Industrial Control Systems

Chapter Six:
Dynamic Behavior and Stability of Closed-Loop Control Systems

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* effect of Zeros
  - LHP : increase the overshoot
    the more the Zero was negative the more the effect of the Zero decrease.
  - RHP : time delay
    the more the Zero was positive the more the effect decrease

* Stability :

- Dominent poles : the closest poles to the imagienary axis, and has the biggest effect on the dinamic syst
  they should be Complex conjugate

- Check for Stability:
  - input bounded → output bounded
  - the integration is to ∞
  - all poles in the left Hand side

in Higher order syst

ex: $G(s) = \dfrac{}{s^2 - s + 4} \cdot \dfrac{1}{s+4} \cdot \dfrac{1}{s+1}$

unstable for the second order syst

but for syst > 2 ⇒ can't define from the coeff

Study the stability of the higher order syst.

Effect of Zeros on the Transient Response

Reminder: for $H(s) = \dfrac{q(s)}{p(s)}$, zeros are the roots of $q(s) = 0$

Example: start with $H_1(s) = \dfrac{1}{s^2 + 2\zeta s + 1} \qquad (\omega_n = 1)$

Let's add a zero at $s = -a$, $a > 0$ – LHP zero

To keep DC gain = 1, let's take the numerator to be $\dfrac{s}{a} + 1$:

$$H_2(s) = \frac{s+a}{s^2+2\zeta s+1}$$

a: common factor

$$H_2(s) = \frac{\frac{s}{a} + 1}{s^2 + 2\zeta s + 1}$$

$$= \underbrace{\frac{1}{s^2 + 2\zeta s + 1}}_{\text{this is } H_1(s)} + \frac{1}{a} \cdot \underbrace{\frac{s}{s^2 + 2\zeta s + 1}}_{\text{call this } H_d(s)}$$

$$= H_1(s) + \frac{1}{a} H_d(s), \qquad H_d(s) = sH_1(s)$$

Effect of a LHP Zero

in left hand sid

$$H_1(s) = \frac{1}{s^2 + 2\zeta s + 1} \xrightarrow{\text{add zero at } s=-a} H_2(s) = H_1(s) + \frac{1}{a} \cdot sH_1(s)$$

Step response:

$$\begin{aligned} Y_1(s) &= \frac{H_1(s)}{s} \\ Y_2(s) &= \frac{H_2(s)}{s} \\ &= \frac{H_1(s)}{s} + \frac{1}{a}\frac{sH_1(s)}{s} \\ &= Y_1(s) + \frac{1}{a} sY_1(s) \end{aligned}$$

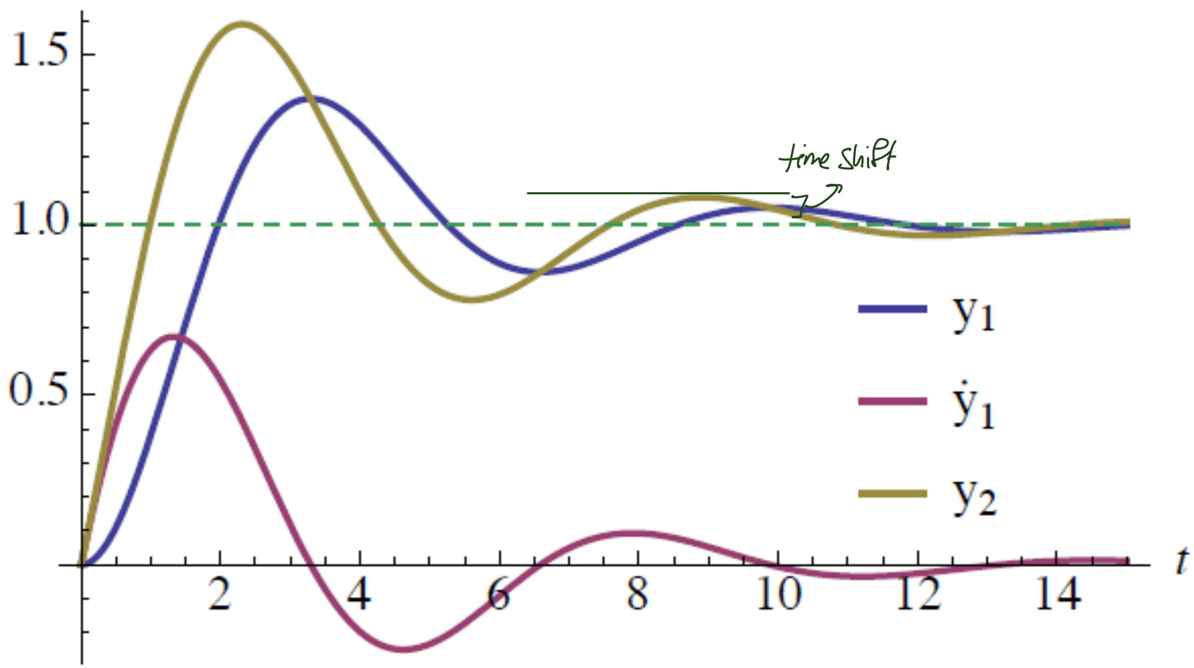
$$y_2(t) = \mathscr{L}^{-1}\{Y_2(s)\} = \mathscr{L}^{-1}\left\{Y_1(s) + \frac{1}{a} \cdot sY_1(s)\right\} = y_1(t) + \frac{1}{a}\dot{y}_1(t)$$

