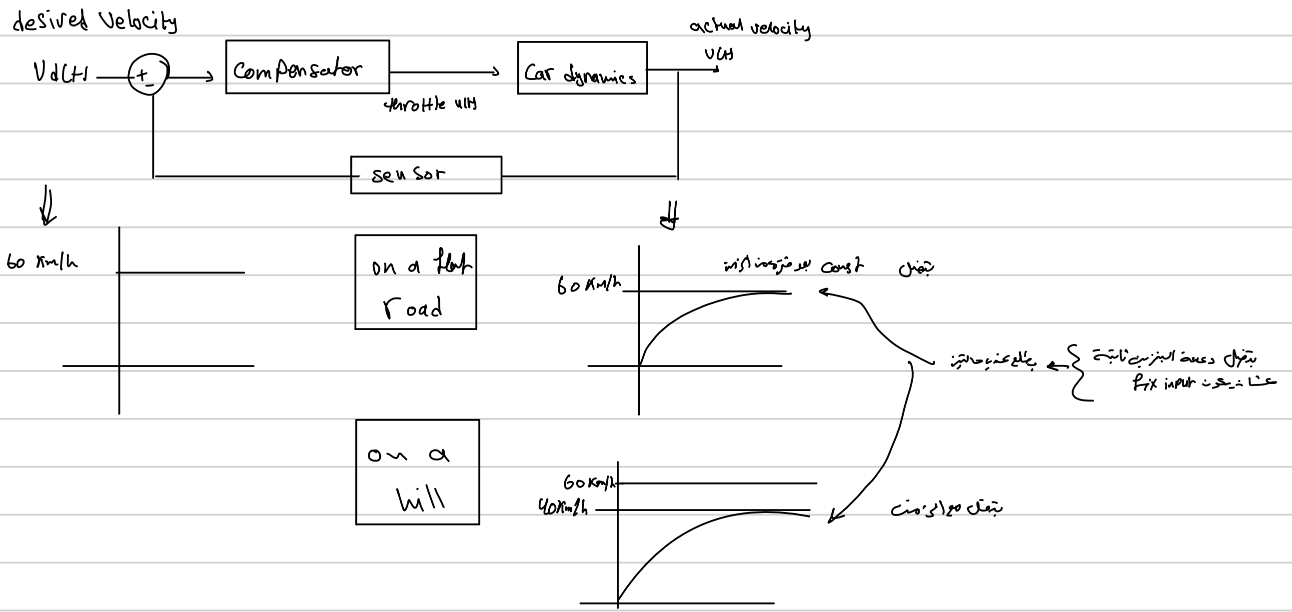
Industrial Control Systems

Chapter Seven:
Feedback Controllers

Dr. Eng. Baha'eddin Alhaj Hasan
Department of Industrial Engineering

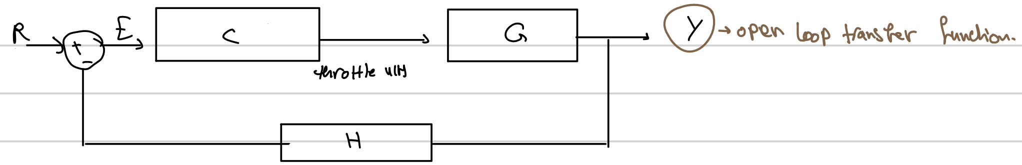
Closed-loop controller



→ to keep track of input – output it must be closed loop to give a perfect control tracking.

(*) Steady state error : $E = V_d(t) - V(t)$

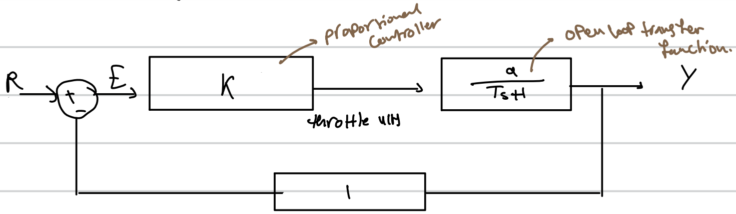
⇒



(*) $E(s) = R(s) - Y(s)$

(*) $\dfrac{Y(s)}{R(s)} = \dfrac{CG}{1 + HCG}$ → closed loop transfer function.

first order transfer function



$$\frac{\frac{Ka}{Ts+1}}{1+\frac{Ka}{Ts+1}} = \frac{Ka}{Ts+1+aK}$$

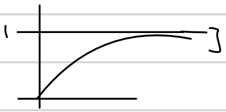
$$= \boxed{\frac{aK / 1+aK}{\left(\frac{T}{1+aK}\right)s+1}}$$ closed loop transfer function.

↳ Standard form of the first order syst.

$$G(s) = \frac{K_{dc}}{\tau s + 1}$$

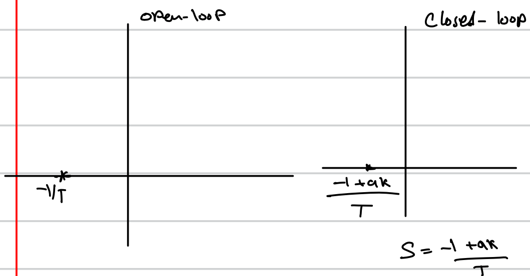
Comp between open and closed loop.

	open loop TF	Closed loop TF
when the syst is steady, $s \to 0$, $t \to \infty$ ⇐ gain	a	$\frac{aK}{1+aK} < 1$
Time const (T)	T	> $\frac{T}{1+aK}$ ↓ fast response.



Dc gain $s \to 0$

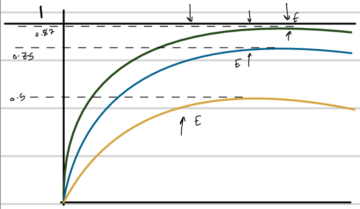
steady state value $t \to \infty$



$S = \frac{-1+aK}{T}$

Increase K more negative pole ⇒ more stable syst
⇒ less steady state error

ex: if $a=1$ find DC gain at $K = 7, 3, 1$

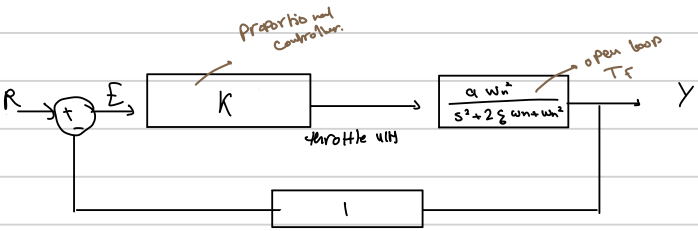


— $K = 7$ → Dc gain $= \frac{7}{8}$

— $K = 3$ → DC gain $= \frac{3}{4}$

— $K = 1$ → DC gain $= 0.5$

Second order transfer function



$$\frac{\dfrac{aK\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}}{1+\dfrac{aK\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}} = \frac{aK\omega_n^2}{s^2+2\zeta\omega_n s+(1+aK)\omega_n^2}$$

$$\left[\frac{aK}{1+aK}\right]\left[\frac{(1+aK)\omega_n^2}{s^2+2\zeta\omega_n s+(1+aK)\omega_n^2}\right]$$

	open loop TF	closed loop TF	
gain	a	$\frac{aK}{1+aK}$	always less than (1)
native freq	ω_n	$\sqrt{(1+aK)\omega_n}$	Faster (as we increase K)
damping ratio	ζ	$\frac{\zeta}{\sqrt{1+aK}}$	Less damping.
poles	$-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$	$\underbrace{-\zeta\omega_n}_{\text{real part } (t_r, t_s)} \pm \underbrace{j\omega_n\sqrt{1+aK-\zeta^2}}_{\text{imaginary part } (t_p)}$	No changes at t_s, t_r less t_p as K increase.

only for step input.

DC Gain ⟹ bounded ⟹ output ≤ input (stable) ⟹ finite DC value

$-5e^{-t}$

$\lim_{t\to\infty} 5e^{-t} = 0$

if no DC Gain ⟹ unbounded ⟹ output > input (unstable). ⟹ ∞ steady state value

$-\lim 5(1-e^{-t}) = 5$

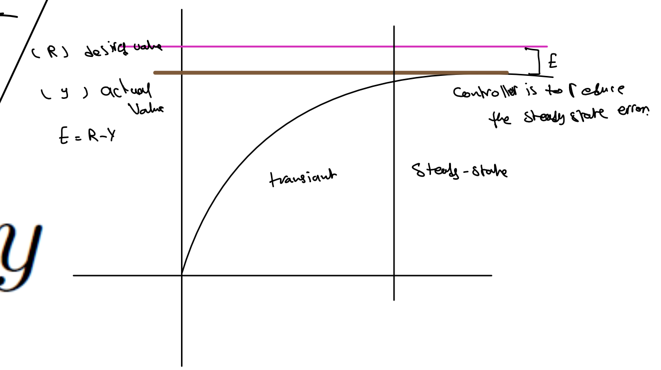
$y = uh$

$u = 1(t)$

$y = h$

$$u \longrightarrow \boxed{h} \longrightarrow y$$

input is step function



Definition: the steady-state value of the step response is called the *DC gain* of the system.

steady state value.

$$\text{DC gain} = y(\infty) = \lim_{t\to\infty} y(t) \qquad \text{for } u(t) = 1(t)$$

In our example above, the step response is

$$y(t) = \frac{1}{2}1(t) + (2\alpha + \beta - 1)e^{-t} + (1/2 - \alpha - \beta)e^{-2t}$$

therefore, DC gain $= y(\infty) = 1/2$

for step input

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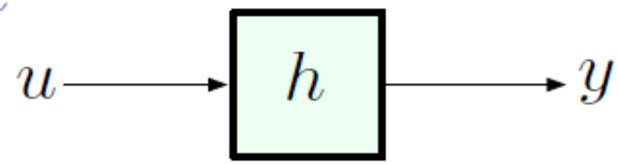
Steady-State Value

in frequency domain.



$y \; \lim_{t \to \infty} y(t) = \frac{1}{s} \lim_{s \to 0} s\,Y(s)$

لانه لما حولت للـ s-domain
أنا أخذت الـ integral ⇐ when $t \to \infty$

$$u(t) = 1(t) \qquad U(s) = \frac{1}{s} \qquad \Longrightarrow \qquad Y(s) = \frac{H(s)}{s}$$

— can we compute $y(\infty)$ from $Y(s)$?

Let's look at some examples:

- $Y(s) = \dfrac{1}{s+a}$, $a > 0$ (pole at $s = -a < 0$)
  $y(t) = e^{-at} \quad \Longrightarrow \quad y(\infty) = 0$ Stable
- $Y(s) = \dfrac{1}{s+a}$, $a < 0$ (pole at $s = -a > 0$)
  $y(t) = e^{-at}$ $-(-a)$ $\quad \Longrightarrow \quad y(\infty) = \infty$ unbounded unstable.
- $Y(s) = \dfrac{1}{s^2 + \omega^2}$, $\omega \in \mathbb{R}$ (poles at $s = \pm j\omega$, purely imaginary)
  $y(t) = \sin(\omega t) \quad \Longrightarrow \quad y(\infty)$ does not exist unstable.
- $Y(s) = \dfrac{c}{s}$ (pole at the origin, $s = 0$)
  $y(t) = c1(t) \quad \Longrightarrow \quad y(\infty) = c$ Stable
  (DC value of (c))

The Final Value Theorem

We can now deduce the Final Value Theorem (FVT):

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> If all poles of $sY(s)$ are *strictly stable* or lie in the *open left half-plane* (OLHP), i.e., have $\mathrm{Re}(s) < 0$, then
>
> $$y(\infty) = \lim_{s \to 0} sY(s).$$

In our examples, multiply $Y(s)$ by s, check poles:

- $Y(s) = \dfrac{1}{s+a} \qquad sY(s) = \dfrac{s}{s+a}$

  if $a > 0$, then $y(\infty) = 0$; if $a < 0$, FVT does not give correct answer

  syst is stable — syst is unstable

- $Y(s) = \dfrac{1}{s^2+\omega^2} \qquad sY(s) = \dfrac{s}{s^2+\omega^2}$

  poles are purely imaginary (not in OLHP), FVT does not give correct answer

- $Y(s) = \dfrac{c}{s} \qquad sY(s) = c \rightarrow$ Stable syst.

  poles at infinity, so $y(\infty) = c$ – FVT gives correct answer

the final value theorem (FVT)

- to find the system's final value without solving the differential equation.

find $y(\infty)$ for $Y(s)$

⊛ How to use it?

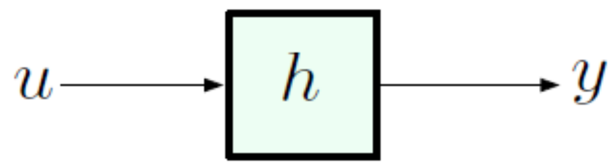
step 1: the stability check

- find the poles they must be -ve (Real part < 0) $\longrightarrow$ the syst is stable
- if one of the poles is positive or purely imaginary ($\pm j\omega$) $\longrightarrow$ syst is unstable $\Rightarrow$ Stop!! in this case FVT cannot be applied.

step 2: apply the final value theorem formula

- Multiply $Y(s)$ by S
- set all S's to zero ($\lim_{s \to 0} sY(s)$)
- the value of the limi is the final value $y(\infty)$

Back to DC Gain



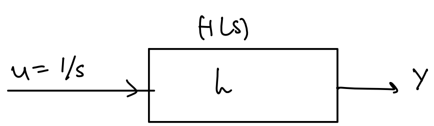
Step response: $$Y(s) = \frac{H(s)}{s}$$

— if all poles of $sY(s) = H(s)$ are strictly stable, then

$$y(\infty) = \lim_{s \to 0} H(s)$$

by the FVT.

Example: compute DC gain of the system with transfer function



$$H(s) = \lim_{s \to 0} \frac{\cancel{s^2} + \cancel{5s} + 3}{\cancel{s^3} + \cancel{4s} + \cancel{2s} + 5} = \frac{3}{5}$$

syst is stable,
- coefs > 0
- 2*4 > 5

All poles of $H(s)$ are strictly stable (we will see this later using the *Routh–Hurwitz criterion*), so

$$y(\infty) = H(s)\Big|_{s=0} = \frac{3}{5}.$$

$Y(s) = H(s)\,\frac{1}{s}$

$\boxed{H(s) = s\,Y(s)}$

$\lim_{t \to \infty} y(t) = \lim_{s \to 0} H(s)$

Steady-State Response Analyses

The steady state performance of a stable control system is generally judged by its steady state error to step, ramp and parabolic inputs. Consider a unity feedback system as shown in the Figure below. The difference between input and output is the error signal E(s).

Open-loop transfer function $G(s)$

Closed-loop TF

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

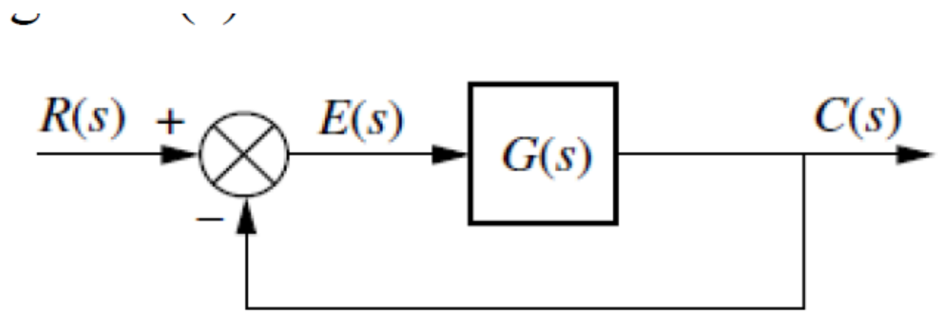


Fig.1 Closed-loop unity feedback control system

$$C(s) = E(s) \cdot G(s)$$

$$E(s) = \frac{C(s)}{G(s)}$$

$$E(s) = R(s) - C(s)$$

$$C(s) = E(s)\, G(s)$$

$$E(s) + E(s)\, G(s) = R(s)$$

$$E(s) = \frac{R(s)}{1+G(s)}$$

The closed loop transfer function (CLTF) is:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

As we know $C(s) = E(s) \times G(s)$

Therefore,

$$E(s) = \frac{1}{1 + G(s)} \times R(s)$$

Steady-state error e_{ss} may be found using the Final Value Theorem (FVT) as follows:

$$e(\infty) = E_{ss} = \lim_{S \to 0} \frac{SR(s)}{1 + G(s)}$$

error

input / Reference value.

The above equation shows that the steady state error depends upon the input R(s) and the forward transfer function G(s).

Steady State Error (Ess) for Step input

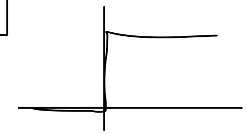
Input is independent on time; $r(t) = 1$ OR $R(s)=1/S$

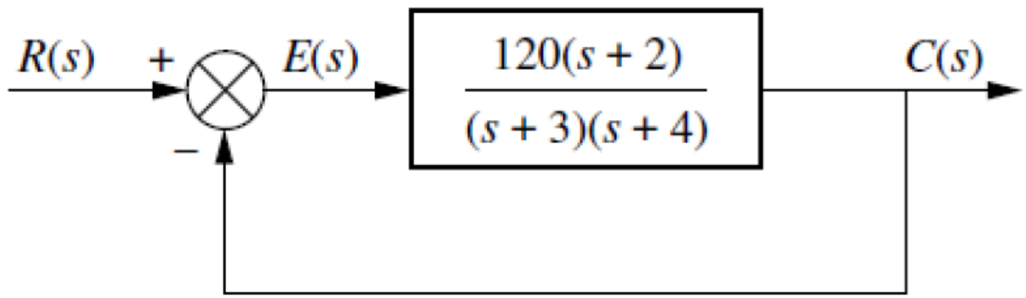
$$E_{ss} = \lim_{S\to 0}\frac{SR(s)}{1+G(s)} = \lim_{S\to 0}\frac{S\times 1/S}{1+G(s)} = \lim_{S\to 0}\frac{1}{1+G(s)}$$

$$E_{ss} = \frac{1}{1+\lim_{S\to 0} G(s)} = \frac{1}{1+K_p}$$

Where K_p is the ***position error coefficient*** and equals G(0) or $K_p = \lim_{S\to 0} G(s)$

Find the steady-state errors for inputs of $\boxed{5u(t)}$ $R(s) = \frac{5}{s}$ to the system shown below. The function u(t) is the unit step.





