Industrial Control Systems

Chapter Seven:
Feedback Controllers

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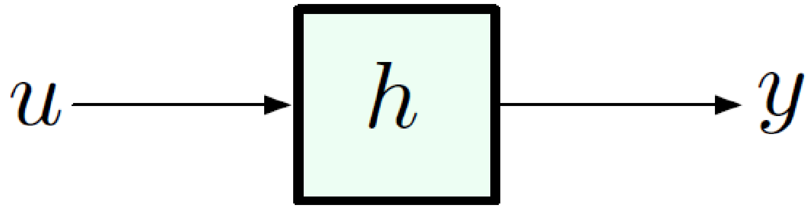
DC Gain



Definition: the steady-state value of the step response is called the *DC gain* of the system.

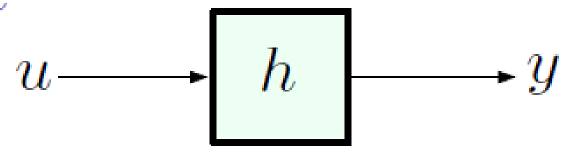
$$\text{DC gain} = y(\infty) = \lim_{t\to\infty} y(t) \qquad \text{for } u(t) = 1(t)$$

In our example above, the step response is

$$y(t) = \frac{1}{2}1(t) + (2\alpha + \beta - 1)e^{-t} + (1/2 - \alpha - \beta)e^{-2t}$$

therefore, DC gain $= y(\infty) = 1/2$

Steady-State Value



$$u(t) = 1(t) \qquad U(s) = \frac{1}{s} \qquad \Longrightarrow \qquad Y(s) = \frac{H(s)}{s}$$

— can we compute $y(\infty)$ from $Y(s)$?

Let's look at some examples:

- $Y(s) = \dfrac{1}{s+a}$, $a > 0$ (pole at $s = -a < 0$)
  $y(t) = e^{-at} \quad \Longrightarrow \quad y(\infty) = 0$
- $Y(s) = \dfrac{1}{s+a}$, $a < 0$ (pole at $s = -a > 0$)
  $y(t) = e^{-at} \quad \Longrightarrow \quad y(\infty) = \infty$
- $Y(s) = \dfrac{1}{s^2+\omega^2}$, $\omega \in \mathbb{R}$ (poles at $s = \pm j\omega$, purely imaginary)
  $y(t) = \sin(\omega t) \quad \Longrightarrow \quad y(\infty)$ does not exist
- $Y(s) = \dfrac{c}{s}$ (pole at the origin, $s = 0$)
  $y(t) = c1(t) \quad \Longrightarrow \quad y(\infty) = c$

The Final Value Theorem

We can now deduce the Final Value Theorem (FVT):

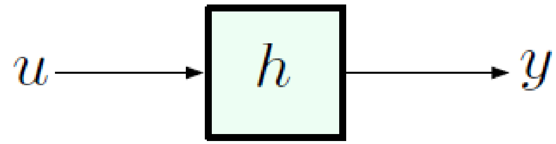
If all poles of $sY(s)$ are *strictly stable* or lie in the *open left half-plane* (OLHP), i.e., have $\mathrm{Re}(s) < 0$, then

$$y(\infty) = \lim_{s \to 0} sY(s).$$

In our examples, multiply $Y(s)$ by s, check poles:

- $Y(s) = \dfrac{1}{s+a} \qquad sY(s) = \dfrac{s}{s+a}$
  if $a > 0$, then $y(\infty) = 0$; if $a < 0$, FVT does not give correct answer
- $Y(s) = \dfrac{1}{s^2+\omega^2} \qquad sY(s) = \dfrac{s}{s^2+\omega^2}$
  poles are purely imaginary (not in OLHP), FVT does not give correct answer
- $Y(s) = \dfrac{c}{s} \qquad sY(s) = c$
  poles at infinity, so $y(\infty) = c$ – FVT gives correct answer

Back to DC Gain



$$\text{Step response:} \qquad Y(s) = \frac{H(s)}{s}$$

— if all poles of $sY(s) = H(s)$ are strictly stable, then

$$y(\infty) = \lim_{s \to 0} H(s)$$

by the FVT.

