transform}} = \frac{Y(s)}{X(s)}$$

Linearity of transform - can multiply by constant

If $\mathcal{L}(f_1(t)) = F_1(s)$ and $\mathcal{L}(f_2(t)) = F_2(s)$

Then $\mathcal{L}(a \cdot f_1(t) + b \cdot f_2(t)) = a \cdot F_1(s) + b \cdot F_2(s)$

Transfer Function:

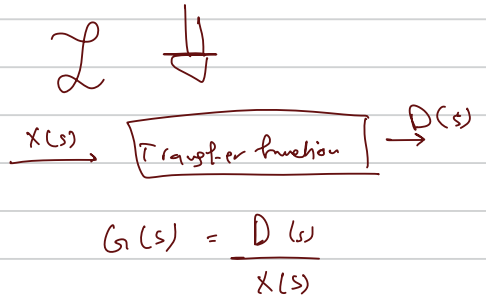
$$G(s) = \frac{U(s)}{F(s)} = \frac{1}{ms + b}$$

$$= \frac{\mathcal{L}[\text{output of the syst.}]}{\mathcal{L}[\text{Input of the syst.}]}$$

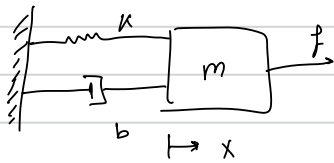
Caplace for linear diff eqn of the syst $(a \frac{d^m x}{dt^m} + b \frac{d^{(k-1)} x}{dt^{(k-1)}} + c x = D)$ LDE: linear diff eqn

Transfer function only defined for **LTI Syst** linear time invariant

Transfer function always represents input-output relation when **all initial conditions are zero**



ex 3:



⊛ Damper (Shock absorber)

slower response than the spring.

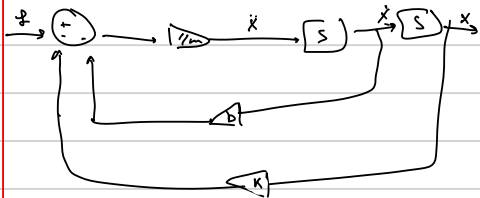
SOL

time domain

$$m \ddot{x} + kx + b\dot{x} = F$$

$$m \ddot{x} = F - b\dot{x} - kx$$

$$\ddot{x} = \frac{1}{m} [F - b\dot{x} - kx]$$



s-domain

$$\mathcal{L} [m \ddot{x} + kx + b\dot{x}] = \mathcal{L} [F]$$

$$m [s^2 X(s) - sX(0) - \dot{X}(0)] + k X(s) + b [sX(s) - X(0)] = F(s)$$

$$X(s) [ms^2 + bs + k] = F(s) + (ms + b) X(0) + m \dot{X}(0)$$

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$



Inverse Laplace Transform

Laplace Transform for Derivatives

Laplace Transform turns derivative into multiplication by s

If $\mathcal{L}(f_1(t)) = F_1(s)$

Then $\mathcal{L}\left(\frac{d}{dt}f_1(t)\right) = s \cdot F_1(s) - f_1(0)$

Subtract any
non-zero
initial
conditions

For higher order derivatives

0 initial conditions reduces formula to

$$\mathcal{L}\left(\frac{d^2}{dt^2}f_1(t)\right) = s \cdot (s \cdot F_1(s) - f_1(0)) - \frac{d}{dt}f_1(0)$$

$$\mathcal{L}\left(\frac{d^2}{dt^2}f_1(t)\right) = s^2 \cdot F_1(s)$$

Laplace Transform for Integrals

Laplace turns integration into division by s

$$\text{If } \mathcal{L}(f_1(t)) = F_1(s)$$

$$\text{Then } \mathcal{L}\left(\int \frac{d}{dt} f_1(t) dt\right) = \frac{1}{s} \cdot F_1(s)$$

*Integration
by parts.*

Examples from circuit analysis:

Capacitor voltage

$$v_C(t) = \frac{1}{C} \cdot \int i_C(t) dt$$

$$\mathcal{L}(v_C(t)) = \mathcal{L}\left(\frac{1}{C} \cdot \int i_C(t) dt\right)$$
$$V_C(s) = \frac{1}{C} \cdot \left(\frac{1}{s}\right) \cdot I_C(s) = \left[\frac{1}{C \cdot s}\right] \cdot I_C(s)$$