For step input 5u(t), we must calculate the position error coefficient (K_p):

$$K_p = \lim_{S\to 0} G(S) = \frac{120 \times 2}{3 \times 4} = 20$$

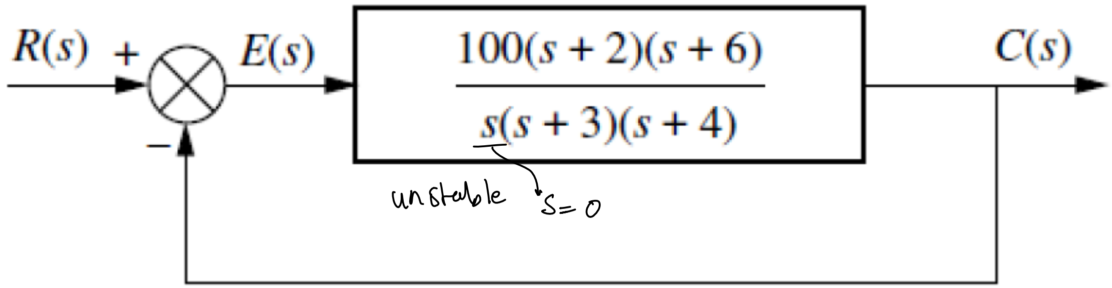
$$E_{ss} = \frac{5}{1+20} = \frac{5}{21}$$

For ramp input 5tu(t), we must calculate the velocity error coefficient (K_v):

$$K_v = \lim_{S\to 0} S\,G(S) = \frac{0 \times 120 \times 2}{3 \times 4} = 0$$

$$E_{ss} = \frac{5}{0} = \infty$$

Find the steady-state errors for inputs of 5u(t) $R(s) = \frac{5}{s}$ to the system shown below. The function u(t) is the unit step.



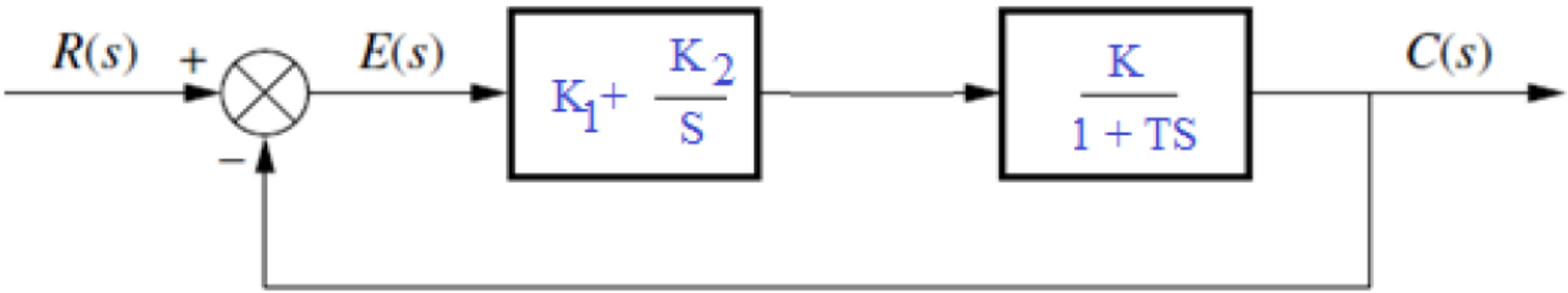
For step input 5u(t), we must calculate the position error coefficient (K_p):

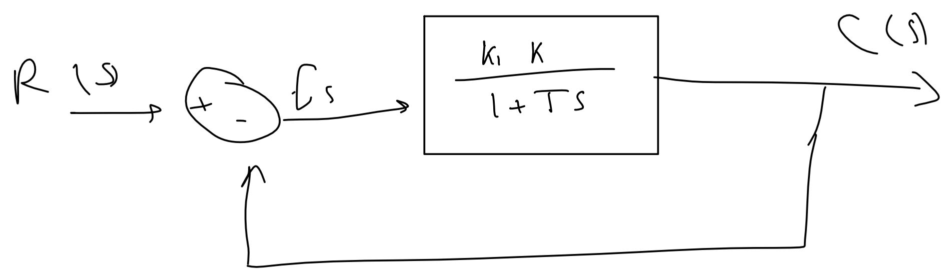
$$K_p = \lim_{S \to 0} G(S) = \frac{100 \times 2 \times 6}{0 \times 3 \times 4} = \infty$$

$$E_{ss} = \frac{5}{1 + \infty} = 0$$

An engine speed control system is shown below. $R(s) = \frac{A}{s}$

1) Calculate E_{ss} for step input with magnitude A when $K_2 = 0$
2) Calculate E_{ss} for step input with magnitude A when $K_2 \neq 0$





$$G(s) = \frac{K(K_1 S + K_2)}{S(1 + TS)}$$

1) When $K_2 = 0$, the above G(s) is reduced to:

$$G(s) = \frac{KK_1}{1 + TS}$$

$$K_p = \lim_{S \to 0} G(S) = KK_1$$

$$E_{ss} = \frac{1}{1 + K_p} = \frac{1}{1 + KK_1}$$

$$K_P = \frac{K_1 K}{1 + T(0)} = K_1 K$$

For a step input with magnitude A;

$$E_{ss} = \frac{A}{1 + KK_1}$$

2) When $K_2 \neq 0$, the open-loop T.F reverts to the original form:

$$G(s) = \frac{K(K_1 S + K_2)}{S(1 + TS)}$$

Which represent a type 1 system.

$$K_p = \lim_{S \to 0} G(S) = \infty$$

$$E_{ss} = \frac{1}{1 + K_p} = \frac{1}{1 + \infty} = 0$$

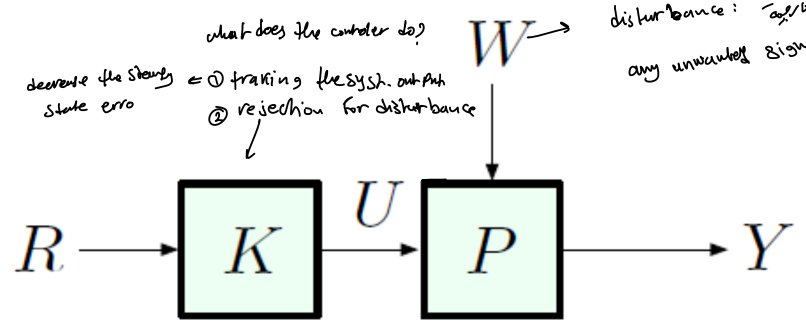
Two Basic Control Architectures

- Open-loop control

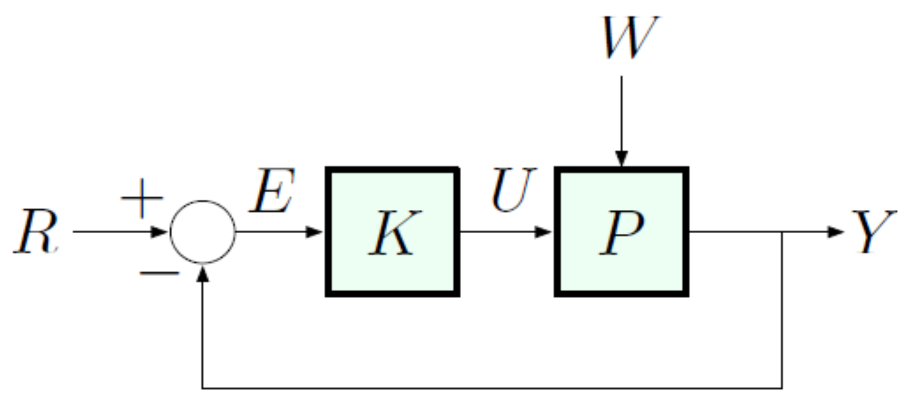
what does the controler do?

deerease the Steady state erro ⟸ ① traking the syst. output
② rejection for disturbance

disturbance : العوامل الخارجية
any unwanted signals

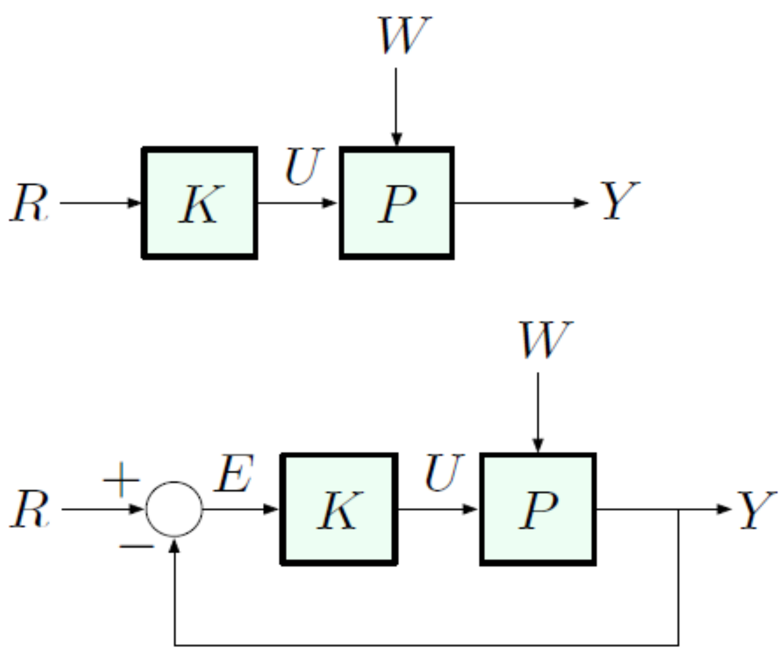


- Feedback (closed-loop) control



Here, W is a *disturbance*; K is *not necessarily* a static gain

Basic Objectives of Control

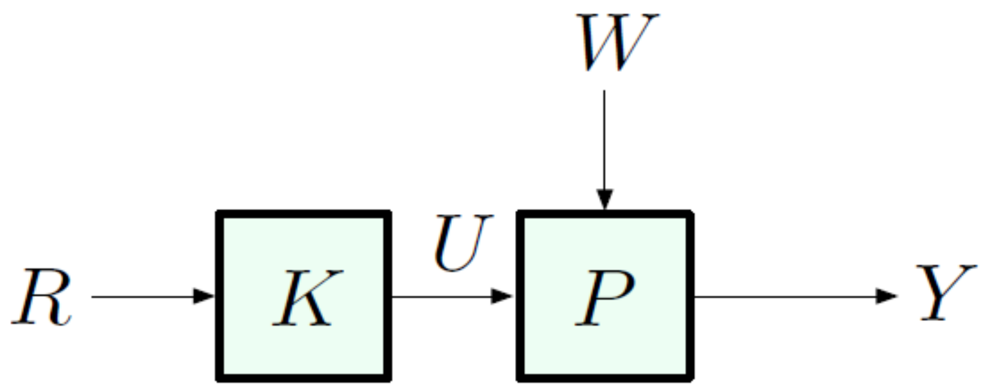


مزایای controler

- track a given reference
- reject disturbances
- meet performance specs

Intuitively, the difference between the open-loop and the closed-loop architectures is clear (think cruise control ...)

Open-Loop Control



- cheaper/easier to implement (no sensor required)
- does not destabilize the system

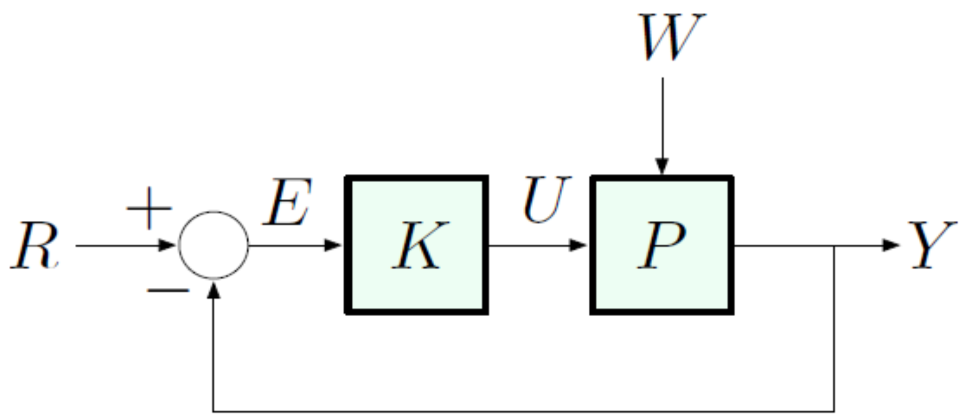
e.g., if both K and P are stable (all poles in OLHP),

$$\frac{Y}{R} = KP$$

is also stable:

$$\{\text{poles of } KP\} = \{\text{poles of } K\} \cup \{\text{poles of } P\}$$

Feedback Control

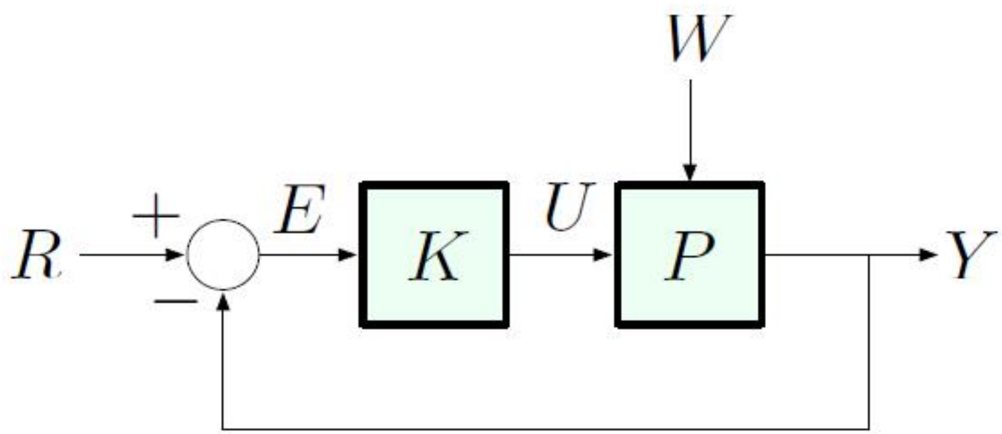


- more difficult/expensive to implement (requires a sensor and an information path from controller to actuator)
- may destabilize the system:

$$\frac{Y}{R} = \frac{KP}{1 + KP}$$

  has new poles, which may be unstable
- but: feedback control is the *only way* to stabilize an unstable plant (this was the Wright brothers' key insight)

Benefits of Feedback Control



Feedback control:

- reduces steady-state error to disturbances
- reduces steady-state sensitivity to model uncertainty (parameter variations)
- improves time response

Summary

Feedback control:

- reduces steady-state error to disturbances
- reduces steady-state sensitivity to model uncertainty (parameter variations)
- improves time response

Word of caution: high-gain feedback only works well for 1st-order systems; for higher-order systems, it will typically cause underdamping and instability.

Thus, we need a more sophisticated design than just static gain. We will take this up in the next lecture with *Proportional-Integral-Derivative* (PID) control.

1. P - Proportional Controller
Think of this as the "Present" error. It reacts to the error happening right now.
• How it works: The output is directly proportional to the size of the error.
• Equation (Transfer Function): $G(s) = K_p$
• Key Benefits:
• Reduces rise time (makes the system respond faster).
• Reduces steady-state error (but does not eliminate it).
• Important Note: If used alone, a steady-state error will always remain.
You cannot achieve perfect tracking with just a P-controller.

2. I - Integral Controller
Think of this as the "Past" error. It accumulates the history of the error.
• How it works: The output is proportional to the accumulation (integral) of the error over time.
• Equation (Transfer Function): $G(s) = K_i/s$
• Key Benefits:
• Eliminates steady-state error completely ($e_{ss} = 0$). This ensures the system eventually reaches the exact target.
• Rejects constant disturbances.
• Important Note: Adding Integral control can make the transient response worse (more oscillatory) and increase settling time. It can also destabilize the system if the gain is too high.

3. D - Derivative Controller
Think of this as the "Future" error. It predicts where the error is going.
• How it works: The output is proportional to the rate of change of the error. It "anticipates" changes.
• Equation (Transfer Function): $G(s) = K_D s$
• Key Benefits:
• Increases stability.
• Reduces overshoot (prevents the system from shooting past the target).
• Improves the transient response.
• Important Note:
• Never used alone: If the error is constant (even if it's huge), the rate of change is zero, so the D-controller does nothing.
• Noise Sensitivity: It amplifies high-frequency noise.

Combined Controllers

In practice, these are combined to get the best of all worlds.