Example: compute DC gain of the system with transfer function

$$H(s) = \frac{s^2 + 5s + 3}{s^3 + 4s + 2s + 5}$$

All poles of $H(s)$ are strictly stable (we will see this later using the *Routh–Hurwitz criterion*), so

$$y(\infty) = H(s)\Big|_{s=0} = \frac{3}{5}.$$

Steady-State Response Analyses

The steady state performance of a stable control system is generally judged by its steady state error to step, ramp and parabolic inputs. Consider a unity feedback system as shown in the Figure below. The difference between input and output is the error signal E(s).

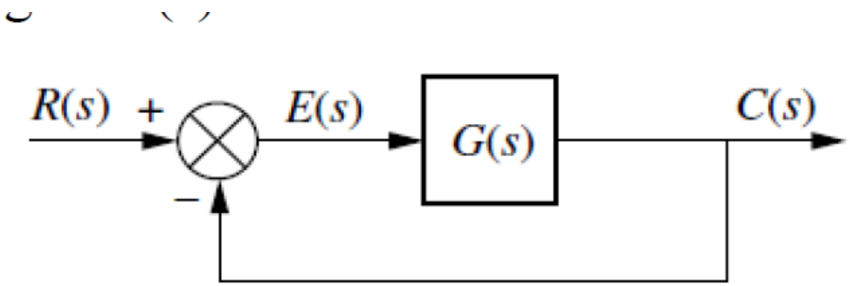


Fig.1 Closed-loop unity feedback control system

The closed loop transfer function (CLTF) is:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

As we know $C(s) = E(s) \times G(s)$

Therefore,

$$E(s) = \frac{1}{1 + G(s)} \times R(s)$$

Steady-state error $\boldsymbol{e_{ss}}$ may be found using the Final Value Theorem (FVT) as follows:

$$e(\infty) = E_{ss} = \lim_{S \to 0} \frac{SR(s)}{1 + G(s)}$$

The above equation shows that the steady state error depends upon the input R(s) and the forward transfer function G(s).

Steady State Error (Ess) for Step input

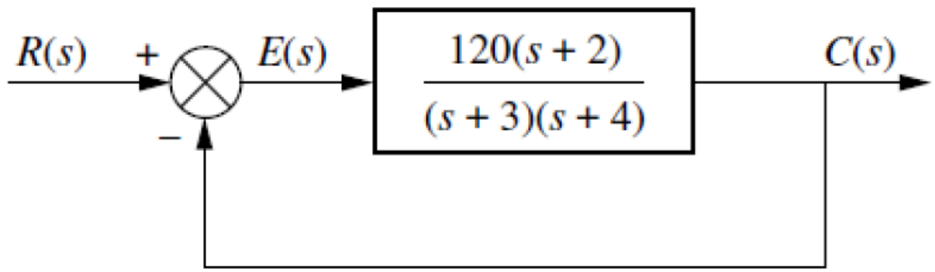
Input is independent on time; $r(t) = 1$ OR $R(s)=1/S$

$$E_{ss} = \lim_{S\to 0} \frac{SR(s)}{1+G(s)} = \lim_{S\to 0} \frac{S \times 1/S}{1+G(s)} = \lim_{S\to 0} \frac{1}{1+G(s)}$$

$$E_{ss} = \frac{1}{1+\lim_{S\to 0} G(s)} = \frac{1}{1+K_p}$$

Where K_p is the ***position error coefficient*** and equals G(0) or $K_p = \lim_{S\to 0} G(s)$

Find the steady-state errors for inputs of 5u(t) to the system shown below. The function u(t) is the unit step.