(assuming zero initial conditions)

Effect of a LHP Zero

Step response (zero at $s = -a$)

$$y_2(t) = y_1(t) + \frac{1}{a}\dot{y}_1(t) \qquad \text{where } y_1(t) = \text{original step response}$$



Effects of a LHP zero:

- increased overshoot (major effect)
- little influence on settling time → almost remained the same.
- what happens as $a \to \infty$? — effects become less significant

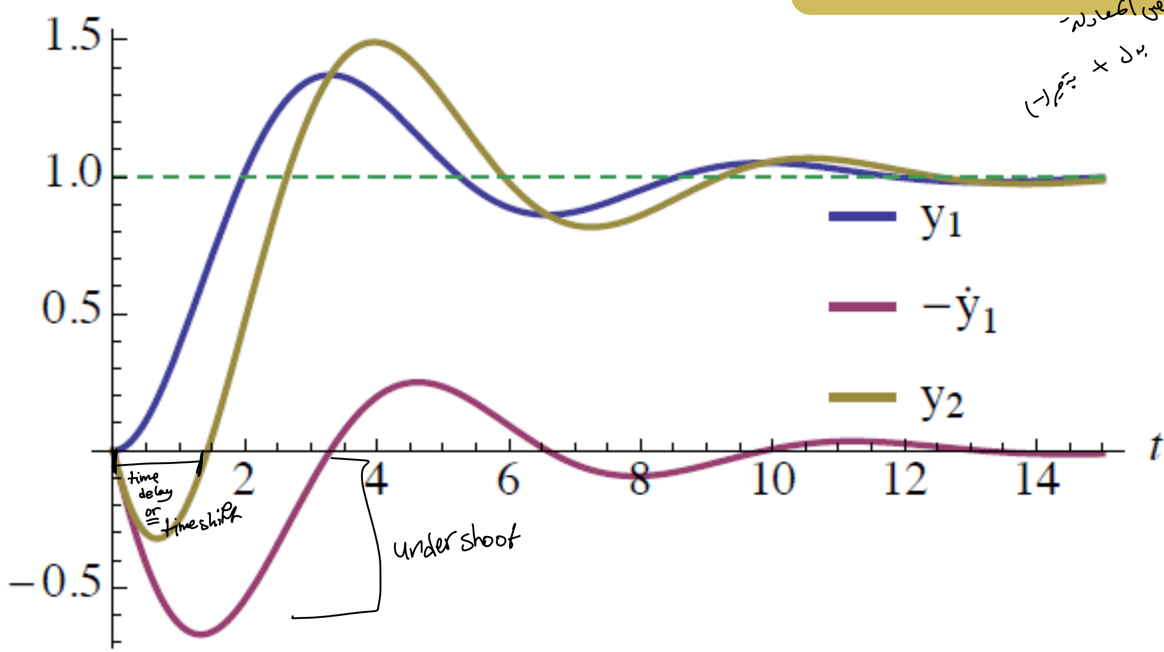
What About a RHP Zero?

$$H_1(s) = \frac{1}{s^2 + 2\zeta s + 1} \xrightarrow{\text{add zero at } s = a} H_2(s) = H_1(s) - \frac{1}{a} \cdot sH_1(s)$$

$$\boxed{y_2(t) = y_1(t) - \frac{1}{a} \cdot \dot{y}_1(t)}$$

نفس المعادلة
بس بدل + بتصير (-)

when $a \uparrow$
the term $\downarrow$
بتصير y_2 أقرب
الى y_1



Effects of a RHP zero:

- slows down (delays) the response
- creates *undershoot* (at least, when a is small enough)

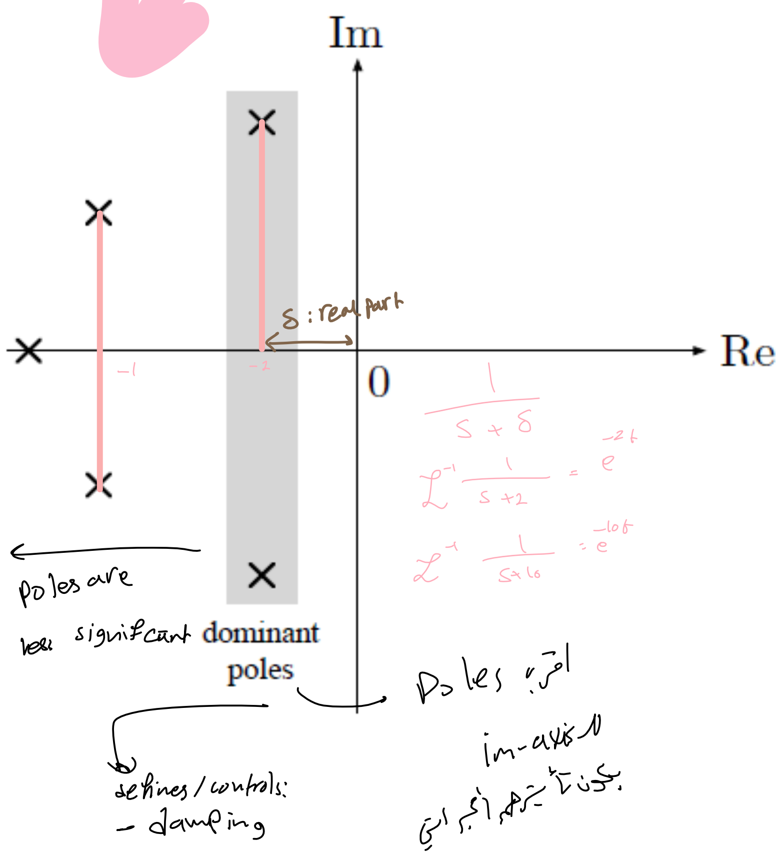
Effect of Extra Poles

1. The "Dominant" vs. "Insignificant" Rule

In a complex system, the poles closest to the imaginary axis ($j\omega$-axis) are the **dominant poles**. This is because they decay the slowest in the time domain.