Laplace Transform for s and t shifting ^{time shift}

$$L[\underline{e^{at}} f(t)](s) = F(s - a)$$

time shift

Example:

$$L[\cos bt] = \frac{s}{s^2 + b^2}$$

$$\Rightarrow L[e^{at} \cos bt] = \frac{s - a}{[(s - a)^2 + b^2]}$$

$$L[H(t - a) f(t - a)](s) = e^{-as} F(s)$$

$$L^{-1}[e^{-as} F(s)](t) = H(t - a) f(t - a)$$

$$L^{-1}[FG] = f * g$$

Laplace Transform Initial and Final Values

Initial and Final Values

The initial-value and final-value properties allow us to find the initial value $f(0)$ and $f(\infty)$ of $f(t)$ directly from its Laplace transform $F(s)$.

$$f(0) = \lim_{s \rightarrow \infty} sF(s)$$

Initial-value theorem

$$f(\infty) = \lim_{s \rightarrow 0} sF(s)$$

Final-value theorem

The Inverse of Laplace Transform

Suppose $F(s)$ has the general form of

$$F(s) = \frac{N(s) \dots \text{numerator polynomial}}{D(s) \dots \text{denominator polynomial}}$$

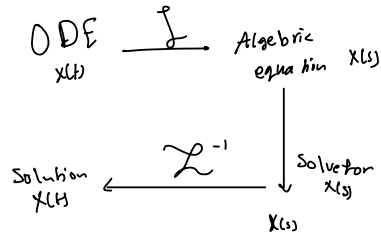
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بمناسبة بعد ذلك
Partial expansion.

The finding the inverse Laplace transform of $F(s)$ involves two steps:



1. Decompose $F(s)$ into simple terms using partial fraction expansion.
2. Find the inverse of each term by matching entries in Laplace Transform Table.



$$Y = \frac{X^3 + X - 1}{X^2 + 3X + 2}$$

درجه آسانتر
من درجه المقام

$$\begin{array}{r} X^2 + 3X + 2 \overline{) X^3 + 0X^2 + X - 1} \\ \underline{-X^3 + 3X^2 + 2X} \\ -3X^2 - X - 1 \\ \underline{-(-3X^2 - 9X - 6)} \\ 8X + 5 \end{array}$$

$X-3$ → Quotient
Remainder

$$Y = X - 3 + \frac{8X + 5}{X^2 + 3X + 2}$$

درجه المقام
من درجه المقام
بقسمه

$\frac{B(x)}{A(x)}$ order of $A(x) >$ order of $B(x)$ → Partial Fraction

Partial Fraction Expansion.

$$\frac{8X + 5}{(X+1)(X+2)} = \frac{a}{X+1} + \frac{b}{X+2}$$

$$= \frac{a(X+2) + b(X+1)}{(X+1)(X+2)}$$

$$\frac{8X + 5}{X^2 + 3X + 2} = \frac{(a+b)X + (2a+b)}{(X+1)(X+2)}$$

$$\frac{X^3 + X - 1}{X^2 + 3X + 2} = (X - 3) - \frac{3}{X+1} + \frac{11}{X+2}$$

بجود ال Linear ال Function
Z

$$a = -3 \quad b = 11$$

The Inverse of Laplace Transform

Example 1

Find the inverse Laplace transform of

$$F(s) = \frac{3}{s} - \frac{5}{s+1} + \frac{6}{s^2+4}$$

Solution:

$$\begin{aligned} f(t) &= L^{-1}\left(\frac{3}{s}\right) - L^{-1}\left(\frac{5}{s+1}\right) + L^{-1}\left(\frac{6}{s^2+4}\right) \\ &= (3 - 5e^{-t} + 3\sin(2t))u(t), \quad t \geq 0 \end{aligned}$$

The Inverse of Laplace Transform

أشهرها الوجهة عنفات البانعة ص باسرة
درجة المتكامل (درجة البسط

PARTIAL FRACTION EXPANSION

$\frac{\text{Polynomial}}{\text{Polynomial}}$

1) Distinct Real Roots of D(s)

$$F(s) = \frac{96(s+5)(s+12)}{s(s+8)(s+6)}$$

درجة المتكامل أكبر من درجة البسط
عنفات: هناك صابغة اعلى قسمة

$$s_1 = 0, s_2 = -8$$

$$s_3 = -6$$

The Inverse of Laplace Transform

بجواب الطمرينة بالمقام لكن جزئياً و بعضاً فية اي س من الحتم .