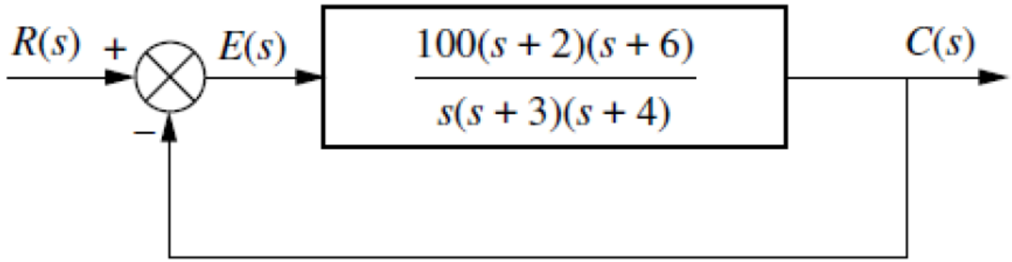


For step input 5u(t), we must calculate the position error coefficient (K_p):

$$K_p = \lim_{S \to 0} G(S) = \frac{120 \times 2}{3 \times 4} = 20$$

$$E_{ss} = \frac{5}{1 + 20} = \frac{5}{21}$$

Find the steady-state errors for inputs of 5u(t) to the system shown below. The function u(t) is the unit step.



For step input 5u(t), we must calculate the position error coefficient (K_p):

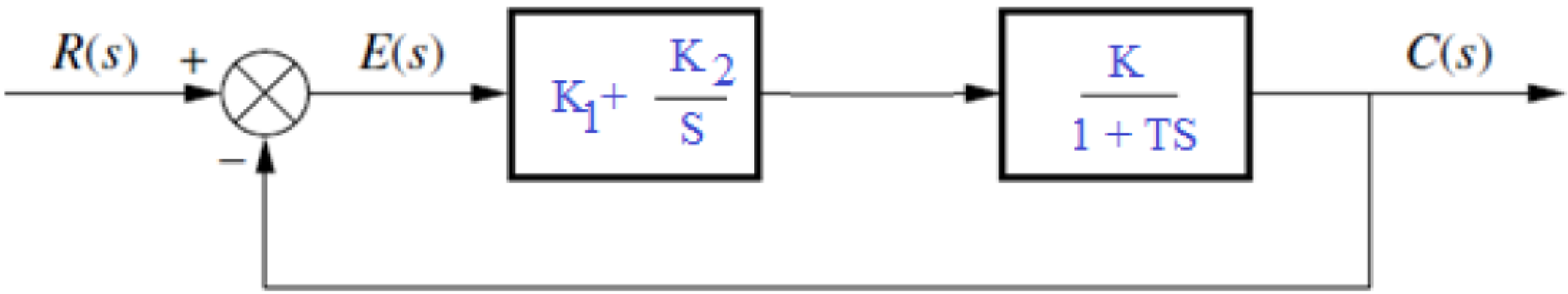
$$K_p = \lim_{S \to 0} G(S) = \frac{100 \times 2 \times 6}{0 \times 3 \times 4} = \infty$$

$$E_{ss} = \frac{5}{1 + \infty} = 0$$

An engine speed control system is shown below.

1) Calculate E_{ss} for step input with magnitude A when $K_2 = 0$

2) Calculate E_{ss} for step input with magnitude A when $K_2 \neq 0$



$$G(s) = \frac{K(K_1S + K_2)}{S(1 + TS)}$$

1) When $K_2 = 0$, the above G(s) is reduced to:

$$G(s) = \frac{KK_1}{1 + TS}$$

$$K_p = \lim_{S\to 0} G(S) = KK_1$$

$$E_{ss} = \frac{1}{1 + K_p} = \frac{1}{1 + KK_1}$$

For a step input with magnitude A;

$$E_{ss} = \frac{A}{1 + KK_1}$$

2) When $K_2 \neq 0$, the open-loop T.F reverts to the original form:

$$G(s) = \frac{K(K_1S + K_2)}{S(1 + TS)}$$

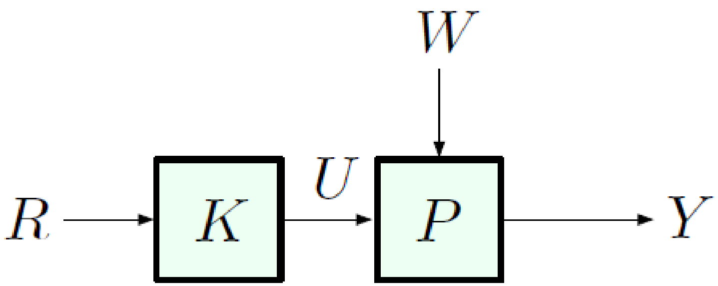
Which represent a type 1 system.