- **The 5x Rule:** A common engineering convention is that if an extra pole is at least **5 times further** to the left (more negative) than the dominant poles, its effect on the transient response is negligible.
- **The Reason:** Poles further left represent faster decay. As shown in your notes, a pole at $s = -2$ produces a term e^{-2t}, while a pole at $s = -10$ produces e^{-10t}. The e^{-10t} term disappears much faster, leaving the e^{-2t} term to "dominate" the shape of the graph.

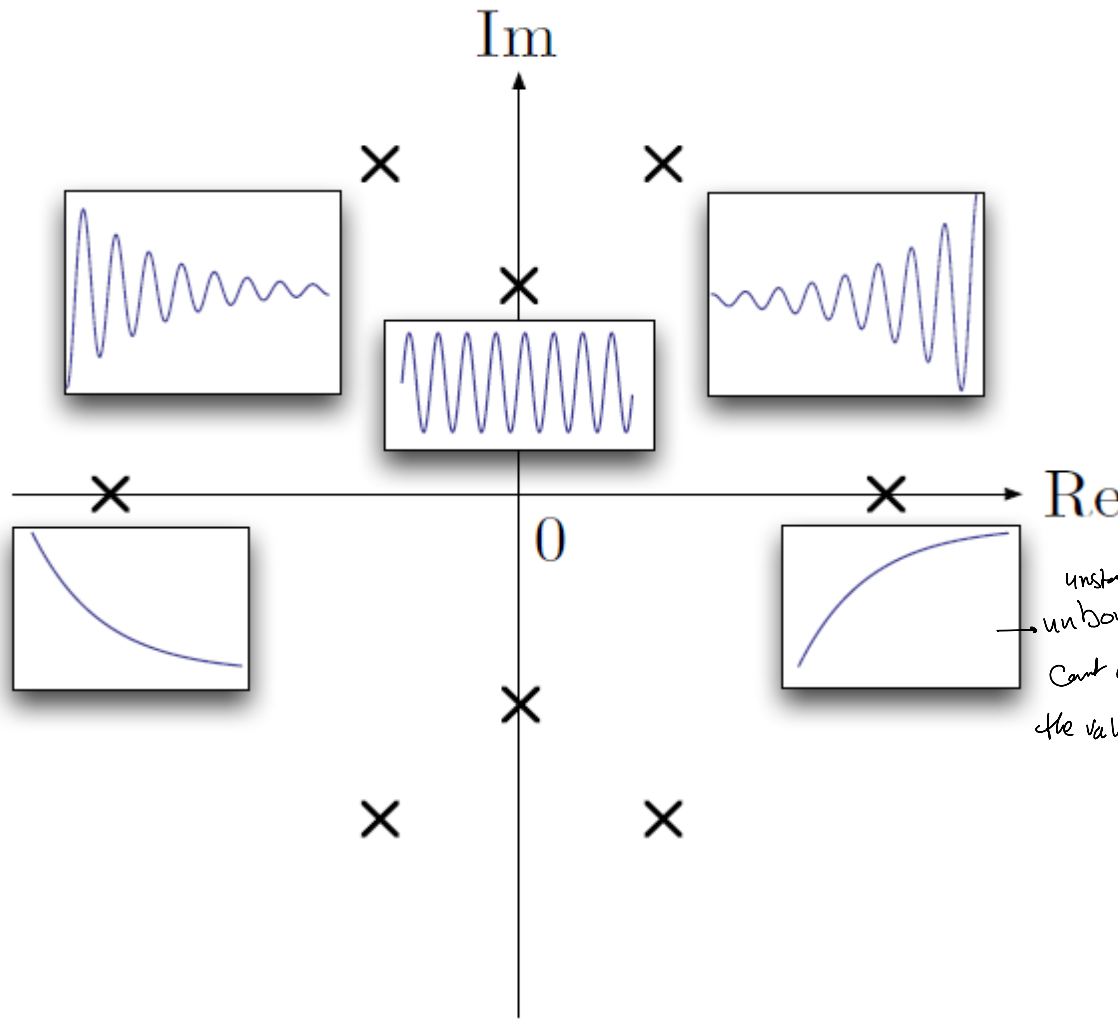
A general nth-order system has n poles



كلما بعدت ال poles عنال im بصير تأثيرها أقل

- extra LHP poles are not significant if their real parts are at least $5\times$ the real parts of dominant LHP poles
- e.g., if dominant poles have $\mathrm{Re}(s) = -2$ and we have extra poles with $\mathrm{Re}(s) = -10$, their time-domain contributions will be e^{-2t} and $e^{-10t} \ll e^{-2t}$
- $5\times$ is just a convention, but we can really see the effect of extra poles that are closer (cf. Lab 2)

Effect of Pole Locations



unstable
unbounded
Cant expect
the values when ∞

- poles in open LHP ($\mathrm{Re}(s) < 0$) — stable response
- poles in open RHP ($\mathrm{Re}(s) > 0$) — unstable response
- poles on the imaginary axis ($\mathrm{Re}(s) = 0$) — tricky case

unstable

explenaition.