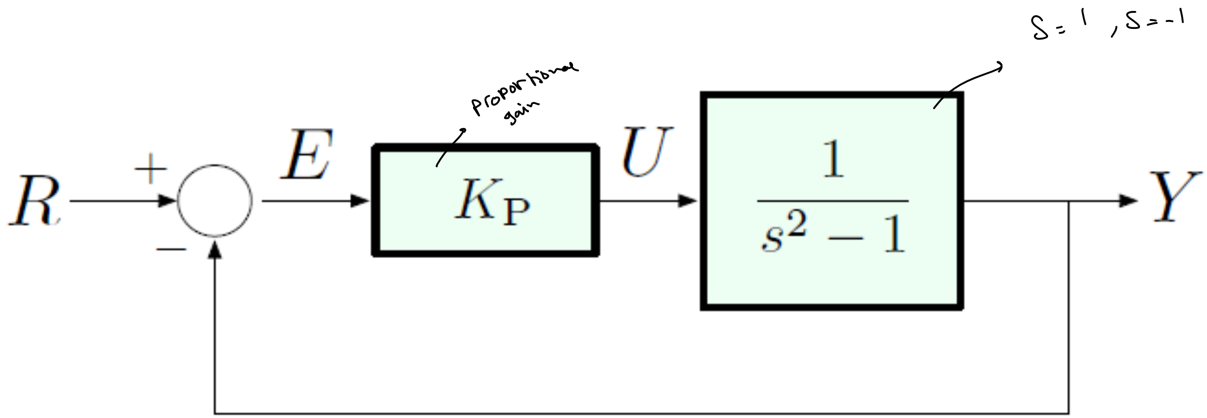
- **PD (Proportional + Derivative):**
  - **Use when:** You need to improve stability and reduce overshoot.
  - **Effect:** It allows for "arbitrary pole placement," meaning you can design the specific response characteristics you want. However, it still leaves a steady-state error. +1
  - **Equation:** $K_P + K_D s$.
- **PI (Proportional + Integral):**
  - **Use when:** You need to eliminate steady-state error.
  - **Effect:** The error disappears, but the system might take longer to settle and be more oscillatory. +1
  - **Equation:** $K_P + \frac{K_I}{s}$.
- **PID (Proportional + Integral + Derivative):**
  - **Use when:** You need to meet all design specifications: fast rise time, low overshoot, and zero error.
  - **Effect:** Can achieve perfect tracking and stability.
  - **Equation:** $K_P + \frac{K_I}{s} + K_D s$.

Practical Tips for Designing a PID Controller

Slide 283 provides a step-by-step logic for "tuning" your controller:

1. **Analyze:** Look at the open-loop response to see what is missing.
2. **Fix Rise Time:** Add P control to make it faster.
3. **Fix Overshoot:** Add D control to dampen the oscillation.
4. **Fix Error:** Add I control to remove the final steady-state error.
5. **Simplify:** Do not use all three if you don't have to. If a PI controller works, you don't need to add D. +1

Proportional Feedback



K_P – "proportional gain" (P-gain) $\qquad U = K_P E$

Let's try to find a value of K_P that would stabilize the system:

$$\frac{Y}{R} = \frac{\frac{K_P}{s^2 - 1}}{1 + \frac{K_P}{s^2 - 1}} = \frac{K_P}{s^2 - 1 + K_P}$$

المستحيل تكون stable
مهما حطيت قيمة لـ Kp

— the polynomial in the denominator has zero coefficient of s

$\Longrightarrow$ necessary condition for stability is not satisfied.

The feedback system is *not stable for* ***any*** *value of* K_P*!!*

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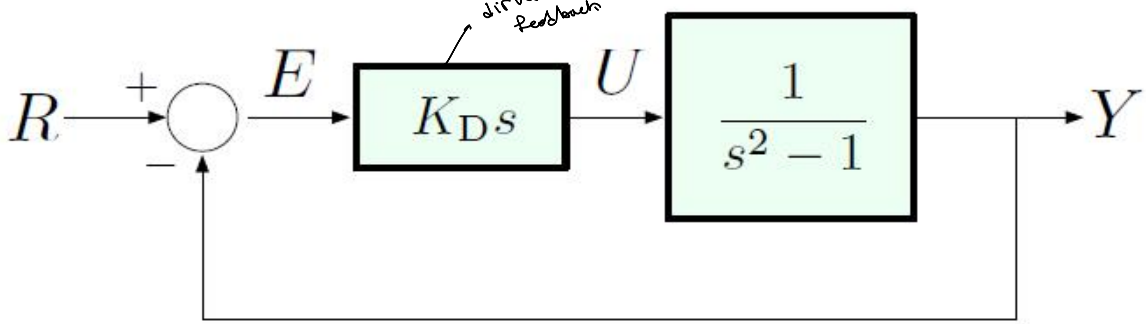
Back to Analysis: Derivative Feedback

dirvative feedback



$$\frac{Y}{R} = \frac{\dfrac{K_D s}{s^2-1}}{1 + \dfrac{K_D s}{s^2-1}} = \frac{K_D s}{\underbrace{s^2 + K_D s - 1}}$$

Not stable because the const coef is -ve

— still not good: the denominator has a negative coefficient $\Longrightarrow$ not stable; also we have picked up a zero at the origin.

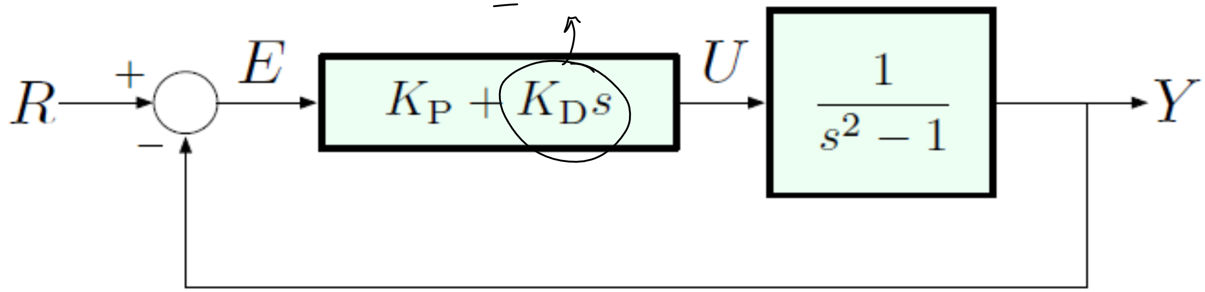
But:

- P-control affected the coefficient of s^0 (constant term)
- D-control affected the coefficient of s

— let's combine them!!

Proportional-Derivative (PD) Control

مشتقة ال KP



$$\frac{Y}{R} = \frac{\frac{K_P + K_D s}{s^2 - 1}}{1 + \frac{K_P + K_D s}{s^2 - 1}} = \frac{K_P + K_D s}{s^2 + \underbrace{K_D s + K_P - 1}}$$

بامكاني اخلي ال system stable من خلال قيمهم لانه قيمة ال KP

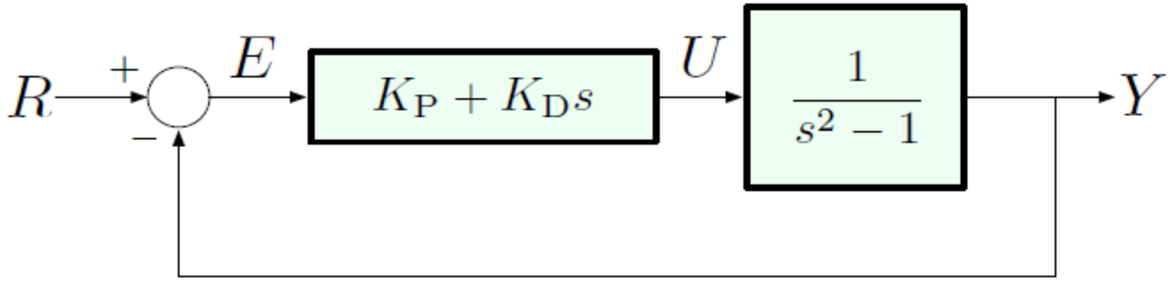
عكس = زيادة ال cost

— now, if we set $K_D > 0$ and $K_P > 1$, then the transfer function will be stable.

Even more: by choosing K_P and K_D, we can *arbitrarily* assign coefficients of the denominator, and therefore the poles of the transfer function:

PD control gives us arbitrary pole placement!!

Proportional-Derivative (PD) Control



$$\frac{Y}{R} = \frac{K_P + K_D s}{s^2 + K_D s + K_P - 1}$$

By choosing K_P, K_D, we can achieve arbitrary pole placement!!

Also note that the addition of P-gain moves the zero:

$$K_D s + K_P = 0 \qquad \text{LHP zero at } -\frac{K_P}{K_D}$$

$$\lim_{s \to 0} \frac{K_P + K_D s}{s^2 + K_D s + K_P - 1} \neq 1$$

But what's missing? DC gain $= \left.\frac{Y}{R}\right|_{s=0} = \frac{K_P}{K_P - 1} \neq 1$

دائماً القيمة أقل من ١ عشان هيك
دائماً يكون في steady state error.

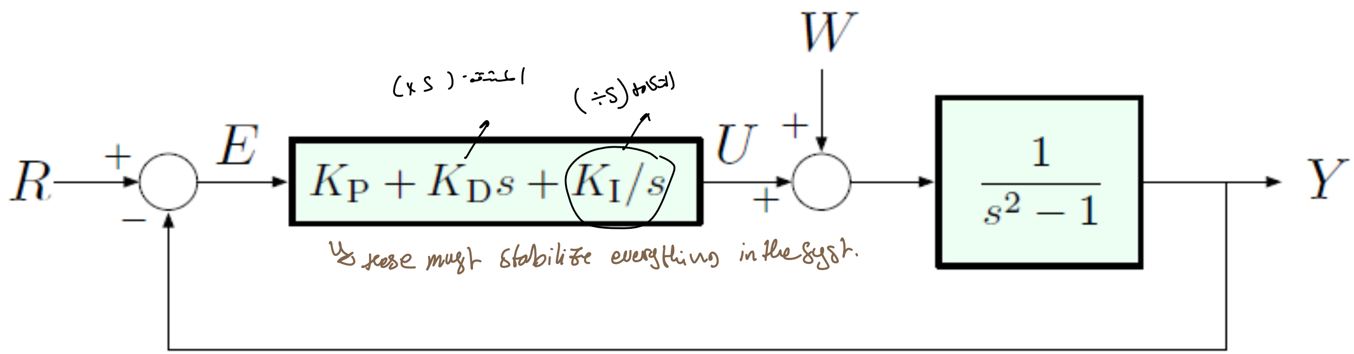
— can't have perfect tracking of constant reference.

Proportional-Integral-Derivative (PID) Control

Let us try

$$U = \left(K_{\mathrm{P}} + K_{\mathrm{D}} s + \frac{K_{\mathrm{I}}}{s} \right) E \qquad \text{– the classic three-term controller}$$

In fact, let's also throw in a constant disturbance:

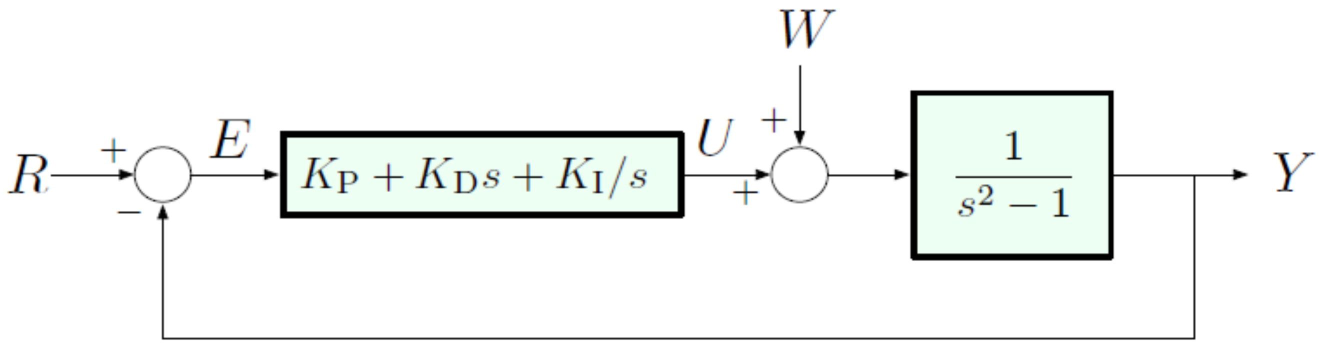


We will see that, with PID control, the goals of

- tracking a constant reference r
- rejecting a constant disturbance w

can be accomplished in one shot.

Proportional-Integral-Derivative (PID) Control



$$Y = \frac{1}{s^2-1}(U+W), \qquad U = \left(K_P + K_D s + \frac{K_I}{s}\right)(R-Y)$$

$$\text{so } Y = \frac{K_P + K_D s + \frac{K_I}{s}}{s^2-1}(R-Y) + \frac{1}{s^2-1}W$$

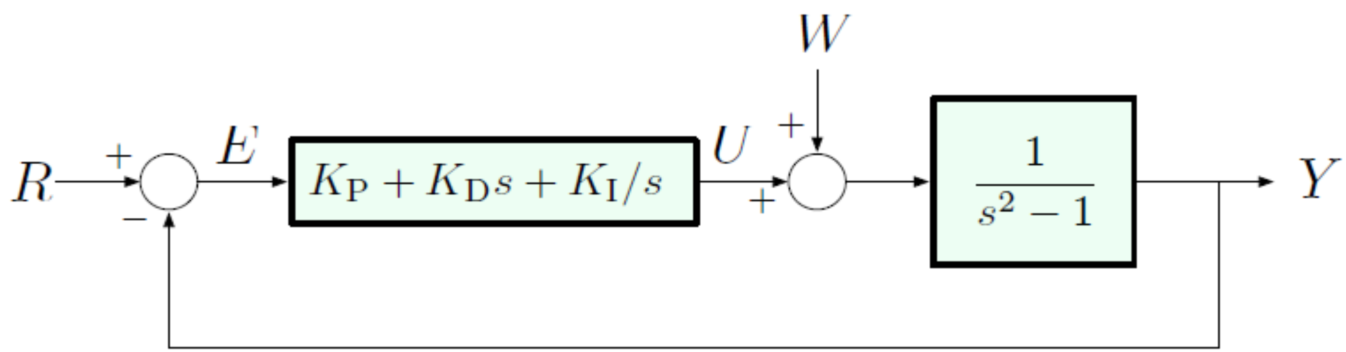
Simplify:

$$(s^2-1)Y = \left(K_P + K_D s + \frac{K_I}{s}\right)(R-Y) + W$$

$$\left(s^2 - 1 + K_P + K_D s + \frac{K_I}{s}\right)Y = \left(K_P + K_D s + \frac{K_I}{s}\right)R + W$$

$$(s^3 - s + K_P s + K_D s^2 + K_I)Y = (K_P s + K_D s^2 + K_I)R + W s$$

Proportional-Integral-Derivative (PID) Control

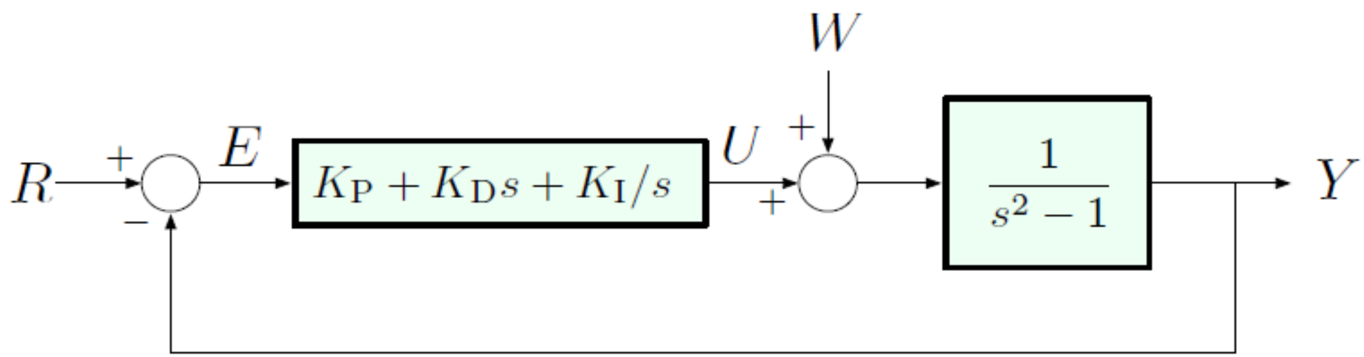


$$(s^3 - s + K_P s + K_D s^2 + K_I)Y = (K_P s + K_D s^2 + K_I)R + Ws$$

Therefore,

$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

Proportional-Integral-Derivative (PID) Control

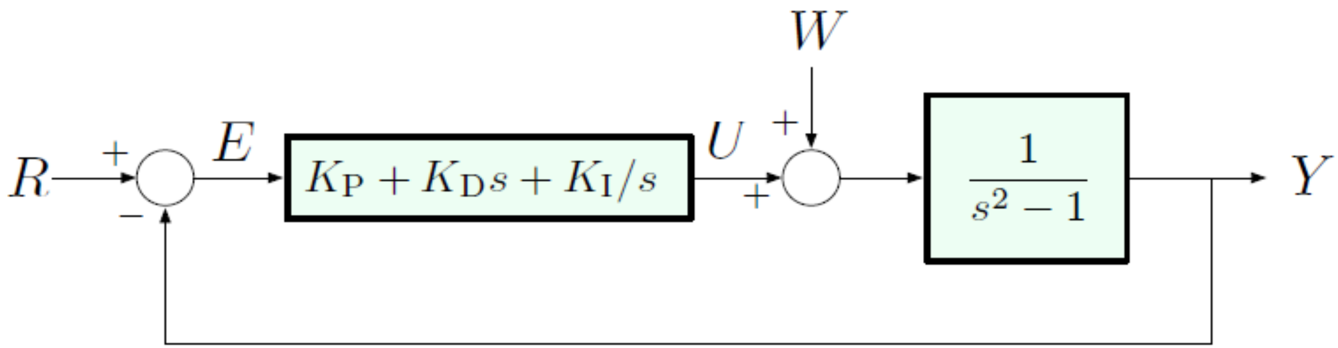


$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

Stability:

- need $K_D > 0$, $K_P > 1$, $K_I > 0$ (necessary condition) and $K_D(K_P - 1) > K_I$ (Routh–Hurwitz for 3rd-order)
- can still assign coefficients arbitrarily by choosing K_P, K_I, K_D

Proportional-Integral-Derivative (PID) Control



$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R$$

→ traking

فقط هو المسؤول عن ال stability

$$+ \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

rejection

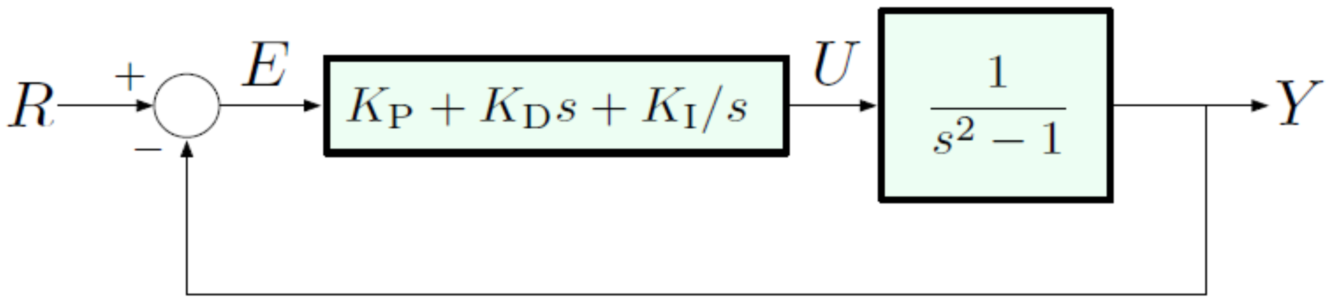
stability ال تحدد ال disturbance

Reference tracking:

$$\text{DC gain}(R \to Y) = \left. \frac{K_D s^2 + K_P s + K_I}{s^3 + (K_P - 1)s + K_D s^2 + K_I} \right|_{s=0} = 1$$

— so, with the addition of I-feedback, we remove earlier limitation and achieve *perfect tracking*!