1) DISTINCT REAL ROOTS

$$F(s) = \frac{96(s+5)(s+12)}{s(s+8)(s+6)} \equiv \frac{K_1}{s} + \frac{K_2}{s+8} + \frac{K_3}{s+6}$$

- To find K_1 : multiply both sides by s and evaluates both sides at $s=0$
- To find K_2 : multiply both sides by $s+8$ and evaluates both sides at $s=-8$
- To find K_3 : multiply both sides by $s+6$ and evaluates both sides at $s=-6$

The Inverse of Laplace Transform

FIND K_1

$$\frac{96(s+5)(s+12)}{(s+8)(s+6)} \Big|_{s=0} \equiv K_1 + \cancel{\frac{K_2 s}{s+8} \Big|_{s=0}} + \cancel{\frac{K_3 s}{s+6} \Big|_{s=0}}$$

$$\therefore K_1 = \frac{96(5)(12)}{(8)(6)} = 120$$

The Inverse of Laplace Transform

FIND K_2

$$\frac{96(s+5)(s+12)}{s(s+6)} \Big|_{s \Rightarrow 8} \equiv \frac{K_1(s+8)}{s(s+6)} \Big|_{s \Rightarrow 8} + K_2 + \frac{K_3(s+8)}{s(s+6)} \Big|_{s \Rightarrow 8}$$

$$\therefore K_2 = \frac{96(-3)(4)}{(-8)(-2)} = -72$$

The Inverse of Laplace Transform


FIND K_3

$$\frac{96(s+5)(s+12)}{s(s+8)} \Big|_{s=6} \equiv \frac{K_1(s+6)}{s(s+8)} \Big|_{s=6} + \frac{K_2(s+6)}{s(s+8)} \Big|_{s=6} + K_3$$

$$\therefore K_3 = \frac{96(-1)(6)}{(-6)(2)} = 48$$

The Inverse of Laplace Transform

INVERSE LAPLACE OF F(S)

$$F(s) = \frac{120}{s} - \frac{72}{s+8} + \frac{48}{s+6}$$


$$L^{-1} \left[\frac{120}{s} - \frac{72}{s+8} + \frac{48}{s+6} \right]$$
$$f(t) = (120 - 72e^{-8t} + 48e^{-6t})u(t)$$

The Inverse of Laplace Transform

2) DISTINCT COMPLEX ROOTS

$$F(s) = \frac{100(s+3)}{(s+6)(s^2+6s+25)}$$

Num (handwritten) above the numerator and *Den* (handwritten) below the denominator.

DE $n >$ Num degree
↓
PFE



$$s_1 = -6$$

$$s_2 = -3 + j4$$

$$s_3 = -3 - j4$$

conjugat. (handwritten) with an arrow pointing to s_2 and s_3

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The Inverse of Laplace Transform

PARTIAL FRACTION EXPANSION

$$\begin{aligned} F(s) &= \frac{100(s+3)}{(s+6)(s^2+6s+25)} \\ &\equiv \frac{K_1}{s+6} + \frac{K_2}{s+3-j4} + \frac{K_3}{s+3+j4} \\ &\equiv \frac{K_1}{s+6} + \frac{K_2}{s+3-j4} + \frac{K_2^*}{s+3+j4} \end{aligned}$$

✓ Complex roots appears in conjugate pairs.

The Inverse of Laplace Transform

FIND K_1

$$\begin{aligned} K_1 &= \frac{100(s+3)}{s^2+6s+25} \Big|_{s=-6} \\ &= \frac{100(-3)}{25} \\ &= -12 \end{aligned}$$

The Inverse of Laplace Transform

FIND K_2 AND K_2^*

$$\begin{aligned} K_2 &= \frac{100(s+3)}{(s+6)(s+3+j4)} \Big|_{s=3+j4} \\ &= \frac{100(j4)}{(3+j4)(j8)} \\ &= 6 - j8 = 10e^{-j53.13^\circ} \end{aligned}$$

$$K_2^* = 6 + j8 = 10e^{j53.13}$$

Coefficients associated with conjugate pairs are themselves conjugates.