Marginal Case: Poles on the Imaginary Axis

bonded output = input

Let's consider the case of a pole at the origin: $H(s) = \dfrac{1}{s}$

s=0

Is this a stable system?

stable

- impulse response: $Y(s) = \dfrac{1}{s} \Longrightarrow y(t) = 1(t)$ (OK)
- step respone: $Y(s) = \dfrac{1}{s^2} \Longrightarrow y(t) = t,\ t \geq 0$ — *unit ramp!!*

No real part.

What about purely imaginary poles? $H(s) = \dfrac{\omega^2}{s^2 + \omega^2}$

- impulse response: $Y(s) = \dfrac{\omega^2}{s^2 + \omega^2} \Longrightarrow y(t) = \boxed{\omega \sin(\omega t)}$ osilations
- step respone: $Y(s) = \dfrac{\omega^2}{s(s^2 + \omega^2)} \Longrightarrow y(t) = \boxed{1 - \cos(\omega t)}$ osilations

not stable

Systems with poles on the imaginary axis are *not stable*.

What Is Stability?

$$u \longrightarrow \boxed{h} \longrightarrow y$$

One reasonable definition is as follows:

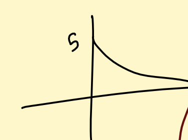
A linear time-invariant system is *Bounded-Input, Bounded-Output* (BIBO) *stable* provided either one of the following three equivalent conditions is satisfied: → اذا واحدة تحققت بيكونو الثلاث بتحققوا مع بعض by defolt → كلهم نفس الاشي بس time او freq او Math او

time domain ← 1. If every bounded input $u(t)$ results in a bounded output $y(t)$, regardless of initial conditions.

Mathmatically ← 2. If the impulse response $h(t)$ is absolutely integrable:

ex: $5e^{-2t}$

$$\int_{-\infty}^{\infty} |h(t)|\, \mathrm{d}t < \infty.$$



$\frac{1}{s+2}$

✓ – pole $\boxed{s=-2}$ stable

✓ – can be done mathmatically

✓ – bonded

sysh is stable

frequensy domain ← 3. If all poles of the transfer function $H(s)$ are *strictly stable* (lie in open LHP).

Checking for Stability?

Consider a general transfer function:

$$H(s) = \frac{q(s)}{p(s)}$$

where q and p are polynomials, and $\deg(q) \le \deg(p)$.

We need tools for checking stability: whether or not all roots of $p(s) = 0$ lie in OLHP.

For simple polynomials, can just factor them "by inspection" and find roots.

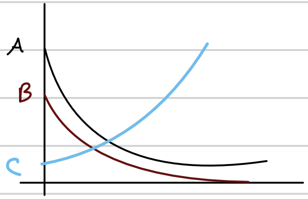
Now, this is hard to do for high-degree polynomials — it's computationally intensive, especially symbolically.

But: often we *don't need to know* precise pole locations, just need to know that they are strictly stable.

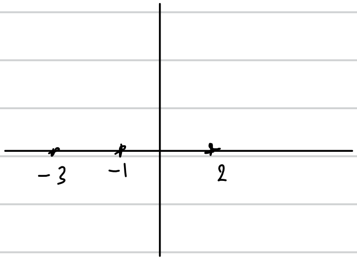
check for stability

(*) $H(s) = \frac{1}{s+1} \cdot \frac{1}{s+3} \cdot \frac{1}{s-2}$ → Not stable because of the sign.

$$\mathcal{L}^{-1}(H(s)) = Ae^{-t} + Be^{-3t} + Ce^{2t}$$



(*) if a single root in the RHP all the syst is unstable



(*) $G(s) = \frac{1}{s^4 + 3s^3 - 5s^2 + s + 2}$

↓ -ve coef

(*) If any coefficient is negative then the entire syst is unstable

The negative must come from a root in the RHP hence the syst is unstable

(*) $G(s) = \frac{1}{s^2 - s + 4} \cdot \frac{1}{s+2} \cdot \frac{1}{s+1}$ unstable

roots: $0.5 \pm j\sqrt{3.75}$, -2, -1

حاصل ضرب الطلعات

$$G(s) = \frac{1}{s^4 + 2s^3 + 3s^2 + 6s + 8}$$

Routh stability criteria

↳ Some calculations are done to the eq^n to check for stability

Checking for Stability

Problem: given an nth-degree polynomial

general form charictaristic equation for stabilitycheck

$$p(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s + a_n$$

with real coefficients, check that the roots of the equation $p(s) = 0$ are strictly stable (i.e., have negative real parts).

Terminology: we often say that the polynomial p is (strictly) stable if all of its roots are.

A Necessary Condition for Stability

Terminology: we say that A is a necessary condition for B if

$$A \text{ is false} \quad \Longrightarrow \quad B \text{ is false}$$

Important!! Even if A is true, B may still be false.

Necessary condition for stability: a polynomial p is strictly stable only if all of its coefficients are strictly positive.