Proportional-Integral-Derivative (PID) Control



$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

Disturbance rejection:

$$\text{DC gain}(W \to Y) = \left. \frac{s}{s^3 + (K_P - 1)s + K_D s^2 + K_I} \right|_{s=0} = 0$$

— so, integral gain also gives *complete attenuation* of *constant* disturbances!!

PID Control: Summary & Further Comments

P-gain simplest to implement, but not always sufficient for stabilization

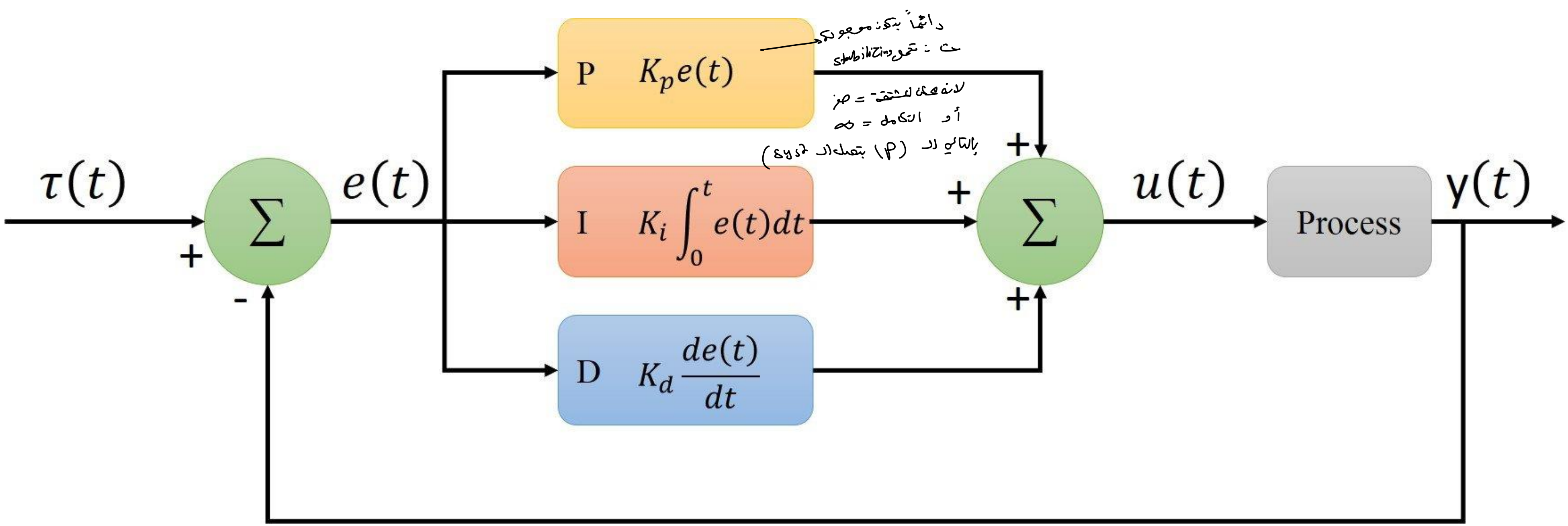
D-gain helps achieve stability, improves time response (more control over pole locations)

- arbitrary pole placement only valid for 2nd-order response; in general, we still have control over two *dominant poles*
- cannot be implemented directly, so need approximate implementation; D-gain also amplifies noise

I-gain essential for perfect steady-state tracking of constant reference and rejection of constant disturbance

PID Controller

in time domain.



PID Controller

- The output of the controller is proportional to the error signal (P-term), the integral signal (I-term), and the derivative of the error signal (D-term) through the gains K_p, K_i, and K_d, respectively.

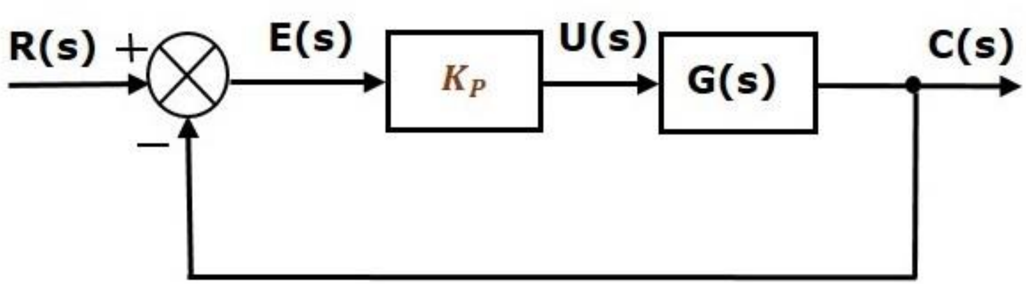
- In practice, variations of the above control law are also implemented, such as a PI controller, which has only the P and I terms, or a PD controller, which has only the P and D terms.

P - Controller

- With the proportional mode, the size of the controller output is proportional to the size of the error.

- This means that the correction elements of the control system (eg: valve), will receive a signal which is proportional to the size of the correction required.

- A gain element with transfer function Kp in series with the forward-path element G(s)



$$U(s) = K_P E(s)$$

$$\frac{U(s)}{E(s)} = K_P$$

D- Controller

- With the derivative mode of control the controller output is proportional to the rate of change with time of the error signal. This can be represented by the equation

$$\textit{controller}\ \text{output} = K_D \frac{de}{dt}$$

Kp is the constant of proportionality. The transfer function is obtained by taking Laplace transforms, thus

$$\textit{controller}\ \text{output(s)} = K_D sE(s)$$

Hence the transfer function is $K_D s$

D- Controller

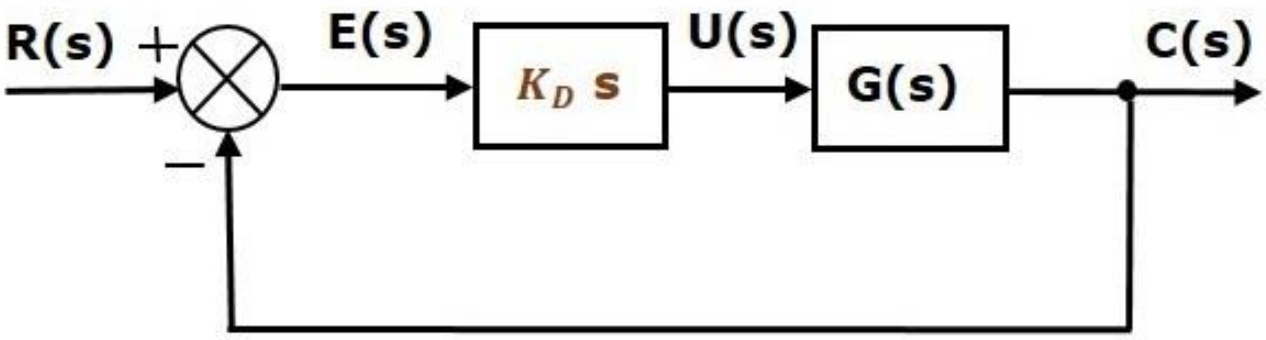
- With derivative control, as soon as the error signal begins to change, there can be quite a large controller output since it is proportional to the rate of change of the error signal and not its value.
  - Rapid initial responses to error signals thus occur.

- The controller output is constant because the rate of change is constant and occurs immediately the deviation occurs.

- Derivative controllers do not, however, respond to steady-state error signals, since with a steady error the rate of change of error with time is zero.

- Because of this, derivative control is always combined with proportional; the proportional part gives a response to all error signals, including steady signals, while the derivative part response to the rate of change.

D- Controller



$$U(s) = K_D s E(s)$$

$$\frac{U(s)}{E(s)} = K_D s$$

PD- Controller

- Derivative control is never used alone because it is not capable of giving an output when there is a steady state error signal and so no correction is possible.

- It is thus invariably used in conjuction with proportional control so that this problem can be resolved.

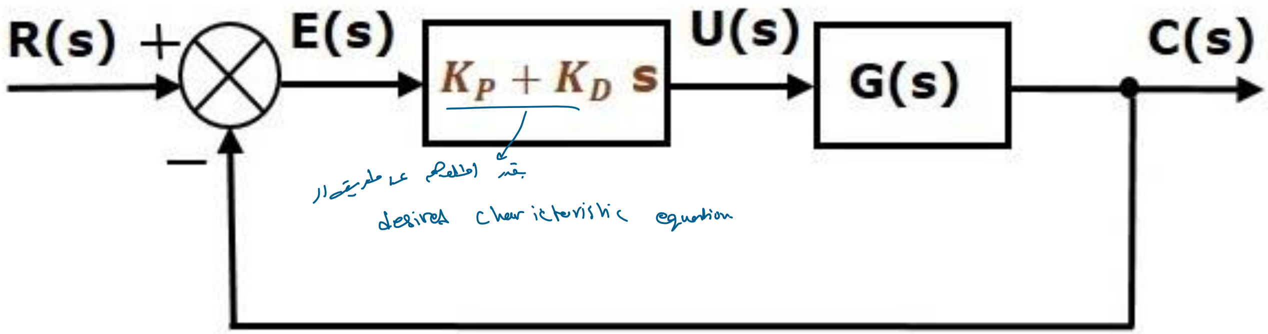
- With proportional plus derivative control the controller output is given by

$$\mathit{controller}\ \text{output} = K_p e + K_D \frac{de}{dt}$$

K_P is the proportionality constant and K_D the derivative constant, de/dt is the rate of change of error.

PD- Controller

- closed loop charcteristic equation



بقدر اطلعهم عطريقة ال

desired charicteristic equation

$$U(s) = (K_P + K_D s)E(s)$$

Char. eqn

$$\frac{U(s)}{E(s)} = K_P + K_D s$$

I - Controller

- The integral mode of control is one where the rate of change of th control output I is proportional to the input error signal e:

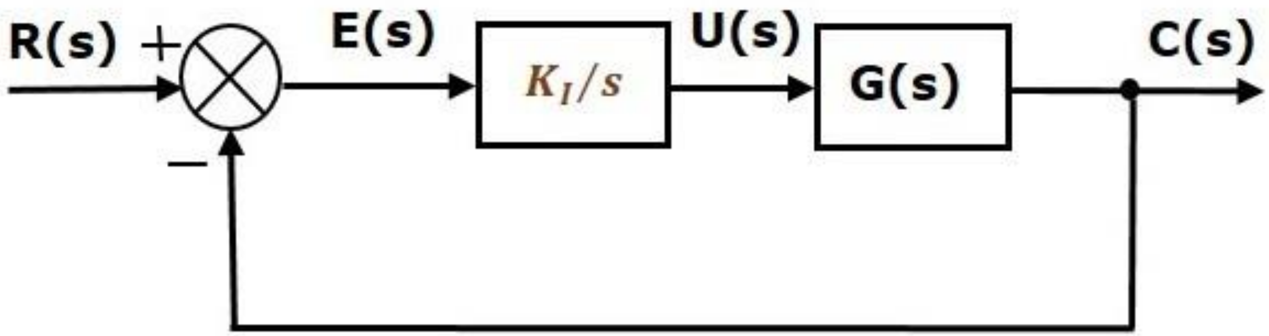
$$\frac{dI}{dt} = K_I e$$

K_I is the constant of proportionality and has units of $1/s$. integrating the above equations gives

$$\int_{Io}^{Iout} dI = \int_0^t K_I e \, \mathrm{d}t \qquad\qquad I_{out} - I_0 = \int_0^t K_I e \, \mathrm{d}t$$

I_0 is the controller output at zero time, I_{out} is the output at time t.

I - Controller



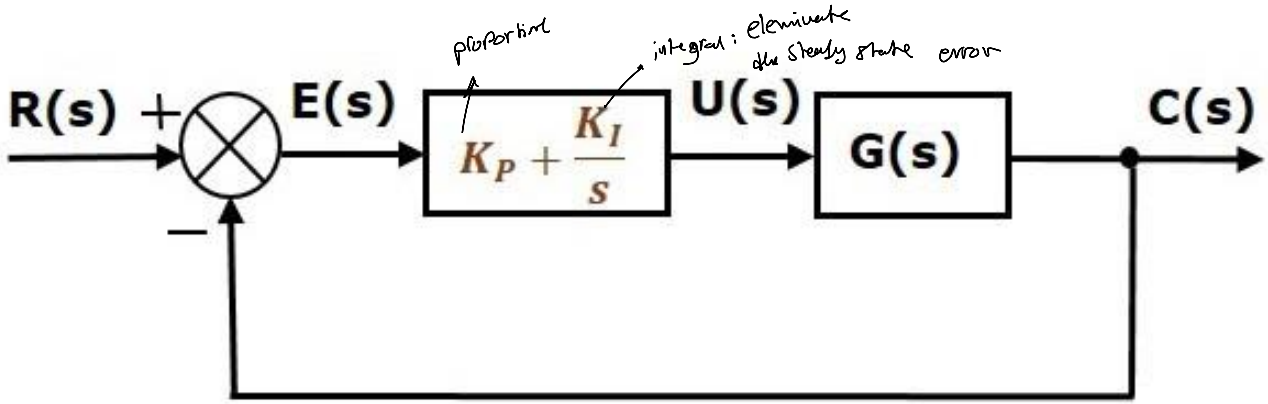
$$U(s) = \frac{K_I E(s)}{s}$$

$$\frac{U(s)}{E(s)} = \frac{K_I}{s}$$

PI - Controller

- The integral mode of control is not usually alone but is frequently used in conjuction with the proportional mode. When integral action is added to a proportional control system the controller output is given by

$$\textit{controller output} = K_p e + K_I \int e \, \mathrm{d}t$$

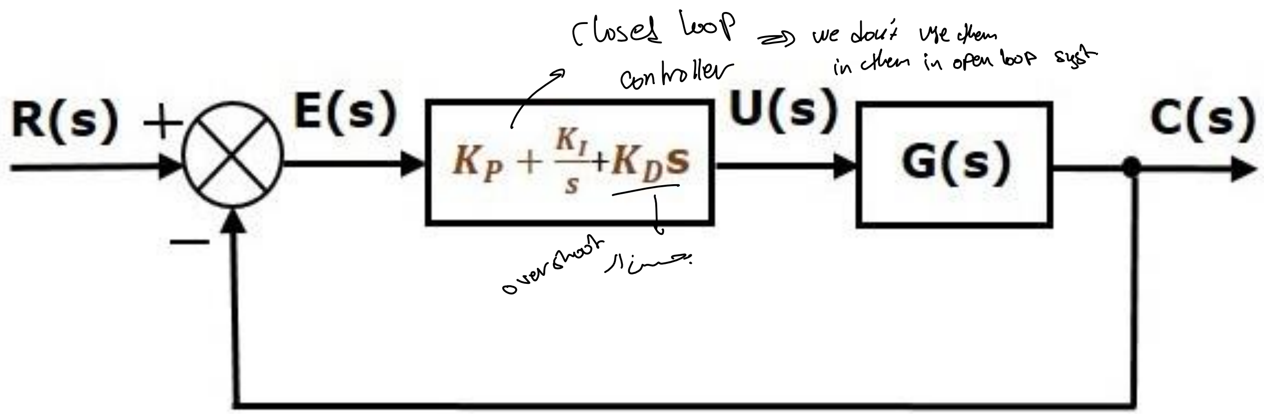


$$U(s) = \left(K_P + \frac{K_I}{s}\right) E(s)$$

$$\frac{U(s)}{E(s)} = K_P + \frac{K_I}{s}$$

PID - Controller

- ❑ Combining all three modes of control gives a controller known as a three-mode controller of PID controller.



$$u(t) = K_P e(t) + K_I \int e(t)dt + K_D \frac{\mathrm{d}e(t)}{\mathrm{d}t}$$

Apply Laplace transform on both sides -

$$U(s) = \left(K_P + \frac{K_I}{s} + K_D s\right) E(s)$$

$$\frac{U(s)}{E(s)} = K_P + \frac{K_I}{s} + K_D s$$

The Characteristics of P, I, and D controllers

A proportional controller (Kp) will have the effect of reducing the rise time and will reduce, but never eliminate, the steady-state error.

An integral control (Ki) will have the effect of eliminating the steady-state error, but it may make the transient response worse.

A derivative control (Kd) will have the effect of increasing the stability of the system, reducing the overshoot, and improving the transient response.

Proportional Control

By only employing proportional control, a steady state error occurs.

Proportional and Integral Control

The response becomes more oscillatory and needs longer to settle, the error disappears.

Proportional, Integral and Derivative Control

All design specifications can be reached.