The Inverse of Laplace Transform

INVERSE LAPLACE OF F(S)

$$F(s) = \frac{100(s+3)}{(s+6)(s^2+6s+25)}$$
$$= \frac{-12}{s+6} + \frac{10\angle -53.13^\circ}{s+3-j4} + \frac{10\angle 53.13^\circ}{s+3+j4}$$

The Inverse of Laplace Transform

INVERSE LAPLACE OF F(S)

$$L^{-1} \left\{ \frac{-12}{s+6} + \frac{10e^{-j53.13^\circ}}{s+3-j4} + \frac{10e^{j53.13^\circ}}{s+3+j4} \right\}$$
$$= \left(-12e^{-6t} + 10e^{-j53.13^\circ} e^{-(3-j4)t} + 10e^{j53.13^\circ} e^{-(3+j4)t} \right) u(t)$$

The Inverse of Laplace Transform

USEFUL TRANSFORM PAIRS

$$1) \frac{K}{s+a} \Leftrightarrow Ke^{-at}u(t)$$

$$2) \frac{K}{(s+a)^2} \Leftrightarrow Kte^{-at}u(t)$$

$$3) \frac{K}{s+\alpha-j\beta} + \frac{K}{s+\alpha+j\beta} \Leftrightarrow 2|K|e^{-\alpha t} \cos(\beta t + \theta)u(t)$$

$$4) \frac{K}{(s+\alpha-j\beta)^2} + \frac{K}{(s+\alpha+j\beta)^2} \Leftrightarrow 2t|K|e^{-\alpha t} \cos(\beta t + \theta)u(t)$$

Summary

$$F(s) = \frac{B(s)}{A(s)} \quad \text{degree of } A(s) > B(s)$$

$B(s), A(s)$ Polynomials with real coefficients

$$A(s) = (s-A)(s-B)(s-C)$$

$A, B, C \rightarrow$ roots of the poly $A(s)$

- n roots, where n is the degree of the polynomial.
- if $(a+ib)$ is a root, so is $(a-ib)$ (roots occur in complex conjugate pairs).
- if the roots are z_1, z_2, \dots, z_n ($z \rightarrow$ real, imag)

$$B(s) = K (s-z_1)(s-z_2)(s-z_3) \dots (s-z_n)$$

$$z = a + jb$$

$$\frac{1}{s-z} = \frac{1}{s-a-jb} = \frac{1}{(s-a-jb) * [(s-a)+jb]}$$

$$= \frac{(s-a)+jb}{(s-a)^2+b^2} \quad [j^2 = -1]$$

$$\frac{s-a}{(s-a)^2+b^2} + j \frac{b}{(s-a)^2+b^2}$$

$$\mathcal{L}^{-1} [e^{at} \cos bt + j e^{at} \sin bt] = \left[\frac{s-a}{(s-a)^2+b^2} + j \frac{b}{(s-a)^2+b^2} \right]$$

Operational Laplace Transform

OPERATIONAL TRANSFORMS

- Indicate how mathematical operations performed on either $f(t)$ or $F(s)$ are converted into the opposite domain.
- The operations of primary interest are:
 1. **Multiplying by a constant**
 2. **Addition/subtraction**
 3. **Differentiation**
 4. **Integration**
 5. **Translation in the time domain**
 6. **Translation in the frequency domain**
 7. **Scale changing**

Operational Laplace Transform

OPERATION	$f(t)$	$F(s)$
Multiplication by a constant	$Kf(t)$	$KF(s)$
Addition/ Subtraction	$f_1(t) + f_2(t) - f_3(t) + \dots$	$F_1(s) + F_2(s) - F_3(s) + \dots$
First derivative (time)	$\frac{df(t)}{dt}$	$sF(s) - f(0^-)$
Second derivative (time)	$\frac{d^2(t)}{dt^2}$	$s^2F(s) - sf(0^-) - \frac{df(0^-)}{dt}$

Operational Laplace Transform

OPERATION	$f(t)$	$F(s)$
n th derivative (time)	$\frac{d^n(t)}{dt^n}$	$s^n F(s) - s^{n-1} f(0^-) - s^{n-2} \frac{df(0^-)}{dt} - s^{n-3} \frac{df^2(0^-)}{dt} - \dots - \frac{df^{n-1}(0^-)}{dt^{n-1}}$
Time integral	$\int_0^t f(x) dx$	$\frac{F(s)}{s}$
Translation in time	$f(t - a)u(t - a),$ $a > 0$	$e^{-as} F(s)$
Translation in frequency	$e^{-at} f(t)$	$F(s + a)$