Proof: suppose that p has roots at $r_1, r_2, \ldots, r_n$ with $\text{Re}(r_i) < 0$ for all i. Then

$$p(s) = (s - r_1)(s - r_2)\ldots(s - r_n)$$

— multiply this out and check that all coefficients are positive.

exp $\dfrac{1}{s^4 + 2s^3 + 3s^2 + 10s + 8}$

Can't define if it's stable.

So we use Ruth's craitiria

for syst > 4th. order

Routhe's Array

s^{n}	:	1	a_2	a_4	a_6
s^{n-1}	:	a_1	a_3	a_5	a_7
s^{n-2}	:	b_1	b_2	b_3	
s^{n-3}	:	c_1	c_2		
s^{n-4}	:	D_1			

* $b_1 = \frac{-1}{a_1} \cdot \det\begin{pmatrix} 1 & a_2 \\ a_1 & a_3 \end{pmatrix} = -\frac{1}{a_1}(a_3 - a_1$

* $b_2 = -\frac{1}{a_1} \det\begin{pmatrix} 1 & a_4 \\ a_1 & a_5 \end{pmatrix} = -\frac{1}{a_1}(a_5 - a_1$

* $b_3 = -\frac{1}{a_1} \det\begin{pmatrix} 1 & a_6 \\ a_1 & a_7 \end{pmatrix} = -\frac{1}{a_1}(a_7 - a$

* $c_1 = -\frac{1}{b_1} \det\begin{pmatrix} a_1 & a_3 \\ b_1 & b_2 \end{pmatrix} = -\frac{1}{b_1}(a_1 b_2 - b_1$

* $c_2 = -\frac{1}{b_1} \det\begin{pmatrix} a_1 & a_5 \\ b_1 & b_3 \end{pmatrix} = -\frac{1}{b_2}(a_1 b_3 - b_1 a_5$

* $D_1 = -\frac{1}{c_1} \det\begin{pmatrix} b_1 & b_2 \\ c_1 & c_2 \end{pmatrix} = -\frac{1}{c_1}(b_1 c_2 - b_2 c_1$

Five Apple

$\longrightarrow$

Routh–Hurwitz Criterion

Necessary & Sufficient Condition for Stability

Terminology: we say that A is a sufficient condition for B if

$$A \text{ is true} \quad \Longrightarrow \quad B \text{ is true}$$

Thus, A is a necessary *and* sufficient condition for B if

$$A \text{ is true} \quad \Longleftrightarrow \quad B \text{ is true}$$

— we also say that A is true if and only if (iff) B is true.

We will now introduce a necessary and sufficient condition for stability: the *Routh–Hurwitz Criterion*.

Routh's Test

Problem: check whether the polynomial

$$p(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s + a_n$$

is strictly stable.

We begin by forming the Routh array using the coefficients of p:

even ⇒ even coef

odd ⇒ odd coef

هوا مفترض انه n even

s^n :	1	a_2	a_4	a_6	$\ldots$	(if necessary, add zeros in the
s^{n-1} :	a_1	a_3	a_5	a_7	$\ldots$	second row to match lengths)

Note that the very first entry is always 1, and also note the order in which the coefficients are filled in.

Routh's Test

even $\Rightarrow a_{even}$
odd $\Rightarrow a_{odd}$

$$
\begin{array}{llllll}
s^{n}: & 1 & a_2 & a_4 & a_6 & \dots \\
s^{n-1}: & a_1 & a_3 & a_5 & a_7 & \dots \\
s^{n-2}: & b_1 & b_2 & b_3 & \dots &
\end{array}
$$

Next, we form the third row marked by s^{n-2}:

$$s^{n-2}: \quad b_1 \quad b_2 \quad b_3 \quad \dots$$

where
$$b_1 = -\frac{1}{a_1}\det\begin{pmatrix} 1 & a_2 \\ a_1 & a_3 \end{pmatrix} = -\frac{1}{a_1}(a_3 - a_1 a_2)$$

$$b_2 = -\frac{1}{a_1}\det\begin{pmatrix} 1 & a_4 \\ a_1 & a_5 \end{pmatrix} = -\frac{1}{a_1}(a_5 - a_1 a_4)$$

$$b_3 = -\frac{1}{a_1}\det\begin{pmatrix} 1 & a_6 \\ a_1 & a_7 \end{pmatrix} = -\frac{1}{a_1}(a_7 - a_1 a_6)$$ and so on ...

Note: the new row is 1 element shorter than the one above it

Routh's Test, continued

$$
\begin{array}{rccccc}
s^n: & 1 & a_2 & a_4 & a_6 & \dots \\
s^{n-1}: & a_1 & a_3 & a_5 & a_7 & \dots \\
s^{n-2}: & b_1 & b_2 & b_3 & \dots & \\
s^{n-3}: & c_1 & c_2 & \dots & &
\end{array}
$$

Next, we form the fourth row marked by s^{n-3}:

$$
s^{n-3}: \quad c_1 \quad c_2 \quad \dots
$$

$$
\text{where} \quad c_1 = -\frac{1}{b_1}\det\begin{pmatrix} a_1 & a_3 \\ b_1 & b_2 \end{pmatrix} = -\frac{1}{b_1}(a_1 b_2 - a_3 b_1)
$$

$$
c_2 = -\frac{1}{b_1}\det\begin{pmatrix} a_1 & a_5 \\ b_1 & b_3 \end{pmatrix} = -\frac{1}{b_1}(a_1 b_3 - a_5 b_1)
$$

and so on ...