Geo Gebra

The Characteristics of P, I, and D controllers

CL RESPONSE	RISE TIME τ	OVERSHOOT	SETTLING TIME	S-S ERROR
initial designs – rise time – time responce **Kp**	Decrease	Increase	Small Change	small effect Decrease
steady state error **Ki**	Decrease	Increase	Increase	Eliminate
time performance **Kd**	Small Change	Decrease	Decrease	Small Change

- PD to decrease the overshoot.
- P i → steady state
P id

Tips for Designing a PID Controller

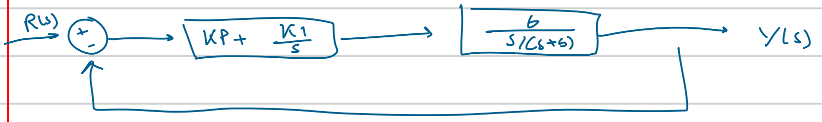
1. Obtain an open-loop response and determine what needs to be improved
2. Add a proportional control to improve the rise time
3. Add a derivative control to improve the overshoot
4. Add an integral control to eliminate the steady-state error
5. Adjust each of Kp, Ki, and Kd until you obtain a desired overall response.

Lastly, please keep in mind that you do not need to implement all three controllers (proportional, derivative, and integral) into a single system, if not necessary. For example, if a PI controller gives a good enough response (like the above example), then you don't need to implement derivative controller to the system. Keep the controller as simple as possible.

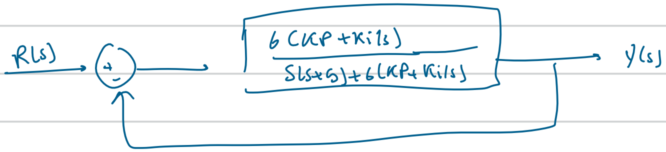
example

if $G(s) = \frac{6}{s(s+5)}$, Designe PI Controller: + $E_{ss} = 0$

+ $T_s < 2$ sec



$\Rightarrow$



① find → closed loop transfer function

$$\frac{Y(s)}{R(s)} = \frac{PI * G(s)}{1 + PI * G(s)} = \frac{6(KP + Ki/s)}{\boxed{s(s+5) + 6(KP + Ki/s)}}$$

characteristic equation.

② characteristic equation

standard form.

$$s^2 + (5 + 6KP)s + 6Ki = 0 \iff s^2 + 2\xi w_n s + w_n^2$$

$$T_s = \frac{4}{\xi w_n}$$

$$T_s < 2$$

$$\xi w_n > 2$$

$$2\xi w_n = 5 + 6KP$$

$$w_n^2 = 6Ki$$

$$\xi w_n = \cancel{(\sqrt{6Ki})} \cdot \boxed{\frac{5 + 6KP}{2\cancel{\sqrt{6Ki}}}}$$

ξ

قبل ما نحذف

$\sqrt{16 Ki}$

$$\xi w_n = \frac{5 + 6KP}{2} > 2$$

Since it's a design

we assume one of the values $\Rightarrow$ the assumption must satisfy the condition given in the question

assume $KP = 2$

$$\Rightarrow \xi w_n = \frac{5 + 6(2)}{2} = \frac{17}{2}$$

بطلع قيمتها من المعادلة

$$\Rightarrow Ki = 8.5$$

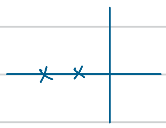
$$s^2 + (5 + 6KP)s + 6Ki$$

$$s^2 + (5 + 12)s + (6)(8)$$

$$s^2 + 17s + 48$$

$$\Rightarrow s = \frac{-17 \pm \sqrt{(17)^2 - 192}}{2}$$

$$= -8.061, -8.99$$



$$T_s = \frac{4}{\xi w_n}$$

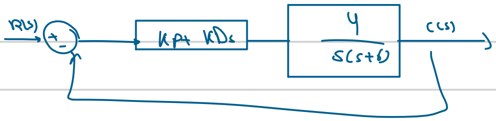
$$\xi w_n = \frac{17}{2}$$

$$T_s = \frac{4}{17/2} = \frac{4}{8.5} \simeq 0.5 s$$

ex:

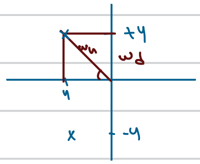
$G(s) = \dfrac{4}{s(s+6)}$, design a PD Controller for dominant Poles

$s = -4 \pm j4$



$$\frac{C(s)}{R(s)} = \frac{\dfrac{4(KP+KDs)}{s(s+6)}}{1+\dfrac{4(KP+KDs)}{s(s+6)}} = \frac{4(KP+KDs)}{s(s+6)+4(KP+KDs)}$$

$$= \frac{4(KP+KDs)}{\boxed{s^2+(6+4KDs)+4KP}} \rightarrow \text{closed loop charactaristic equation.}$$



Dominant Poles:

- The nearest Poles to the Jw axis.
- The poles responsible for Performance Characteristics of the sys.

جذور ال Poles بعوض

$(s+4-j4)(s+4+4j) = \boxed{s^2+8s+32} \rightarrow$ Desired charactaristic equation.

$\boxed{s^2+8s+32} \Longleftrightarrow \boxed{s^2+(6+4KDs)+4KP}$

$4KP = 32 \Rightarrow \boxed{KP = 8}$

$6+4KD = 8 \Rightarrow \boxed{KD = 0.5}$

لو أعطاني السؤال Ts بدلالة

$\Rightarrow T_s = \dfrac{4}{\zeta w_n} = 8s$

KP, Ks > zero

$8\zeta w_n = 4$

$\boxed{\zeta w_n = 0.5}$

تعويض

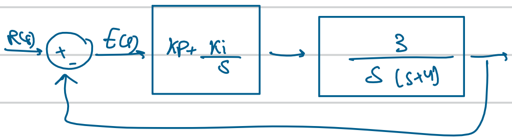
$s^2 + 2\zeta w_n s + w_n^2$

$\boxed{s^2 + s + w_n^2}$

ex: if the steady state error = 0, what controller should be used? PI

H.w: $G(s) = \frac{3}{s(s+4)}$, $E_{ss} = 0$, $T_s < 3s$
$\downarrow$
PI

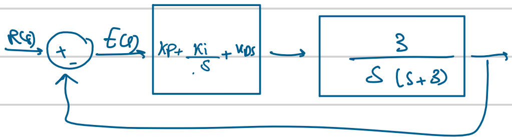
* اذا ما أعطاك بالسؤال قيمة R(s)
افترض انه step-resp

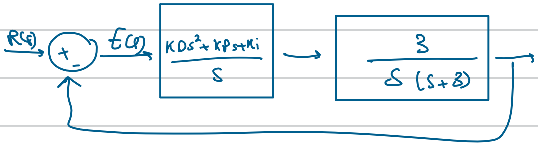


ex: Design a PID Controller $G(s) = \frac{3}{s(s+3)}$, find KP, KD, Ki

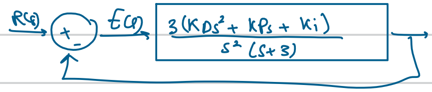
Dominant Poles

$S = -5 \pm j5$





$\Rightarrow$



closed loop characteristic equ: $s^2(s+3) + 3(KDs^2 + KPs + Ki)$

$= s^3 + s^2(3 + 3KD) + 3KPs + 3Ki$

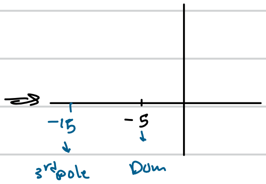
desired characterist equ: $(s+5+j5)(s+5-j5)$

$= s^2 + 10s + 50$

$\rightarrow$ ما بقدر اقارن بين معاملات S^3, S^2

عشان هيك بفترض 3rd pole بكون

بعيد عن ال Dominant poles



* كلما كان ال Pole ابعد بال -ve كان تأثيره اقل

* بالعادة بنجيب Dominant x 10

بس الدكتور افترض Dominant x 3

$\Rightarrow (S^2+10S+50)(S+15) = S^3+10S^2+50S+750+15S^2+150S$

$= \boxed{S^3 + 25S^2 + 200S + 750}$

$\boxed{S^3 + s^2(3+3KD) + 3KPs + 3Ki}$

$25 = 3 + 3KD \Rightarrow \boxed{KD = \frac{22}{3}}$

$200 = 3KPs \Rightarrow \boxed{KPs = \frac{200}{3}}$

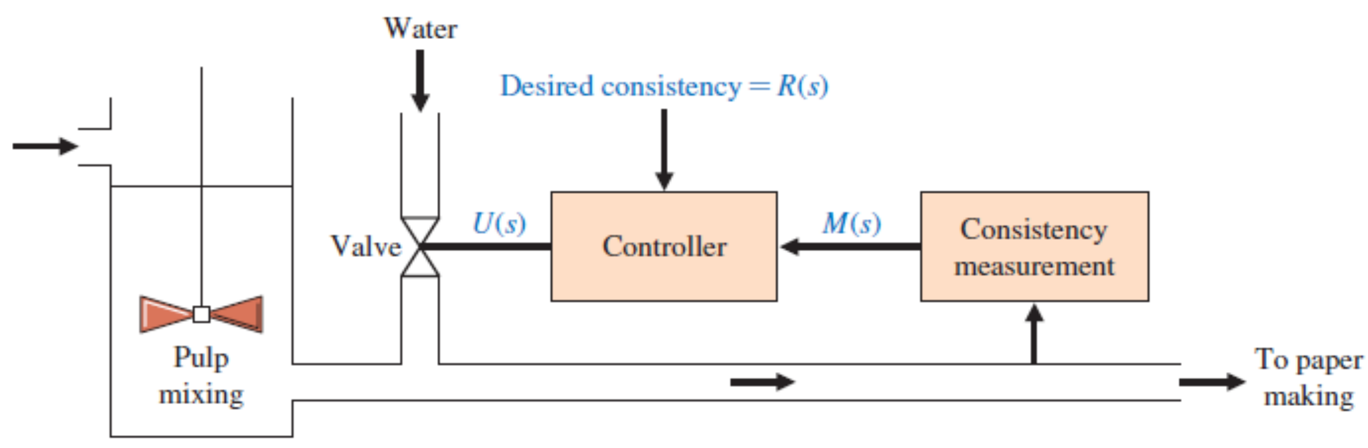
$750 = 3Ki \Rightarrow \boxed{KPi = 250}$

Questions

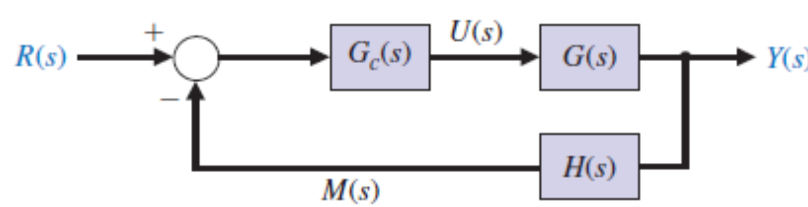
P4.11 One important objective of the paper-making process is to maintain uniform consistency of the stock output as it progresses to drying and rolling. A diagram of the thick stock consistency dilution control system is shown in Figure P4.11(a). The amount of water added determines the consistency. The block diagram of the system is shown in Figure P4.11(b). Let $H(s) = 1$ and

$$G_c(s) = \frac{K}{8s+1}, \qquad G(s) = \frac{1}{3s+1}.$$

Determine (a) the closed-loop transfer function $T(s) = Y(s)/R(s)$, (c) the steady-state error for a step change in the desired consistency $R(s) = A/s$. (d) Calculate the value of K required for an allowable steady-state error of 2%.



(a)



(b)

Questions

P4.11 (a) The closed-loop transfer function is

$$T(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)} = \frac{K}{(8s+1)(3s+1)+K} = \frac{K}{24s^2+11s+1+K} .$$

(c) Define $E(s) = R(s) - Y(s)$. Then

$$E(s) = \frac{R(s)}{1 + G_c(s)G(s)} = \left[\frac{24s^2+11s+1}{24s^2+11s+K+1}\right] R(s) .$$

With

$$R(s) = \frac{A}{s} ,$$

we have

$$e_{ss} = \lim_{s \to 0} sE(s) = \frac{A}{1+K} .$$

(d) We want $|e(t)| \leq 0.02A$ as $t \to \infty$. So,

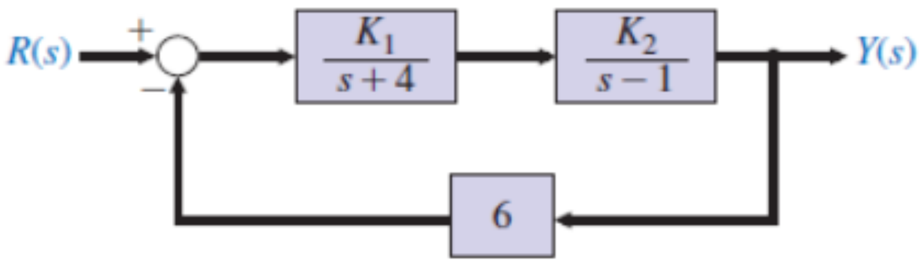
$$0.02A \geq \frac{A}{K+1}$$

implies

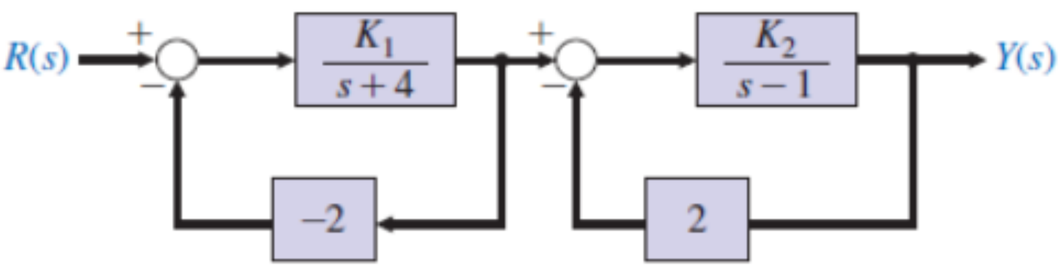
$$K \geq 49 .$$

Questions

P4.12 Two feedback systems are shown in Figures P4.12(a) and (b). (a) Evaluate the closed-loop transfer functions T_1 and T_2 for each system.



(a)



(b)

Questions

(a) The two transfer functions are

$$T_1(s) = \frac{K_1K_2}{s^2 + 3s - 4 + 6K_1K_2}$$

and

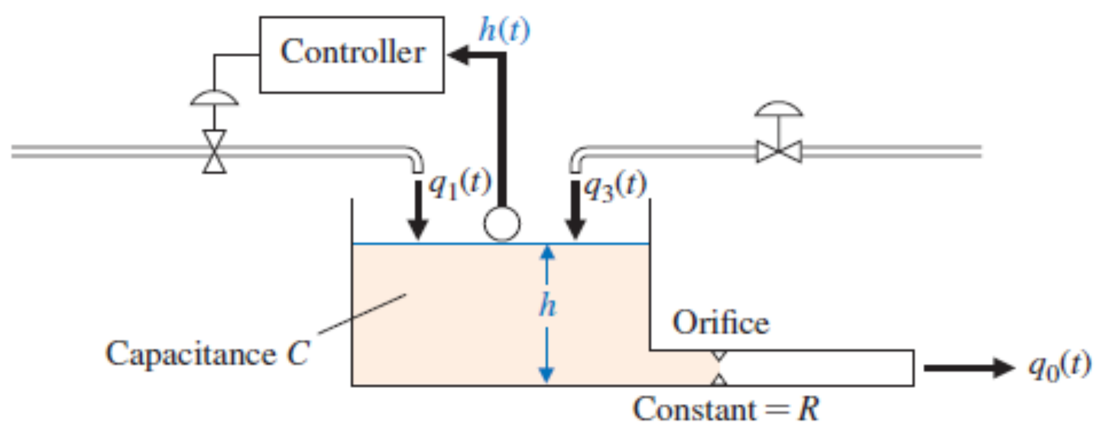
$$T_2(s) = \frac{K_1K_2}{s^2 + (3 - 2K_1 + 2K_2)s - 4 + 8K_2 + 2K_1 - 4K_1K_2} .$$

When $K_1 = K_2 = 1$, we find that

$$T_1(s) = T_2(s) = \frac{1}{s^2 + 3s + 2} .$$

Questions

AP4.1 A tank level regulator control is shown in Figure AP4.1(a). It is desired to regulate the level $H(s)$ in response to a disturbance change $Q_3(s)$. The block diagram shows small variable changes about the equilibrium conditions so that the desired $H_d(s) = 0$. Determine the equation for the error $E(s)$, and determine the steady-state error for a unit step disturbance when (a) $G(s) = K$ and (b) $G(s) = K/s$.



(a)

$R(s) = \frac{1}{s}$

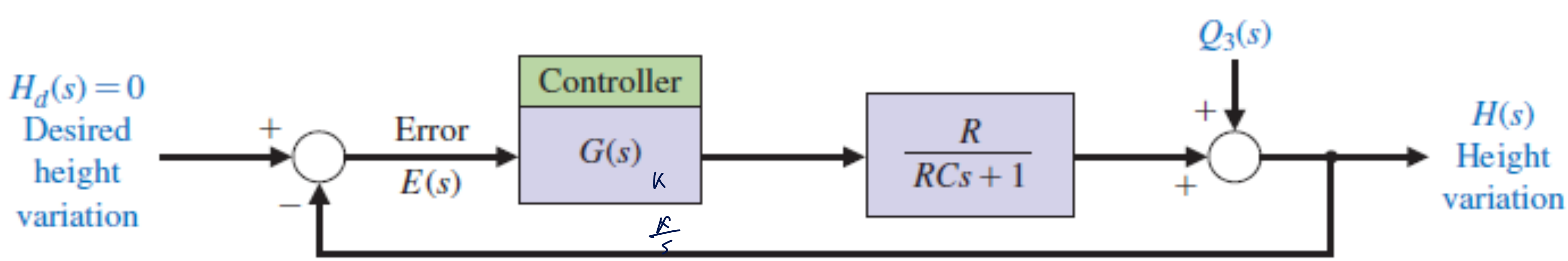
$e_{ss} = \frac{1}{1+K_P}$

$K_P = \lim_{s \to 0} G(s)$

$= \lim_{s \to 0} K$

$= K$

$e_s = \frac{1}{1+K}$



(b)

Questions

AP4.1 The plant transfer function is

$$G_p(s) = \frac{R}{RCs + 1} \ .$$

The closed-loop output is given by

$$H(s) = \frac{1}{1 + GG_p(s)} Q_3(s) + \frac{GG_p(s)}{1 + GG_p(s)} H_d(s) \ .$$

Therefore, with $E(s) = H_d(s) - H(s)$, we have

$$E(s) = \frac{-1}{1 + GG_p(s)} Q_3(s) \ ,$$

since $H_d(s) = 0$.