Operational Laplace Transform

OPERATION	$f(t)$	$F(s)$
Scale changing	$f(at), a > 0$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
First derivative (s)	$tf(t)$	$- \frac{dF(s)}{ds}$
n th derivative	$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$
s integral	$\frac{f(t)}{t}$	$\int_s^\infty F(u) du$

Operational Laplace Transform

TRANSLATION IN TIME DOMAIN

If we start with any function:

$$f(t)u(t)$$

we can represent the same function translated in time by the constant a , as:

$$f(t - a)u(t - a)$$

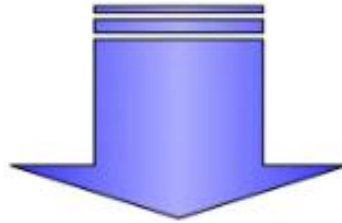
In frequency domain:

$$f(t - a)u(t - a) = e^{-as} F(s)$$

Operational Laplace Transform

EX:

$$L[tu(t)] = 1/s^2$$



$$L[(t - a)u(t - a)] = e^{-as} / s^2$$

Operational Laplace Transform

TRANSLATION IN FREQUENCY DOMAIN

Translation in the frequency domain is defined as:

$$L[e^{-at} f(t)] = F(s + a)$$

Operational Laplace Transform

EX:

$$L[\cos \omega t] = \frac{s}{s^2 + \omega^2}$$



$$L[e^{-at} \cos \omega t] = \frac{s + a}{(s + a)^2 + \omega^2}$$

Operational Laplace Transform

EX:

$$L[\cos t] = \frac{s}{s^2 + 1}$$



Scalder

$$L[\cos \omega t] = \frac{1}{\omega} \frac{s/\omega}{(s/\omega)^2 + 1} = \frac{s}{s^2 + \omega^2}$$

Table 3.1 Laplace Transforms for Various Time-Domain Functions^a

$f(t)$	$F(s)$
1. $\delta(t)$ (unit impulse)	1
2. $S(t)$ (unit step)	$\frac{1}{s}$
3. t (ramp)	$\frac{1}{s^2}$
4. t^{n-1}	$\frac{(n-1)!}{s^n}$
5. e^{-bt}	$\frac{1}{s+b}$
6. $\frac{1}{\tau} e^{-t/\tau}$	$\frac{1}{\tau s + 1}$
7. $\frac{t^{n-1} e^{-bt}}{(n-1)!}$ ($n > 0$)	$\frac{1}{(s+b)^n}$
8. $\frac{1}{\tau^n (n-1)!} t^{n-1} e^{-t/\tau}$	$\frac{1}{(\tau s + 1)^n}$
9. $\frac{1}{b_1 - b_2} (e^{-b_2 t} - e^{-b_1 t})$	$\frac{1}{(s+b_1)(s+b_2)}$

Table 3.1 Laplace Transforms for Various Time-Domain Functions^a

$f(t)$	$F(s)$
10. $\frac{1}{\tau_1 - \tau_2} (e^{-t/\tau_1} - e^{-t/\tau_2})$	$\frac{1}{(\tau_1 s + 1)(\tau_2 s + 1)}$
11. $\frac{b_3 - b_1}{b_2 - b_1} e^{-b_1 t} + \frac{b_3 - b_2}{b_1 - b_2} e^{-b_2 t}$	$\frac{s + b_3}{(s + b_1)(s + b_2)}$
12. $\frac{1}{\tau_1} \frac{\tau_1 - \tau_3}{\tau_1 - \tau_2} e^{-t/\tau_1} + \frac{1}{\tau_2} \frac{\tau_2 - \tau_3}{\tau_2 - \tau_1} e^{-t/\tau_2}$	$\frac{\tau_3 s + 1}{(\tau_1 s + 1)(\tau_2 s + 1)}$
13. $1 - e^{-t/\tau}$	$\frac{1}{s(\tau s + 1)}$
14. $\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
15. $\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
16. $\sin(\omega t + \phi)$	$\frac{\omega \cos \phi + s \sin \phi}{s^2 + \omega^2}$
17. $e^{-bt} \sin \omega t$	$\left\{ \begin{array}{l} \frac{\omega}{(s + b)^2 + \omega^2} \\ \frac{s + b}{(s + b)^2 + \omega^2} \end{array} \right.$
18. $e^{-bt} \cos \omega t$	
$b, \omega \text{ real}$	
19. $\frac{1}{\tau \sqrt{1 - \zeta^2}} e^{-\zeta t/\tau} \sin(\sqrt{1 - \zeta^2} t/\tau)$ ($0 \leq \zeta < 1$)	$\frac{1}{\tau^2 s^2 + 2\zeta \tau s + 1}$