Routh's Test, continued

Eventually, we complete the array like this:

$$
\begin{array}{rccccc}
s^n : & 1 & a_2 & a_4 & a_6 & \dots \\
s^{n-1} : & a_1 & a_3 & a_5 & a_7 & \dots \\
s^{n-2} : & b_1 & b_2 & b_3 & \dots & \\
s^{n-3} : & c_1 & c_2 & \dots & & \\
\vdots & & & & & \\
s^1 : & * & * & & & \\
s^0 : & * & & & &
\end{array}
$$

(as long as we don't get stuck with division by zero: more on this later)

After the process terminates, we will have $n+1$ entries in the first column.

eg:

1) $\overset{a_4}{s^4} + \overset{a_3}{4s^3} + \overset{a_2}{16s^2} + \overset{a_1}{10s} + \overset{a_0}{6} = 0$

Ru this Array:

	s^4	1	16	6 → even Coef
	s^3	4	10	0
b	s^2	13.5	6	
c	s^1	8.2	0	
d	s^0	6		

$b_1 = \frac{-1}{4}(1\times10 - 16\times4) = 13.5$

$b_2 = \frac{-1}{4}(1\times0 - 6\times4) = 6$

$C_1 = \frac{-1}{13.5}(4\times6 - 10\times13.5) = 8.2$

$C_2 = \frac{-1}{13.5}(4\times0 - 13.5\times0) = 0$

$D_1 = \frac{-1}{8.2}(-6\times8.2) = 6$

⇒ Syst is stable.

→ No changes in Coefs signs

Note:

if some S^n is not in the eq n

always assume that its Coef

is zero

$s^2 + 1 = 0$ ⇒ Coef for S is zero

→ بفرض انه قيمة الـ Coeff ε وهي قيمة

قريبة جداً من الصفر

(ابس هاي الحالة ماييجي)

* Number of the sign changes = Number of the poles on R.H.P

2) $s^3 + 10s^2 + 31s + 1030$

3^{rd} order syst.

→ all coefts are +ve

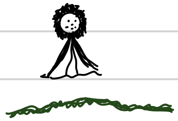
→ $a_1 \cdot a_2 > a_3$??

$31 \times 10 < 1030$

⇒ syst unstable

Ruth's Array:

	a_3	a_1
s^3	1	31
s^2	10	1030
s^1	-72	0
s^0	1030	



⇒ syst unstable.

→ sign changed $+ve \xrightarrow{①} -ve \xrightarrow{②} +ve$

of the Right poles $\underline{\underline{= 2}}$

3) $s^3 + 3s^2 + 7s + K$

Range of K for stable syst.

$3 \times 7 > K$

$21 > K$

$K > 0$

$$\boxed{0 < K < 21}$$

4)

$$\frac{C(s)}{R(s)} = \frac{2(s^2+2s+25)}{s^5+s^4+3s^3+9s^2+16s+10}$$

→ characteristic equation

$$\overset{a_5}{s^5} + \overset{a_4}{s^4} + \overset{a_3}{3s^3} + \overset{a_2}{9s^2} + \overset{a_1}{16s} + \overset{a_0}{10} = 0$$

Ruth's Array

s^5	1	3	16
s^4	1	9	10
s^3	-6	6	0
s^2	10	10	
s^1	12	0	
s^0	10		

→ unstable syst.

+ve → -ve → +ve

2 Right poles.

5) H.W:

$(3s^7 + 9s^6 + 6s^5 + 4s^4 + 7s^3 + 8s^2 + 2s + 6) \div 3$

$s^7 + 3s^6 + 2s^5 + 4/3\, s^4 + 7/3\, s^3 + 8/3\, s^2 + 2/3\, s + 2$

s^7	1	2	7/3	2/3
s^6	3	4/3	8/3	2
s^5				
s^4				
s^3				
s^2				
s^1				
s^0				

6) $\overset{a_4}{s^4} + \overset{a_3}{2s^3} + \overset{a_2}{3s^2} + \overset{a_1}{4s} + \overset{a_0}{5}$

Ruth's array:

s^4	1	3	5
s^3	2	4	0
s^2	1	5	0
s^1	-6	0	0
s^0	5	0	0

$b_1 = \frac{-1}{2}(1\times 4 - 3\times 2)$

$= 1$

$b_2 = \frac{-1}{2}(1\times 0 - 5\times 2)$

$= 5$

$c_1 = \frac{-1}{1}(2\times 5 - 4\times 1)$

unstable

2 sign changes = 2 poles on the RHP

* Note: if all signs are +ve the syst is stable

$\downarrow$

$b_1, c_1, d_1 \dots$

7) $\overset{a_5}{s^5} + \overset{a_4}{s^4} + \overset{a_3}{3s^3} + \overset{a_2}{9s^2} + \overset{a_1}{16s} + \overset{a}{10}$

s^5	1	3	16
s^4	1	9	10
s^3	-6	6	0
s^2	10	10	0
s^1	12	0	0
s^0	10	0	0

$b_1 = \frac{-1}{1}(1\times 9 - 3)$

$= -6$

$b_2 = \frac{-1}{1}(1\times 10 - 16)$

$= 6$

$c_1 = \frac{-1}{-6}(6 - (-54))$

$= 10$

$c_2 = \frac{-1}{-6}(1\times 0 - (-6\times 10))$

unstable

2 sign changes = 2 poles RHP

The Routh–Hurwitz Criterion

Consider degree-n polynomial

$$p(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n$$

and form the Routh array:

$$\begin{array}{rccccc}
s^n: & 1 & a_2 & a_4 & a_6 & \ldots \\
s^{n-1}: & a_1 & a_3 & a_5 & a_7 & \ldots \\
s^{n-2}: & b_1 & b_2 & b_3 & \ldots & \\
s^{n-3}: & c_1 & c_2 & \ldots & & \\
\vdots & & & & & \\
s^1: & * & * & & & \\
s^0: & * & & & &
\end{array}$$