(a) When $G(s) = K$, we have

$$e_{ss} = \lim_{s \to 0} sE(s) = \frac{-1}{1 + KR} \ .$$

(b) When $G(s) = K/s$, we have

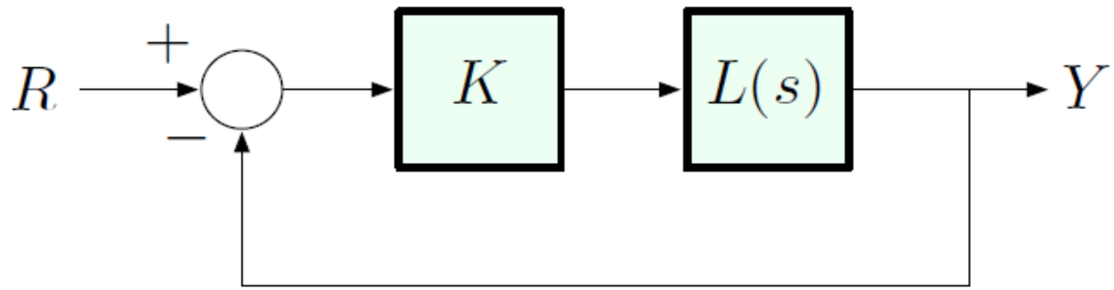
$$e_{ss} = \lim_{s \to 0} sE(s) = 0 \ .$$

largest range Value of K that make the syst stable
best design specification

The Root Locus Design Method

(invented by Walter R. Evans in 1948)

Consider this unity feedback configuration:



where

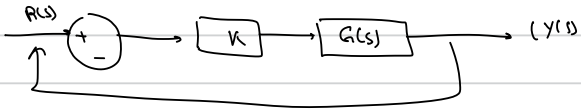
Characteristic equ: $\boxed{1+K\,L(s)=0}$

- K is a constant gain
- $L(s) = \dfrac{b(s)}{a(s)}$, where $a(s)$ and $b(s)$ are some polynomials

2/ يجي عليه ا سؤال
بالفاينل

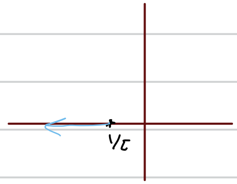
The Root locus design Method:

$G(s) = \frac{1}{\boxed{Ts+1}}$ → open loop transfer function.



$G(s) = \boxed{\frac{K}{Ts+1+K}}$ → closed loop transfer function.

K ↑ T ↓

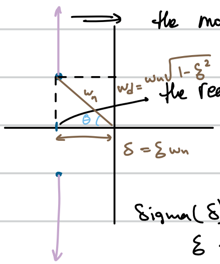


(*) The set of the closed loop poles as K changes from $0 \le K < \infty$

The Root locus:

- $G(s) = \frac{w_n^2}{s^2+2\xi w_n s+w_n^2}$

$G(s) = \frac{K w_n^2}{s^2+2\xi w_n s+(K+1) w_n^2}$



→ the movment of the imajenary part of the complex conjeg (when K increase the poles move in these directions)

$w_d = w_n\sqrt{1-\xi^2}$ → the real part remains const.

$\sigma = \xi w_n$

when K increase ξ

Sigma(σ): const → (T_s) remains const

ξ : change

w_d : change

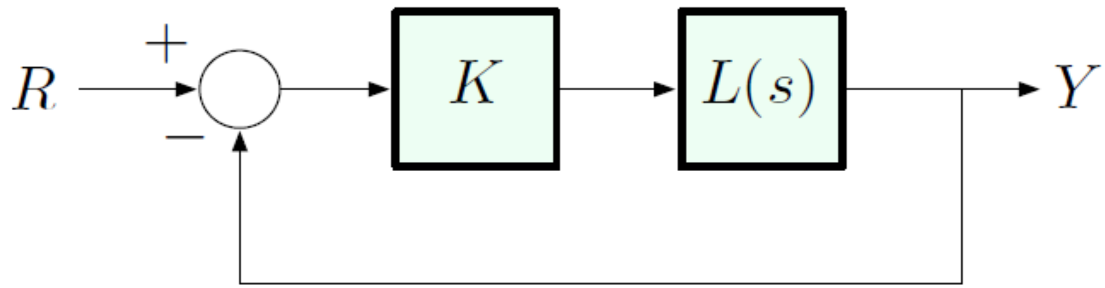
w_n : change

$T_s = \frac{4}{\sigma}$

over shoot : increase because θ increase

$t_p = \frac{\pi}{w_d}$

The Root Locus Design Method



Closed-loop transfer function: $\frac{Y}{R} = \frac{KL(s)}{1+KL(s)}$, $L(s) = \frac{b(s)}{a(s)}$

Closed loop poles are solutions of:

characteristic equation

$$1 + KL(s) = 0 \quad \Leftrightarrow \quad L(s) = -\frac{1}{K}$$

$$\Updownarrow$$

$$1 + \frac{Kb(s)}{a(s)} = 0$$

توحيد المقامات

$$\Updownarrow$$

poles for open loop transfer function

zeros for the closed loop transfer func

$$\underbrace{a(s) + K\boxed{b(s)}}_{\text{characteristic polynomial}} = 0 \qquad \text{characteristic equation}$$

$$\frac{a(s) + Kb(s)}{a(s)}$$

characteristic equation

A Comment on Change of Notation

Note the change of notation:

$$\text{from } H(s) \text{ or } G(s) = \frac{q(s)}{p(s)} \quad \text{to } L(s) = \frac{b(s)}{a(s)}$$

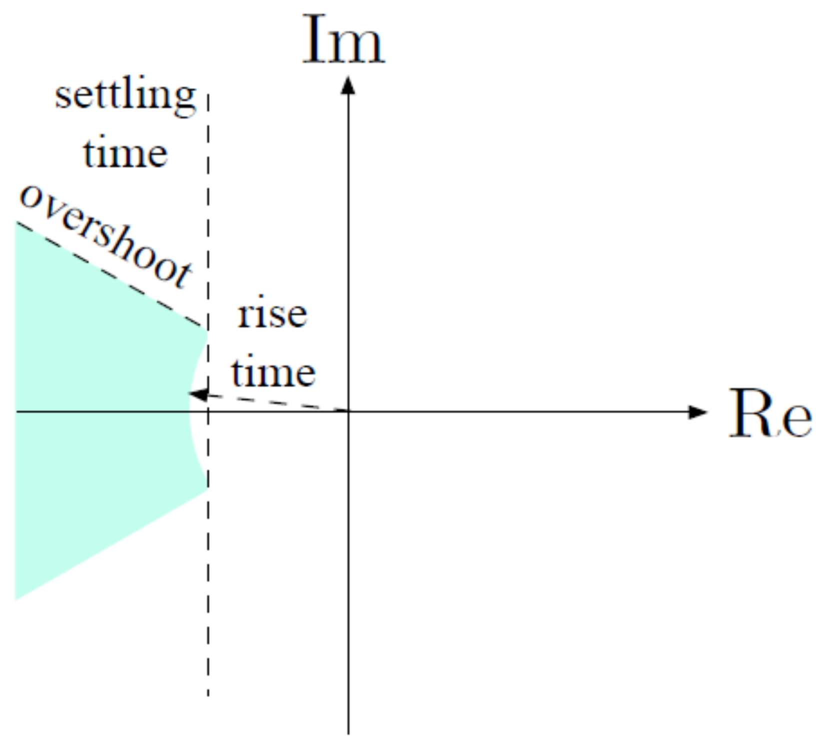
— the RL method is quite general, so $L(s)$ is not necessarily the *plant* transfer function, and K is not necessary *feedback gain* (could be *any parameter*).

E.g., $L(s)$ and K may be related to plant transfer function and feedback gain through some transformation.

As long as we can represent the poles of the closed-loop transfer function as roots of the equation $1 + KL(s) = 0$ for *some choice* of K and $L(s)$, we can apply the RL method.

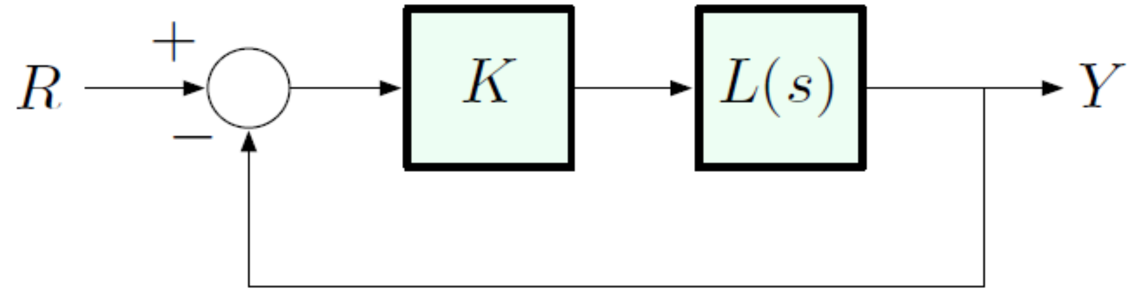
Towards Quantitative Characterization of Stability

Qualitative description of stability: Routh test gives us a range of K to guarantee stability.



For what values of K do we best satisfy given design specs?

Root Locus and Quantitative Stability



Closed-loop transfer function: $\dfrac{Y}{R} = \dfrac{KL(s)}{1 + KL(s)}, \;\; L(s) = \dfrac{b(s)}{a(s)}$

For what values of K do we best satisfy given design specs?

Specs are encoded in pole locations, so:

The *root locus* for $1 + KL(s)$ is the set of all closed-loop poles, i.e., the roots of

$$1 + KL(s) = 0,$$

as K varies from 0 to ∞.

A Simple Example

$$L(s) = \frac{1}{s^2 + s} \qquad b(s) = 1, \; a(s) = s^2 + s$$

unstable ←

For open loop transfer function.

اعقام البسط

Characteristic equation: $\qquad a(s) + Kb(s) = 0$

لازم اعرف عين اطلعها

$$s^2 + s + K = 0$$

Here, we can just use the quadratic formula:

$$s = -\frac{1 \pm \sqrt{1-4K}}{2} = -\frac{1}{2} \pm \frac{\sqrt{1-4K}}{2}$$

Root locus

بحدود ها على رسمة

the set of Closed loop Pole as K Varies from Zero to ∞

$$\text{Root locus} = \left\{ \underbrace{-\frac{1}{2} \pm \frac{\sqrt{1-4K}}{2}}_{\text{Closed-loop Pole equation}} : 0 \le K < \infty \right\} \subset \mathbb{C}$$

$$RL = \frac{-1}{2} \pm \frac{\sqrt{1-4K}}{2}$$

اي قيمة ال K بين صفر و ∞ عشان اعرف ارسمها

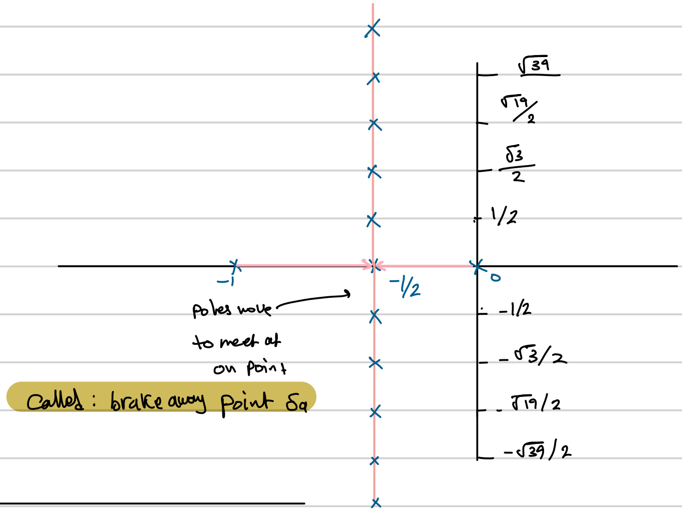
بعوض قيم ال K في معادلة ال RL

K	RL
0	0 , -1
1/4	-1/2
1/2	$-1/2 \pm \frac{1}{2} j$
1	$-1/2 \pm j\frac{\sqrt{3}}{2}$
5	$-1/2 \pm j\frac{\sqrt{19}}{2}$
10	$-1/2 \pm j\frac{\sqrt{39}}{2}$

to draw the Root locus

① start from the open loop poles

$$\frac{1}{s^2+s} \;,\; \boxed{\begin{matrix} s=0 \\ s=-1 \end{matrix}}$$



⟹ how to find the value of δa in a quantitative way?

* open loop poles / for open loop transfer function

Centroid:

$$\delta_A = \frac{\sum p_i - \sum z_i}{n - m}$$

of the poles # of the zeros

angle of asymptotes

$$\Rightarrow \phi_A = \frac{(1 \pm 2K)\pi}{n-m}$$

* starting at K=0

$$\Rightarrow \phi_A = \pm \frac{\pi}{2}$$

TD Specs in Frequency Domain

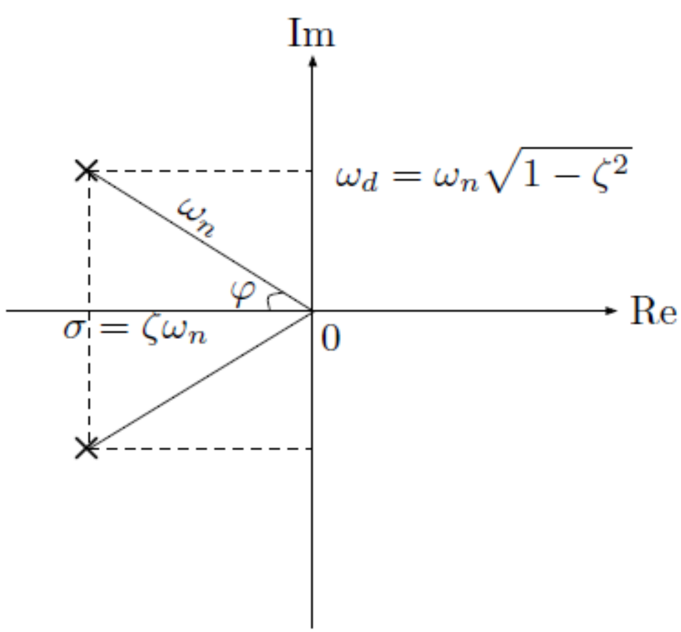
We want to *visualize* time-domain specs in terms of *admissible pole locations* for the 2nd-order system

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\sigma^2 + \omega_d^2}{(s+\sigma)^2 + \omega_d^2}$$

where $\sigma = \zeta\omega_n$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Step response: $y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$



$$\omega_n^2 = \sigma^2 + \omega_d^2$$

$$\zeta = \cos\varphi$$

Formulas for TD Specs

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\sigma^2 + \omega_d^2}{(s+\sigma)^2 + \omega_d^2}$$

$$t_r \approx \frac{1.8}{\omega_n}$$

$$t_p = \frac{\pi}{\omega_d}$$

$$M_p = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

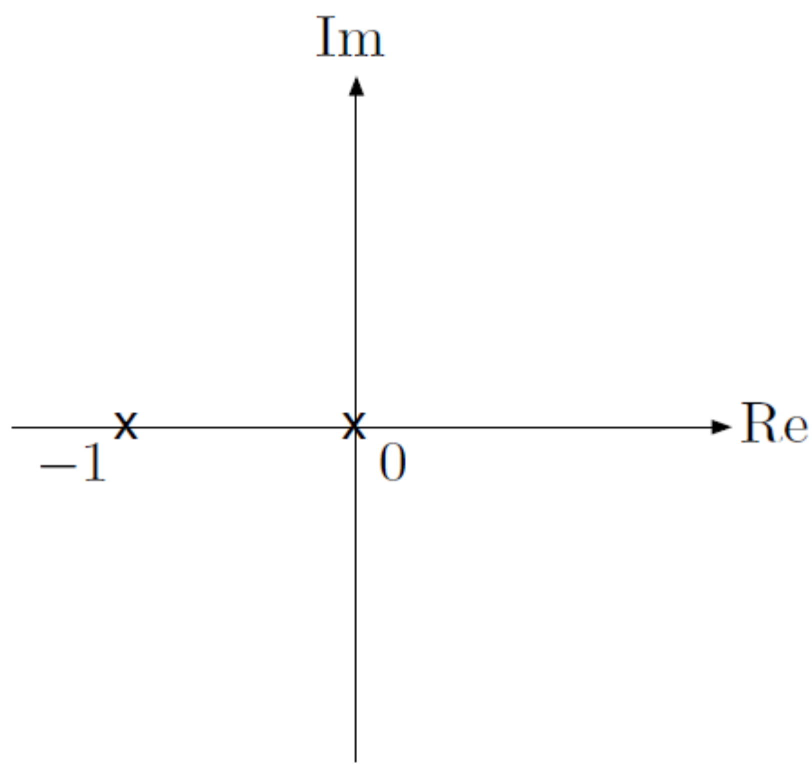
$$t_s \approx \frac{3}{\sigma} \quad 4$$

Example, continued

$$\text{Root locus} = \left\{ -\frac{1}{2} \pm \frac{\sqrt{1-4K}}{2} \; : 0 \le K < \infty \right\} \subset \mathbb{C}$$

Let's plot it in the s-plane:

- start at $K = 0$ the roots are $-\frac{1}{2} \pm \frac{1}{2} \equiv -1, 0$
  note: these are poles of L (open-loop poles)