Table 3.1 Laplace Transforms for Various Time-Domain Functions^a (continued)

$f(t)$	$F(s)$
20. $1 + \frac{1}{\tau_2 - \tau_1} (\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2})$ ($\tau_1 \neq \tau_2$)	$\frac{1}{s(\tau_1 s + 1)(\tau_2 s + 1)}$
21. $1 - \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta t/\tau} \sin [\sqrt{1 - \zeta^2} t/\tau + \psi]$ $\psi = \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{\zeta}, \quad (0 \leq \zeta < 1)$	$\frac{1}{s(\tau^2 s^2 + 2\zeta\tau s + 1)}$
22. $1 - e^{-\zeta t/\tau} [\cos (\sqrt{1 - \zeta^2} t/\tau)$ $+ \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin (\sqrt{1 - \zeta^2} t/\tau)]$ ($0 \leq \zeta < 1$)	$\frac{1}{s(\tau^2 s^2 + 2\zeta\tau s + 1)}$
23. $1 + \frac{\tau_3 - \tau_1}{\tau_1 - \tau_2} e^{-t/\tau_1} + \frac{\tau_3 - \tau_2}{\tau_2 - \tau_1} e^{-t/\tau_2}$ ($\tau_1 \neq \tau_2$)	$\frac{\tau_3 s + 1}{s(\tau_1 s + 1)(\tau_2 s + 1)}$
24. $\frac{df}{dt}$	$sF(s) - f(0)$
25. $\frac{d^n f}{dt^n}$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \dots$ $- s f^{(n-2)}(0) - f^{(n-1)}(0)$
26. $f(t - t_0)S(t - t_0)$	$e^{-t_0 s} F(s)$

^aNote that $f(t)$ and $F(s)$ are defined for $t \geq 0$ only.

Example 3.1

Solve the ODE,

$$5\frac{dy}{dt} + 4y = 2 \quad y(0) = 1 \quad (3-26)$$

First, take \mathcal{L} of both sides of (3-26),

$$5(sY(s) - 1) + 4Y(s) = \frac{2}{s}$$

Rearrange,

$$Y(s) = \frac{5s + 2}{s(5s + 4)} \quad (3-34)$$

Take \mathcal{L}^{-1} ,

$$y(t) = \mathcal{L}^{-1} \left[\frac{5s + 2}{s(5s + 4)} \right]$$

From Table 3.1 (line 11),

$$\boxed{y(t) = 0.5 + 0.5e^{-0.8t}} \quad (3-37)$$

◦ **Example 3.2:** $y' - 4y = 1, y(0) = 1$

$$\begin{aligned} L[y' - 4y](s) &= L[y'](s) - 4L[y](s) = (sY(s) - y(0)) - 4Y(s) \\ &= (s - 4)Y(s) - 1 \end{aligned}$$

$$L[1](s) = \frac{1}{s}$$

$$(s - 4)Y(s) - 1 = \frac{1}{s}, \quad Y(s) = \frac{1}{s - 4} + \frac{1}{s(s - 4)}$$

$$y = L^{-1}[Y] = L^{-1}\left[\frac{1}{s - 4} + \frac{1}{s(s - 4)}\right] = L^{-1}\left[\frac{1}{s - 4}\right] + L^{-1}\left[\frac{1}{s(s - 4)}\right]$$

From Table 3.1, entries (5) and (9)

$$(5) \quad L[e^{at}] = \frac{1}{s - a}, \quad (8) \quad L\left[\frac{e^{at} - e^{bt}}{a - b}\right] = \frac{1}{(s - a)(s - b)}$$

$$y(t) = e^{4t} + \frac{1}{4}(e^{4t} - 1) = \frac{5}{4}e^{4t} - \frac{1}{4}$$

Example 3.3:

$$L^{-1}\left[\frac{4}{s^2 + 4s + 20}\right] = L^{-1}\left[\frac{4}{(s + 2)^2 + 16}\right]$$

$$\because L[\sin at] = \frac{a}{s^2 + a^2}, \quad L[\sin 4t] = \frac{4}{s^2 + 16}$$