The Routh–Hurwitz criterion: Assume that the necessary condition for stability holds, i.e., $a_1, \ldots, a_n > 0$. Then:

- p is stable if and only if all entries in the first column are positive;
- otherwise, #(RHP poles) = #(sign changes in 1st column)

Example

Check stability of

$$p(s) = s^4 + 4s^3 + s^2 + 2s + 3$$

All coefficients strictly positive: necessary condition checks out.

$$\begin{array}{lccc} s^4: & 1 & 1 & 3 \\ s^3: & 4 & 2 & 0 \\ s^2: & 1/2 & 3 & \\ s^1: & -22 & 0 & \\ s^0: & 3 & & \end{array}$$

Answer: p is unstable — it has 2 RHP poles (2 sign changes in 1st column)

Low-Order Cases ($n = 2, 3$)

$n = 2$ $\quad p(s) = s^2 + a_1 s + a_2$

$$
\begin{array}{lll}
s^2 & : 1 & a_2 \\
s^1 & : a_1 & 0 \\
s^0 : & b_1 &
\end{array}
\qquad b_1 = -\frac{1}{a_1} \det \begin{pmatrix} 1 & a_2 \\ a_1 & 0 \end{pmatrix} = a_2
$$

— p is stable iff $a_1, a_2 > 0$ (necessary *and* sufficient).

$n = 3$ $\quad p(s) = s^3 + a_1 s^2 + a_2 s + a_3$

$$
\begin{array}{lll}
s^3 & : 1 & a_2 \\
s^2 & : a_1 & a_3 \\
s^1 : & b_1 & 0 \\
s^0 : & c_1 &
\end{array}
\qquad
\begin{array}{l}
b_1 = -\frac{1}{a_1} \det \begin{pmatrix} 1 & a_2 \\ a_1 & a_3 \end{pmatrix} = \frac{a_1 a_2 - a_3}{a_1} \\
c_1 = -\frac{1}{b_2} \det \begin{pmatrix} a_1 & a_3 \\ b_1 & 0 \end{pmatrix} = a_3
\end{array}
$$

— p is stable iff $a_1, a_2, a_3 > 0$ (necc. cond.) and $a_1 a_2 > a_3$

Stability Conditions for Low-Order Polynomials

The upshot:

- A 2nd-degree polynomial $p(s) = s^2 + a_1 s + a_2$ is stable if and only if $a_1 > 0$ and $a_2 > 0$

  Note: $-5s^2 - 2s - 1$
  Syst is stable
  $(\div -5) \to s^2 + \frac{2}{5} + \frac{1}{5} \Rightarrow a_1, a_2 > 0$

- A 3rd-degree polynomial $p(s) = s^3 + a_1 s^2 + a_2 s + a_3$ (constant.) is stable if and only if $a_1, a_2, a_3 > 0$ and $a_1 a_2 > a_3$

- These conditions were already obtained by Maxwell in 1868.
- In both cases, the computations were *purely symbolic*: this can make a lot of difference in *design*, as opposed to *analysis*.

Stability in:

↳ time domain:
- bounded ($\lim_{t \to \infty}$ has a final value)
- unbounded syst is unstable

↳ s-domain:
- poles must be in LHP
- second order → from the sign of coef must all be +ve.
- 3rd order → coef all +ve
  → $a_1 a_2 > a_3$
- $>$ 4th order → Ruth's

Routh–Hurwitz as a Design Tool

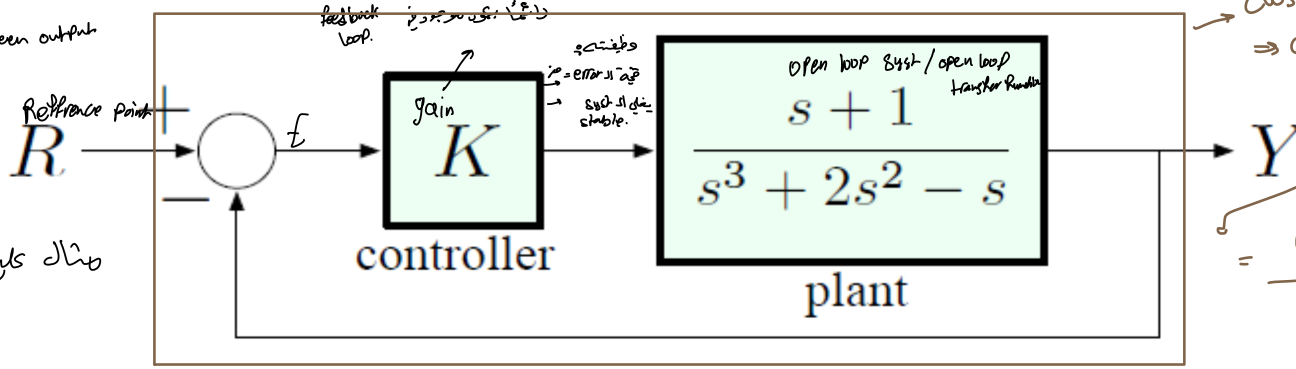
Parametric stability range

We can use the Routh test to determine *parameter ranges* for stability.