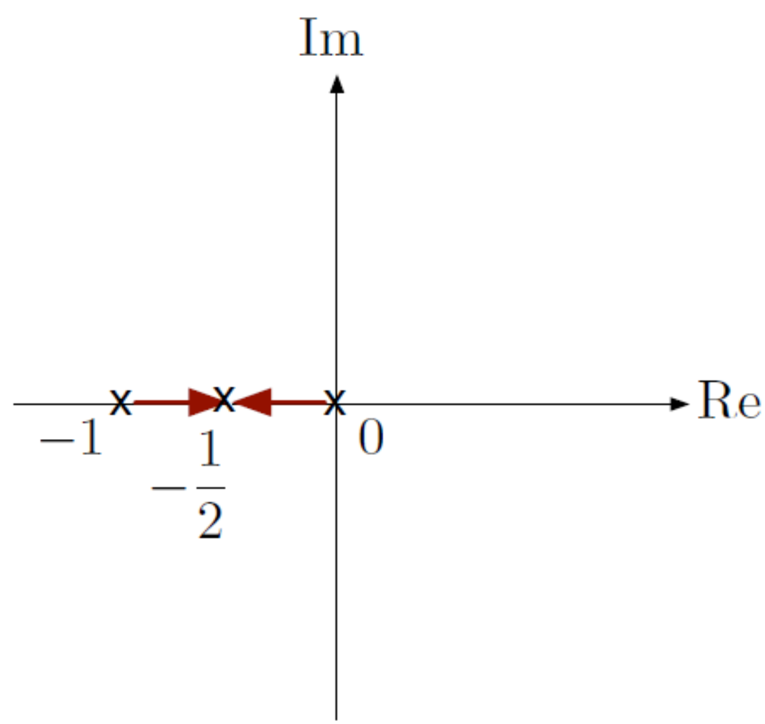


Example, continued

$$\text{Root locus: } \left\{ -\frac{1}{2} \pm \frac{\sqrt{1-4K}}{2} \ : 0 \le K < \infty \right\} \subset \mathbb{C}$$

- as K increases from 0, the poles start to move

$$\begin{aligned} 1 - 4K > 0 \quad &\Longrightarrow \ 2 \text{ real roots} \\ K = 1/4 \quad &\Longrightarrow 1 \text{ real root } s = -1/2 \end{aligned}$$

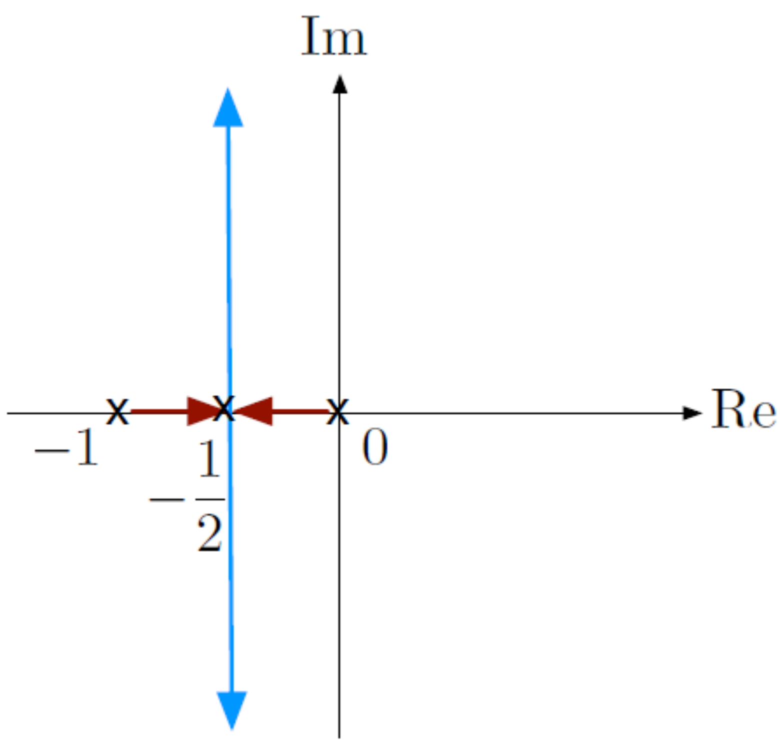


Example, continued

Root locus: $\left\{ -\frac{1}{2} \pm \frac{\sqrt{1-4K}}{2} \; : 0 \le K < \infty \right\} \subset \mathbb{C}$

- as K increases from 0, the poles start to move

$$K > 1/4 \qquad \Longrightarrow \; 2 \text{ complex roots with } \mathrm{Re}(s) = -1/2$$

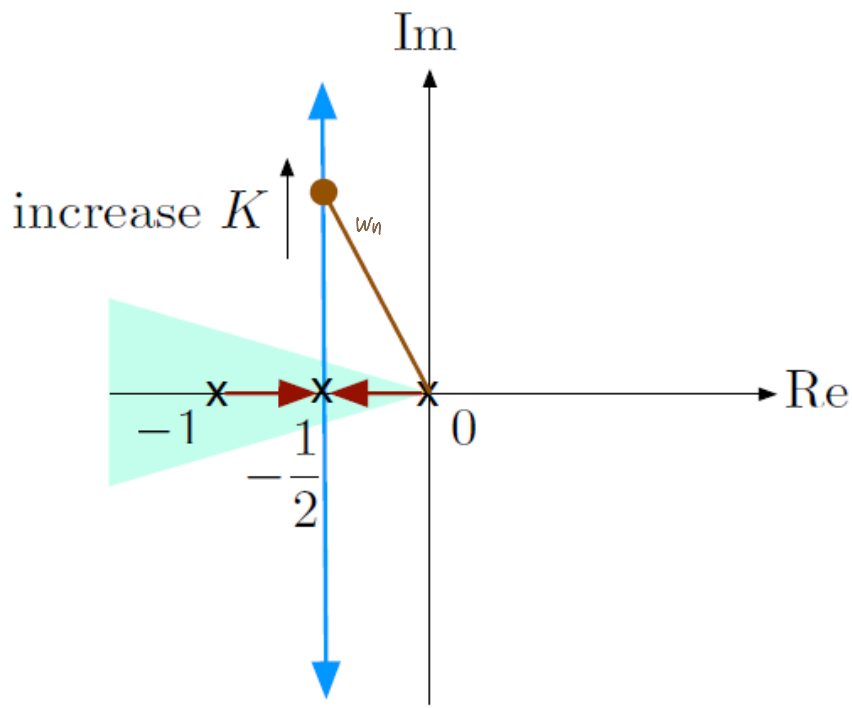


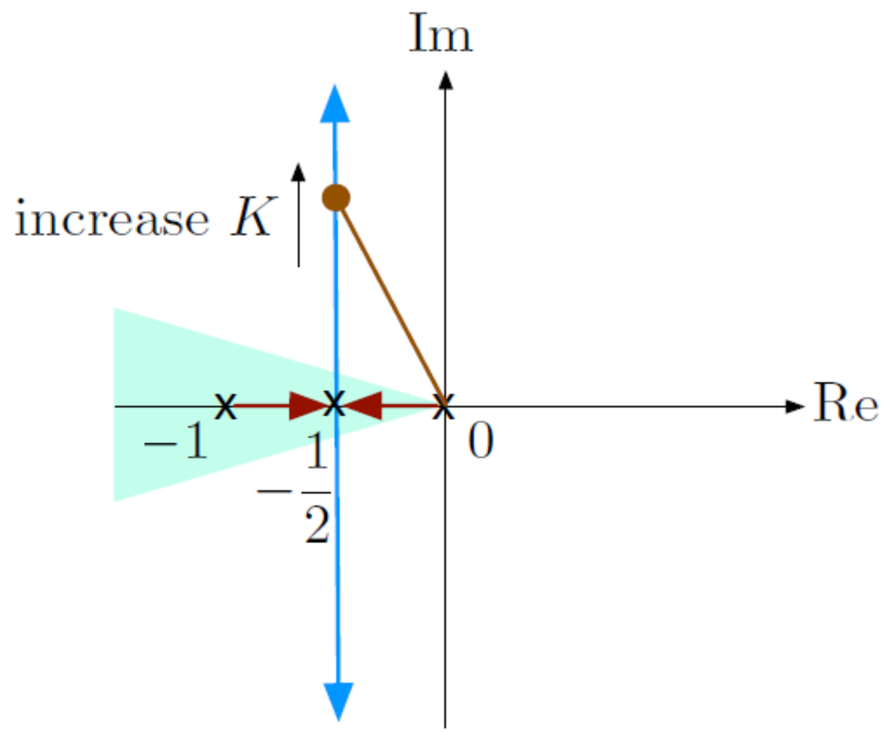
($s = -1/2$ is the *point of breakaway* from the real axis)

Example, continued

Compare this to admissible regions for given specs:

$t_s \approx \dfrac{4}{\sigma}$	want σ large, can only have $\sigma = \dfrac{1}{2}$ $(t_s = 6)$
$t_r \approx \dfrac{1.8}{\omega_n}$	want ω_n large $\Longrightarrow$ want K large
M_p	want to be inside the shaded region $\Longrightarrow$ want K small





Thus, the root locus helps us *visualize the trade-off* between all the specs in terms of K.

However, for order > 2, there will generally be no direct formula for the closed-loop poles as a function of K.

Our goal: develop simple rules for (approximately) sketching the root locus in the general case.

* purpose
to determine syst. specifications.

Summary:

* char equ: $1 + K\, L(s) = 0$

$L(s) = \frac{b(s)}{a(s)}$

(1) $\boxed{a(s) + K\, b(s) = 0}$ $\quad 0 \leq K < \infty$ for closed loop syst.

(2) Find the set of closed-loop poles as K varies from

$0 \leq K < \infty$

by drawing the Root locus (RL)

(3) Start from $K=0$ (open loop poles)

Rules:

* # of branches = # of open loop poles
* Root locus starts from open loop poles and ends at open loop zeros
* $L(s) = \frac{b(s)}{a(s)} \quad \frac{\text{order } m}{\text{order } n}$

$n \geq m$

n: # of the poles, m: # of the zeros

$n - m$ poles $\Rightarrow$ will direct to ∞

Equivalent Characterization of RL: Phase Condition

Recall our original definition: The *root locus* for $1 + KL(s)$ is the set of all closed-loop poles, i.e., the roots of

$$1 + KL(s) = 0,$$

as K varies from 0 to ∞.

A point $s \in \mathbb{C}$ is on the RL if and only if

$$L(s) = \underbrace{-\frac{1}{K}}_{\text{negative and real}} \qquad \text{for some } K > 0$$

This gives us an equivalent characterization:

The phase condition: The root locus of $1 + KL(s)$ is the set of all $s \in \mathbb{C}$, such that $\angle L(s) = 180°$, i.e., $L(s)$ is real and negative.

Rules for Sketching Root Loci

There are *six rules* for sketching root loci. These rules are mainly qualitative, and their purpose is to give intuition about impact of poles and zeros on performance.

These rules are:

- Rule A — number of branches
- Rule B — start points
- Rule C — end points

Rule A: Number of Branches

$$1 + K\frac{b(s)}{a(s)} = 1 + K\frac{s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n} = 0$$

$$\Longrightarrow (s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n) + K(s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m) = 0$$

Since $\deg(a) = n \geq m = \deg(b)$, the characteristic polynomial $a(s) + Kb(s) = 0$ has degree n.

The characteristic polynomial has n solutions (roots), some of which may be repeated. As we vary K, these n solutions also vary to form n branches.

Rule A:

$$\#(\text{branches}) = \deg(a)$$

Rule B: Start Points

The locus starts from $K = 0$. What happens near $K = 0$?

If $a(s) + Kb(s) = 0$ and $K \sim 0$, then $a(s) \approx 0$.

Therefore:

- s is close to a root of $a(s) = 0$, or
- s is close to a pole of $L(s)$

Rule B: branches start at open-loop poles.

Rule C: End Points

What happens to the locus as $K \to \infty$?

$$a(s) + Kb(s) = 0$$
$$b(s) = -\frac{1}{K} a(s)$$

— as $K \to \infty$,

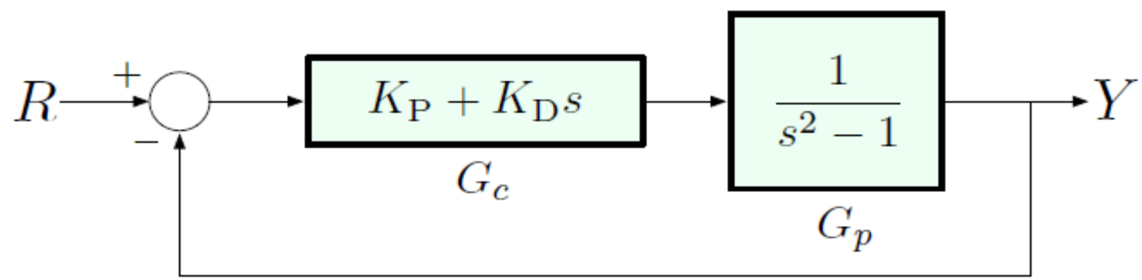
- branches end at the roots of $b(s) = 0$, or
- branches end at zeros of $L(s)$

Rule C: branches end at open-loop zeros.

Note: if $n > m$, we have n branches, but only m zeros. The remaining $n - m$ branches go off to infinity (end at "zeros at infinity").

Example

PD control of an unstable 2nd-order plant



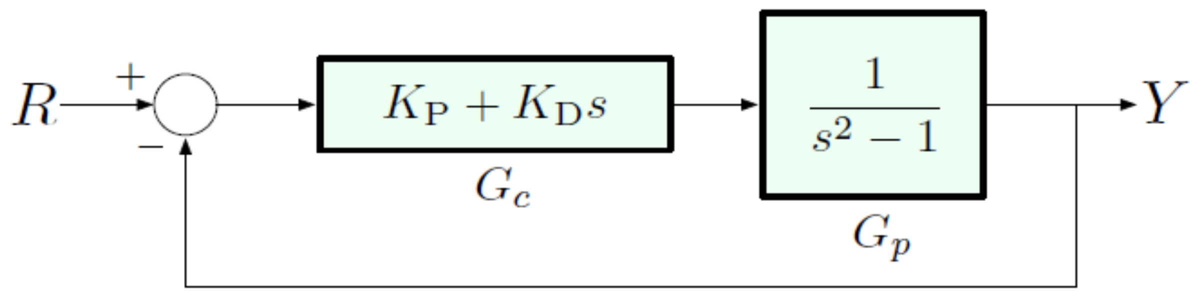
$$\frac{Y}{R} = \frac{G_c G_p}{1 + G_c G_p} \qquad \text{poles: } 1 + G_c(s)G_p(s) = 0$$

$$1 + (K_{\mathrm{P}} + K_{\mathrm{D}}s)\left(\frac{1}{s^2-1}\right) = 0$$

We will examine the impact of varying $K = K_{\mathrm{D}}$, assuming the ratio $K_{\mathrm{P}}/K_{\mathrm{D}}$ *fixed*.

Example

PD control of an unstable 2nd-order plant



We will examine the impact of varying $K = K_D$, assuming the ratio K_P/K_D *fixed.*

Let us write the characteristic equation in *Evans form*:

$$1 + \underbrace{K_D}_{K}\left(s + \frac{K_P}{K_D}\right)\left(\frac{1}{s^2-1}\right) = 1 + K\underbrace{\frac{s + K_P/K_D}{s^2-1}}_{L(s)} = 0$$

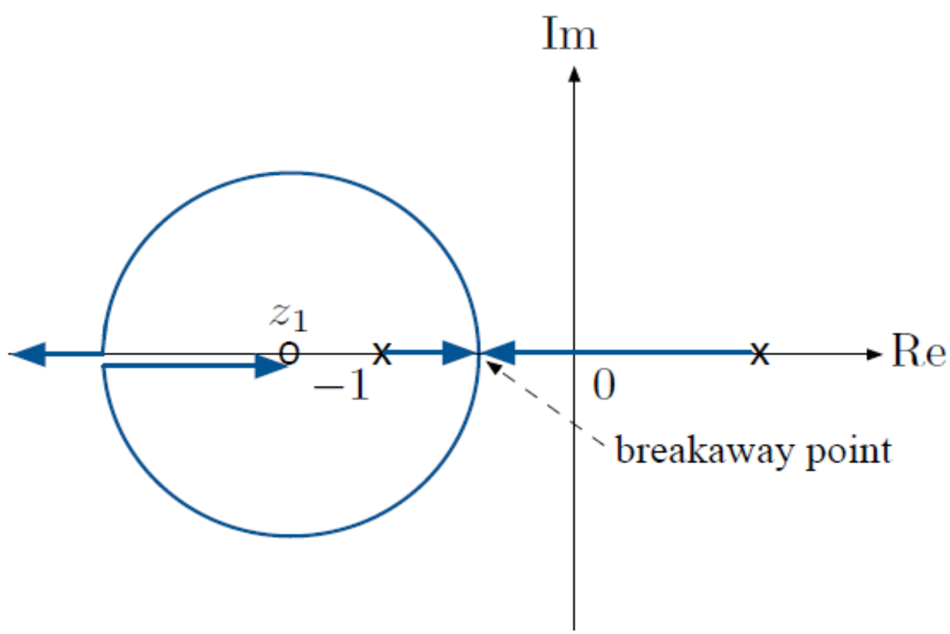
$$L(s) = \frac{s - z_1}{s^2 - 1} \qquad \text{zero at } s = z_1 = -K_P/K_D < 0$$