$$L[e^{-2t} \sin 4t] = \frac{4}{(s + 2)^2 + 16}$$

$$\therefore L^{-1}\left[\frac{4}{(s + 2)^2 + 16}\right] = e^{-2t} \sin 4t$$

Factoring the denominator polynomial

1.
$$\frac{2}{3s^2 + 4s + 1}$$

$$3s^2 + 4s + 1 = (3s + 1)(s + 1) = 3\left(s + \frac{1}{3}\right)(s + 1)$$

From 9 in the table, Transforms to $e^{-t/3} - e^{-t}$

Real roots = no oscillation

$$2. \quad \frac{2+s}{s^2+s+1}$$

$$s^2+s+1 = \left(s + \frac{1}{2} + \frac{\sqrt{3}}{2}j\right)\left(s + \frac{1}{2} - \frac{\sqrt{3}}{2}j\right) = \left(s + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

Transforms to $e^{-0.5t} \sin \frac{\sqrt{3}}{2}t, e^{-0.5t} \cos \frac{\sqrt{3}}{2}t$

Complex roots = oscillation

$$L^{-1}\left[\sqrt{3}\left(\frac{\frac{\sqrt{3}}{2}}{\left(s + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right) + \left(\frac{s + \frac{1}{2}}{\left(s + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right)\right]$$

From Table 3.1, line 17 and 18

$$y(t) = e^{-\frac{t}{2}} \left(\sin\left(\frac{\sqrt{3}}{2}t\right) + \sqrt{3} \cos\left(\frac{\sqrt{3}}{2}t\right)\right)$$

EXAMPLE 3.5

Find the inverse transform of

$$Y(s) = \frac{1 + e^{-2s}}{(4s + 1)(3s + 1)} \quad (3-72)$$

SOLUTION

Equation 3-72 can be split into two terms:

$$Y(s) = Y_1(s) + Y_2(s) \quad (3-73)$$

$$= \frac{1}{(4s + 1)(3s + 1)} + \frac{e^{-2s}}{(4s + 1)(3s + 1)} \quad (3-74)$$

The inverse transform of $Y_1(s)$ can be obtained directly from Table 3.1:

$$y_1(t) = e^{-t/4} - e^{-t/3} \quad (3-75)$$

Because $Y_2(s) = e^{-2s}Y_1(s)$, its inverse transform can be written immediately by replacing t by $(t - 2)$ in (3-75), and then multiplying by the shifted step function:

$$y_2(t) = [e^{-(t-2)/4} - e^{-(t-2)/3}]S(t - 2) \quad (3-76)$$

Thus, the complete inverse transform is

$$\begin{aligned} y(t) &= e^{-t/4} - e^{-t/3} + [e^{-(t-2)/4} - e^{-(t-2)/3}]S(t - 2) \\ &= y_1(t) + y_2(t) \end{aligned} \quad (3-77)$$

3.16 (a) The differential equation

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = \cos t$$

has initial conditions, $y(0) = 1$, $y'(0) = 2$. Find $Y(s)$ and, without finding $y(t)$, determine what functions of time will appear in the solution.

(b) If $Y(s) = \frac{s+1}{s(s^2+4s+8)}$, find $y(t)$.

a) Take the Laplace transform:

$$\left[s^2 Y(s) - sy(0) - y'(0) \right] + 6[sY(s) - y(0)] + 9Y(s) = \frac{s}{s^2 + 1}$$

$$(s^2 + 6s + 9)Y(s) - s(1) - 2 - (6)(1) = \frac{s}{s^2 + 1}$$

$$(s^2 + 6s + 9)Y(s) = \frac{s}{s^2 + 1} + s + 8$$

$$(s^2 + 6s + 9)Y(s) = \frac{s + s^3 + s + 8s^2 + 8}{s^2 + 1}$$

$$Y(s) = \frac{s^3 + 8s^2 + 2s + 8}{(s + 3)^2 (s^2 + 1)}$$

To find $y(t)$ we have to expand $Y(s)$ into its partial fractions

$$Y(s) = \frac{A}{(s + 3)^2} + \frac{B}{s + 3} + \frac{Cs}{s^2 + 1} + \frac{D}{s^2 + 1}$$

$$y(t) = Ate^{-3t} + Be^{-3t} + C \cos t + D \sin t$$

b)
$$Y(s) = \frac{s+1}{s(s^2+4s+8)}$$

Since $\frac{4^2}{4} < 8$, there are complex factors.