Example: consider the unity feedback configuration

$\mathcal{E}$: difference between output and input.

Stedy state error: $\mathcal{E} = R - Y$



مثال كلية قيزر المي بالبيت

closed loop syst

⇒ closed loop transfer function.

$$\frac{Y}{R} = \frac{\text{forward TF}}{1 + (\text{forward TF})(\text{back TF})}$$

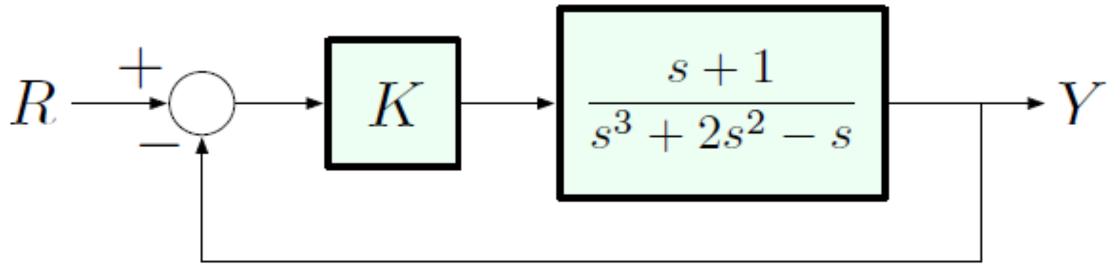
$$= \frac{(K)\left(\frac{s+1}{s^3+2s^2-s}\right)}{1 + (K)\left(\frac{s+1}{s^3+2s^2-s}\right)}$$

$$= \frac{\frac{K(s+1)}{s^3+2s^2-s}}{\frac{s^3+2s^2-s+(Ks+1)}{s^3+2s^2-s}} = \frac{K(s+1)}{s^3+2s^2-s+Ks+K}$$

Note that the plant is *unstable* (the denominator has a negative coefficient and a zero coefficient).

Problem: determine the range of values the scalar gain K can take, for which the closed-loop system is stable.

Example, continued



Problem: determine the range of values the scalar gain K can take, for which the closed-loop system is stable.

Let's write down the transfer function from R to Y:

$$
\begin{aligned}
\frac{Y}{R} &= \frac{\text{forward gain}}{1 + \text{loop gain}} \\
&= \frac{K \cdot \frac{s+1}{s^3+2s^2-s}}{1 + K \cdot \frac{s+1}{s^3+2s^2-s}} = \frac{K(s+1)}{s^3+2s^2-s+K(s+1)} \\
&= \boxed{\frac{Ks+K}{s^3+\underset{a_1}{2s^2}+\underset{a_2}{(K-1)s}+\overset{a_3}{K}}}
\end{aligned}
$$

closed loop transfer function

range of values for K that makes the syst stable.

- $\boxed{K > 1}$
- $2(K-1) > K$
  $2K - 2 > K \Rightarrow \boxed{K > 2}$

Now we need to test stability of $p(s) = s^3 + 2s^2 + (K-1)s + K$.

Example, continued

Test stability of

$$p(s) = s^3 + 2s^2 + (K - 1)s + K$$

using the Routh test.

بقدر استخدم هاي اسهل

$2(K-1) > K$

$\boxed{K > 2}$

Form the Routh array:

مش ضروري اجده

$$\begin{array}{llc} s^3: & 1 & K-1 \\ s^2: & 2 & K \\ s^1: & \frac{K}{2} - 1 & 0 \\ s^0: & K & \end{array}$$

For p to be stable, all entries in the 1st column must be positive:

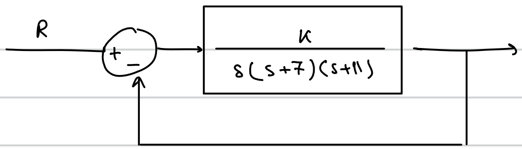
$$K > 2 \quad \text{and} \quad K > 0 \quad \text{(already covered by } K > 1\text{)}$$

Note: The necessary condition requires $K > 1$, but now we actually know that we must have $K > 2$ for stability.

Some Comments on the Routh Test

- The result (#(RHP roots)) is not affected if we multiply or divide any row of the Routh array by an arbitrary *positive* number.
- If we get a zero element in the 1st column, we can't continue. In that case, we can replace the 0 by a small number ε and apply Routh test to that. When we are done with the array, take the limit as $\varepsilon \to 0$. (see Ex. 3.33 in FPE)
- For an *entire row of zeros*, the procedure is a more complicated (see Example 3.34 in FPE) – we will not worry about this too much.

exes



→ Zeros ⇒ $\boxed{s=0}$ ⇒ on the Imagenary axis

$s=-7$ ⇒ unstable

$s=-11$

$$\frac{Y(s)}{R(s)} = \frac{K}{s^3+18s^2+77s+K}$$

- $K>0$
- $18 * 77 > K$
- $1386 > K$

⇒ $0 < K < 1386$