Example

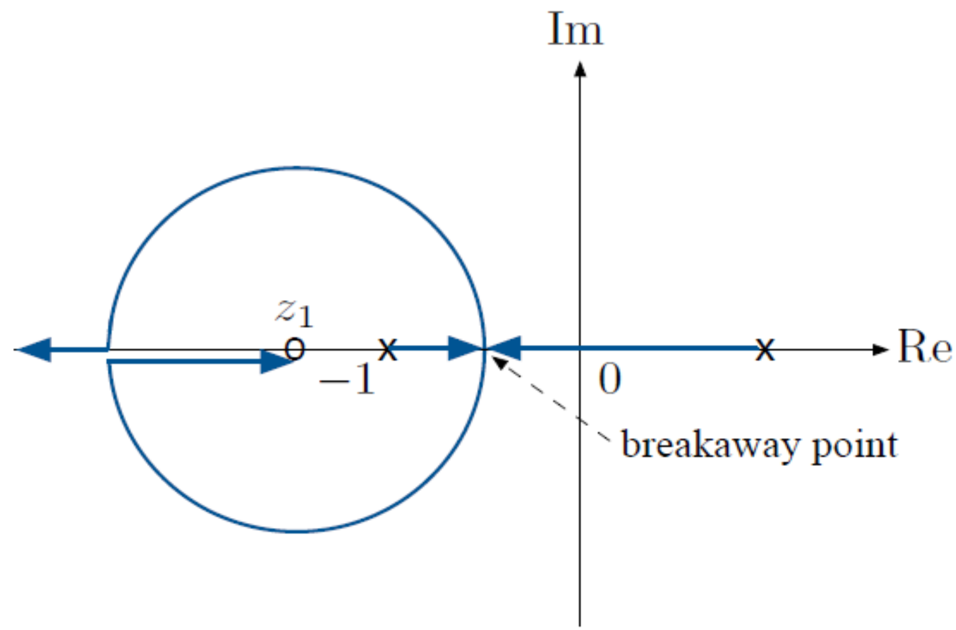
$$L(s) = \frac{s - z_1}{s^2 - 1}$$

- Rule A: $\begin{cases} m = 1 \\ n = 2 \end{cases} \implies$ 2 branches
- Rule B: branches start at open-loop poles $\quad s = \pm 1$
- Rule C: branches end at open-loop zeros $\quad s = z_1, -\infty$
  (we will see why $-\infty$ later)

So the root locus will look something like this:



$$L(s) = \frac{s - z_1}{s^2 - 1}$$



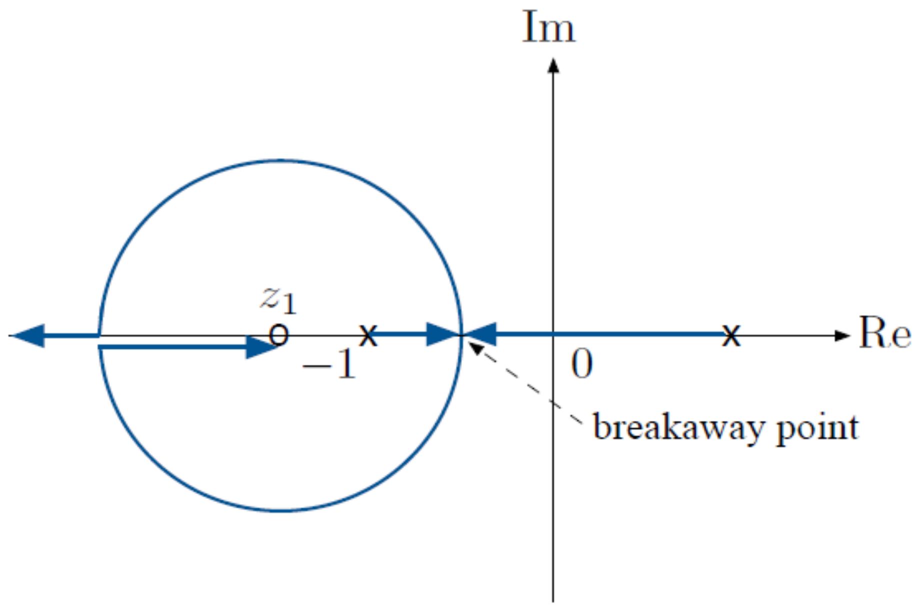
Why does one of the branches go off to $-\infty$?

$s^2 - 1 + K(s - z_1) = 0$

$s^2 + Ks - (Kz_1 + 1) = 0$

$s = -\frac{K}{2} \pm \sqrt{\frac{K^2}{4} + Kz_1 + 1},\ z_1 < 0$ as $K \to \infty$, s will be < 0

$$L(s) = \frac{s - z_1}{s^2 - 1}$$



Is the point $s = 0$ on the root locus?

Let's see if there is any value $K > 0$, for which this is possible:

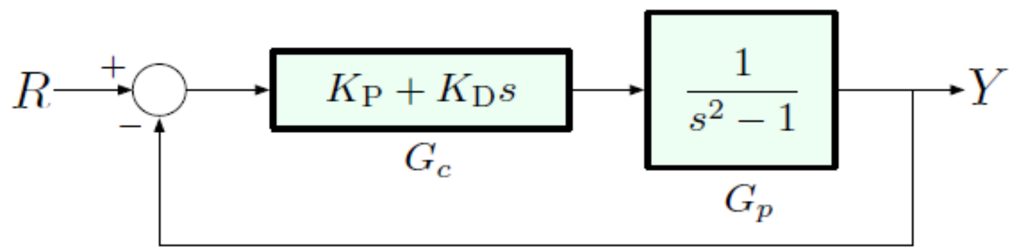
$$1 + KL(0) = 0$$

$$1 + Kz_1 = 0 \qquad K = -\frac{1}{z_1} > 0 \text{ does the job}$$

From Root Locus to Time Response Specs

For concreteness, let's see what happens when

$$K_{\mathrm{P}}/K_{\mathrm{D}} = -z_1 = 2 \qquad \text{and} \qquad K = K_{\mathrm{D}} = 5 \Longrightarrow K_{\mathrm{P}} = 10$$



$$G_c(s) = 10 + 5s$$
$$u = 10e + 5\dot{e}, \qquad e = r - y$$

$$\text{Characteristic equation:} \quad 1 + 5\left(\frac{s+2}{s^2-1}\right) = 0$$
$$s^2 + 5s + 9 = 0$$

Relate to 2nd-order response: $\omega_n^2 = 9,\ 2\zeta\omega_n = 5 \Longrightarrow \zeta = 5/6$

exp: $\dfrac{1}{S(S+1)(S+2)}$, solve for Root locus.

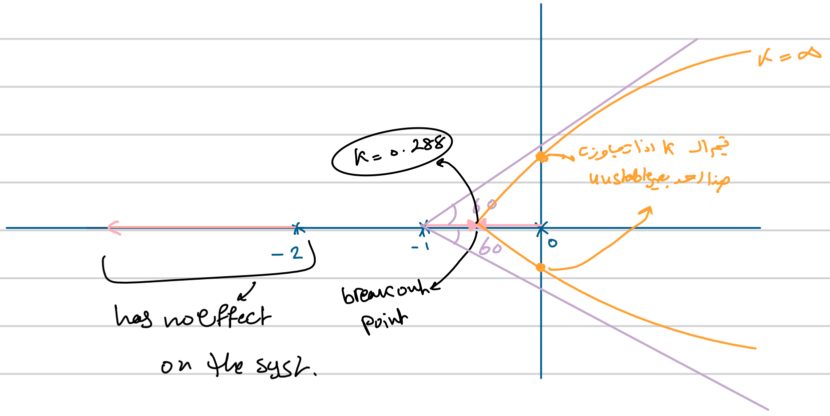
↳ open loop T.F

$a(s) + K\,b(s)$

$S(S+1)(S+2) + K = 0$ characteristic equ.

→ 3 branches

→ no zeros



Rule D:
all poles on the real axis to $\boxed{\text{the left}}$ of an odd # of open loop pole, Poles and/or zeros are on the root locus

Rule E:

$$\delta_A = \frac{\sum P_i - \sum Z_i}{n-m}$$

$$\phi_A = \frac{(1 \pm 2K)\pi}{n-m}$$

⇒ $\text{Centroid} = \dfrac{-3-0}{3} = -1$

$\phi_A = \pm 60$

Rule F :

Break points

$$\frac{d\, L(s)}{ds} = 0$$

$$y = \frac{u(x)}{v(x)}$$

First derivative ← $\bar{y} = \frac{v(x)\, \bar{u}(x) - u(x)\, \bar{v}(x)}{(v(x))^2}$

حـ2: اطلع القيمة العظمى لـ K

بوخذ المشتقة الاولى لل

open-loop transfer function

$$\Rightarrow \frac{s\,(s+1)\,(s+2)\,(0) - (1)\,(3s^2+6s+2)}{[s\,(s+1)\,(s+2)]^2}$$

$$3s^2+6s+2=0$$

$$s_{1,2} = \frac{-2 \pm \sqrt{4-8/3}}{2}$$

$$s_{1,2} = -1 \pm 0.8$$

$$= 0.2\,,\ -1.8$$

$\Rightarrow$ تعويض

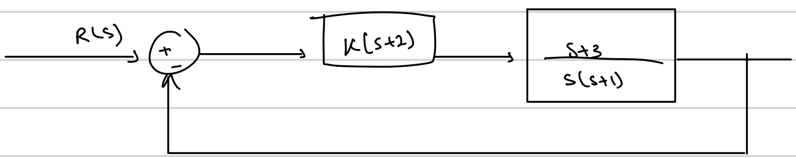
$$(-0.2)\,(-0.2+1)\,(-0.2+2) = -K$$

$$K = (0.2)\,(0.8)\,(1.8)$$

$$\boxed{K = 0.288}$$

---

ex :



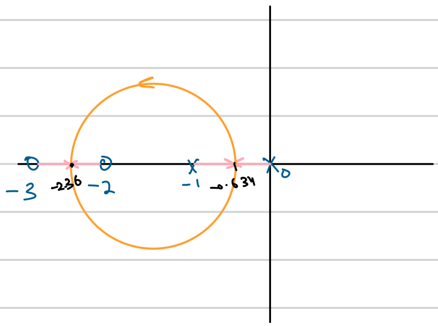
$$\frac{K\,(s+2)\,(s+3)}{s\,(s+1)} \rightarrow \text{open loop transfer function}$$

$$s(s+1) + K(s+2)\,(s+3) = 0$$

number of asymptotes $= (n-m)$

$= 2-2$

$= 0$



$$d = \frac{(s+2)\,(s+3)}{s\,(s+1)} = 0$$

$$s^2 + 3s + 1.5 = 0$$

$$s_1 = -0.634$$

$$s_2 = -2.356$$

$$\frac{K\,(s+2)\,(s+3)}{s\,(s+1)} + 1 = 0$$

$$\frac{K(s+2)\,(s+3)}{s(s+1)} = -1$$

$$K = \frac{-s(s+1)}{(s+2)\,(s+3)}$$

$s = -0.634$ ↓ 0.0718

$s = -2.366$ ↓ 14

example :-

$L(s) = \dfrac{1}{s(s+1)(s+2)}$ → open loop transfer funct

$b(s)$, $a(s)$

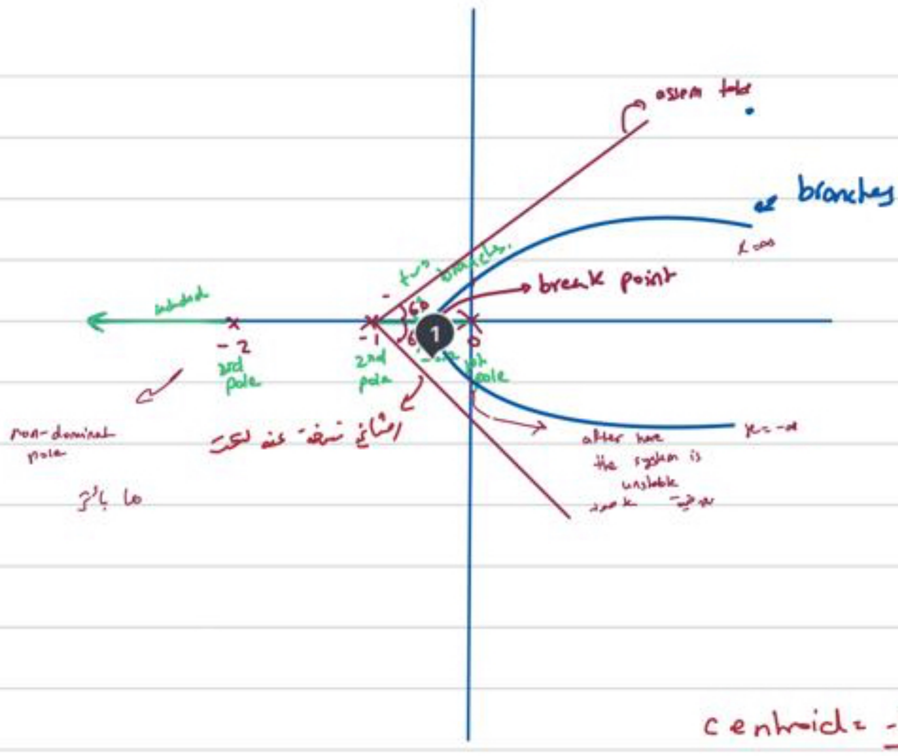
characteristic eq ⇒ $a(s) + k\,b(s) = 0$ → characteristic equa for the closed loop.

$s(s+1)(s+2) + k = 0$

$(s^2+s)(s+2) = s^3 + s^2 + 2s^2 + 2s$

$s^3 + 3s^2 + 2s + K$

root locus → poles of this to $0 \le K < \infty$ graphically.



→ we have 3 open loop poles → 3 branches

→ we start @ the " " poles → @ $0, -1, -2$

→ no zeros → branches end @ ∞ or $-\infty$

centroid $= \dfrac{-3-0}{3} = (-1)$

Rule D: All points on the real axis to the left of an odd number of open-loop poles and/or zeros are on the root locus

branches و asem tote

$\phi_A = \pm \dfrac{\pi}{3} = \pm 60^\circ$

max k → charach eq → break point

Rule E: $\sigma A = \dfrac{\sum P_i - \sum Z_i}{n-m}$

centroid

$\phi_A = \dfrac{(1 \pm 2k)\pi}{n-m}$, $k = 0, 1, \ldots$, $n-m$

the assem totes begin from the Centroid. (σA)

but the branches begin from the break point. → open loop $= 0$

Rule F : break points

$$\frac{dL(s)}{ds} = 0$$

$$\frac{s(s+1)(s+2)(0) - 1(3s^2+6s+2)}{[s(s+1)(s+2)]^2}$$

$$y = \frac{u(x)}{v(x)}$$

$$\bar{y} = \frac{v(x)\bar{u}(x) - u(x)\bar{v}(x)}{(v(x))^2}$$

$s^2(s^2+s+1)(s^2+4s+4)$

$(s^4+s^3+s^2)(s^2+4s+4)$

$s^6+4s^5+4s^4+s^5+4s^4+4s^3+s^4+4s^3+4s^2$

$s^6+5s^5+9s^4+8s^3+4s^2$

$$\frac{3s^2+6s+2}{s^6+5s^5+9s^4+8s^3+4s^2}$$

$$\bar{y} = \frac{3s^2+6s+2=0}{s}$$

$s = -1 \pm 0.8$

-1.8, -0.2

because our break point is between $\boxed{0, -1}$

characteristic equa: $s(s+1)(s+2) + k = 0$

$s = -0.2$

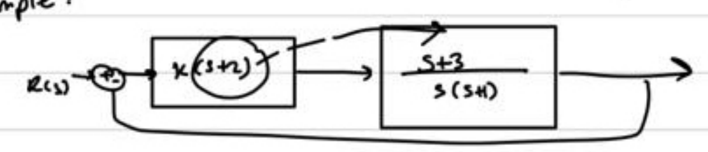
break point : $-0.2(1-0.2)(2-0.2) + k = 0$

$k = 0.288$

---

start $0, -1$; end -3

example :

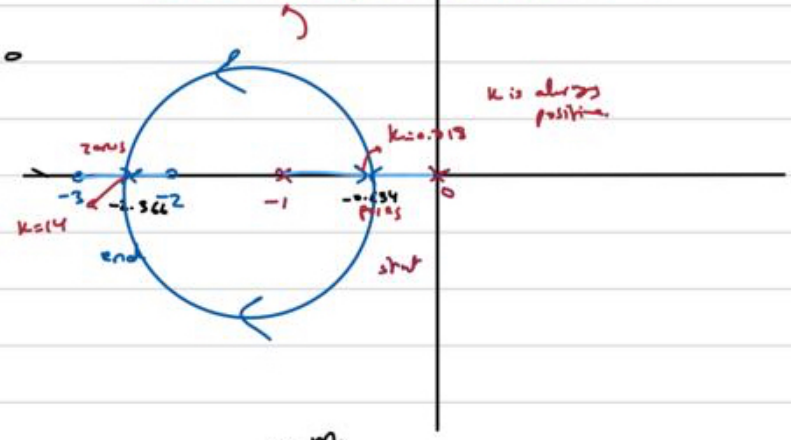


open loop TF : $\frac{k(s+2)(s+3)}{s(s+1)}$ $\frac{b(s)}{a(s)}$

$s(s+1) + k(s+2)(s+3) = 0$

closed loop charac equ: $s(s+1) + k(s+2)(s+3) = 0$

it looks like this because branches start @ poles and end @ zeros



poles & zeros → تقاطع rule D

two poles → two branches

break points :

$$\frac{d}{ds}\frac{(s+2)(s+3)}{s(s+1)} = 0$$

$n-m$

$$k = -\frac{s(s+1)}{(s+2)(s+3)}$$

k

$s = -0.634 \rightarrow 0.019$

$s = -2.36 \rightarrow 14$

$\vdots$

$s^2 + 3s + 1.5 = 0$

$s_1 = -0.634$

$s_2 = -2.366$

$-4s^2 - 12s$

$(2s^3+7s^2+5s) - (2s^3+11s^2+17s)$

$s^2+3s+2s+6 \rightarrow s^2+5s+6$ / s^2+s

$$\frac{(s^2+s)(2s+5) - (s^2+5s+6)(2s+1)}{(s^2+s)^2}$$

$(s^2+s)(s^2+s)$

s^4

of the asymptotes $= n-m$

here $= 2-2 = 0$

no asymptotes means no need to get the centroid.

no asymp— , مش angle و مش بدنا