∴ complete the square in denominator

$$s^2 + 4s + 8 = s^2 + 4s + 4 + 8 - 4$$

$$= s^2 + 4s + 4 + 4 = (s+2)^2 + (2)^2 \quad \{ b = 2, \omega = 2 \}$$

∴ Partial fraction expansion gives

$$Y(s) = \frac{A}{s} + \frac{B(s+2)}{s^2+4s+8} + \frac{C}{s^2+4s+8} = \frac{s+1}{s(s^2+4s+8)}$$

Multiply by s and let $s \rightarrow 0$

$$A = 1/8$$

Multiply by $s(s^2+4s+8)$

$$A(s^2+4s+8) + B(s+2)s + Cs = s + 1$$

$$As^2 + 4As + 8A + Bs^2 + 2Bs + Cs = s + 1$$

$$s^2: \quad A + B = 0 \quad \rightarrow \quad B = -A = -\frac{1}{8}$$

$$s^1: \quad 4A + 2B + C = 1 \quad \rightarrow \quad C = 1 + 2\left(\frac{1}{8}\right) - 4\left(\frac{1}{8}\right) = \frac{3}{4}$$

$$s^0: \quad 8A = 1 \quad \rightarrow \quad A = \frac{1}{8} \quad (\text{This checks with above result})$$

$$Y(s) = \frac{1/8}{s} + \frac{(-1/8)(s+2)}{(s+2)^2 + 2^2} + \frac{3/4}{(s+2)^2 + 2^2}$$

$$y(t) = \left(\frac{1}{8}\right) - \left(\frac{1}{8}\right)e^{-2t} \cos 2t + \left(\frac{3}{8}\right)e^{-2t} \sin 2t$$

Example 8

$$G(s) = \frac{3s^3 + 3s^2 + 5s - 7}{s^4 + s^3 + s^2 + 9s + 10} \quad \begin{matrix} B(s) \\ A(s) \end{matrix}$$

$B(x) < A(x) \rightarrow$ PFB

$$A(x) = s^4 + s^3 + s^2 - 9s - 10$$

ببقیم التقریب Try and error $s=0, -1, -2, 1$

$$A(-1) = (-1)^4 + (-1)^3 + (-1)^2 - 9(-1) - 10 = 1 - 1 + 1 + 9 - 10 = 0$$

$$\Rightarrow (s+1)$$

$$\begin{array}{r} s^3 + s - 10 \\ s+1 \overline{) s^4 + s^3 + s^2 - 9s - 10} \\ \underline{-s^4 - s^2} \\ s^2 - 9s - 10 \\ \underline{-s^2 - 10} \\ -10s - 10 \\ \underline{+10s + 10} \\ 0 \end{array}$$

$$A(x) = (s+1) \underbrace{(s^2 + s - 10)}_{C(x)}$$

$$C(x) = s^2 + s - 10$$

$s=2 \Rightarrow (s-2)$ try and error.

$$\begin{array}{r} s^2 + 2s + 6 \\ s-2 \overline{) s^2 + s - 10} \\ \underline{-s^2 + 2s^2} \\ 2s^2 + s - 10 \\ \underline{-2s^2 + 4s} \\ 5s - 10 \\ \underline{-5s + 10} \\ 0 \end{array}$$

$$C(x) = (s-2)(s^2 + 2s + 6)$$

$$A(x) = (s+1)(s-2)(s^2 + 2s + 6)$$

المميز $\Delta = b^2 - 4ac$ \Rightarrow جذور المعادلة التربيعية

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$s = -1 \pm j2$$

$$(s+1+j2)(s+1-j2)$$

$$A(x) = (s+1)(s-2)(s+1+j2)(s+1-j2)$$

$$\Rightarrow G(s) = \frac{3s^3 + 3s^2 + 5s - 7}{(s+1)(s-2)(s+1+j2)(s+1-j2)} = \frac{K_1}{s+1} + \frac{K_2}{s-2} + \frac{K_3}{s+1+j2} + \frac{K_4}{s+1-j2}$$

Solve for K_1

$$\frac{3s^3 + 3s^2 + 5s - 7}{(s+1)(s-2)(s+1+j2)(s+1-j2)} \Big|_{s=-1} = K_1$$

$$K_1 = 1$$