$$K_p = \lim_{S\to 0} G(S) = \infty$$

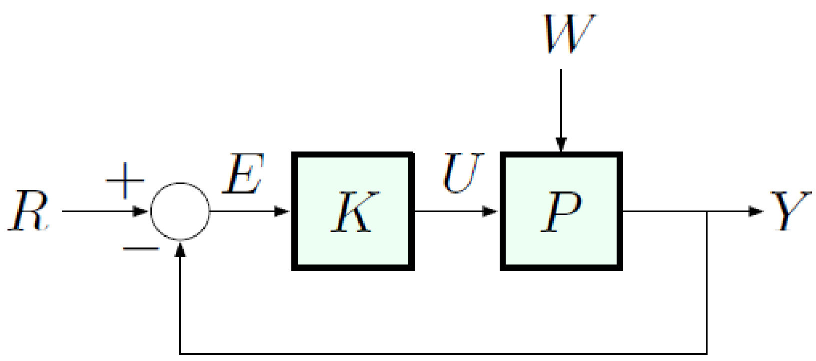
$$E_{ss} = \frac{1}{1 + K_p} = \frac{1}{1 + \infty} = 0$$

Two Basic Control Architectures

- Open-loop control

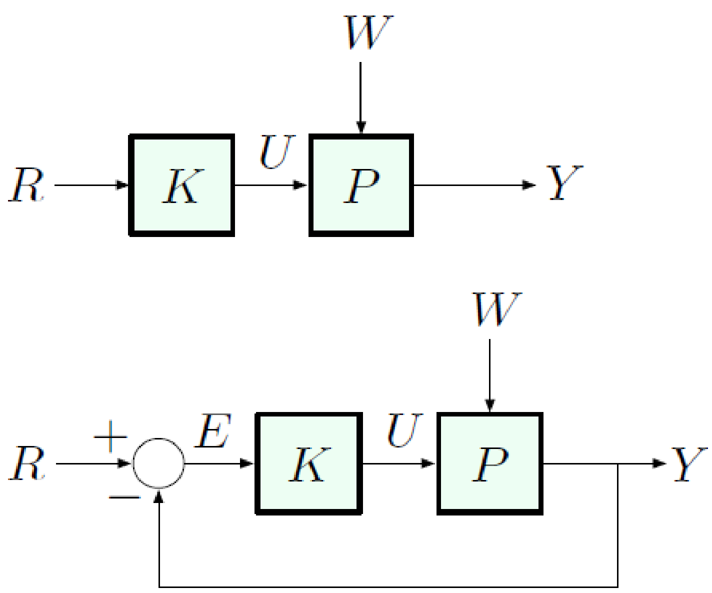


- Feedback (closed-loop) control



Here, W is a *disturbance*; K is *not necessarily* a static gain

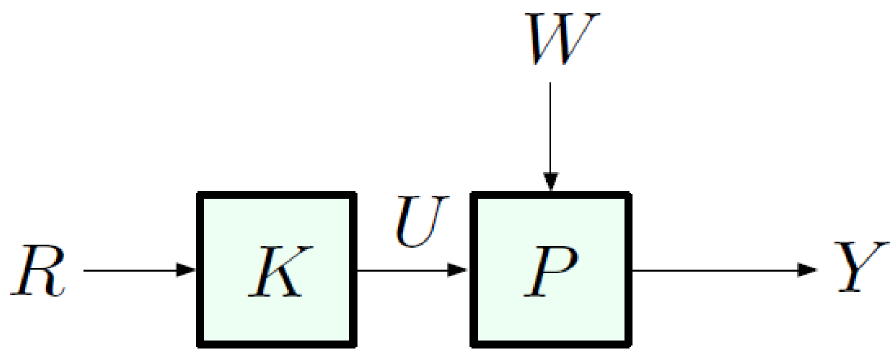
Basic Objectives of Control



- track a given reference
- reject disturbances
- meet performance specs

Intuitively, the difference between the open-loop and the closed-loop architectures is clear (think cruise control ...)

Open-Loop Control



- cheaper/easier to implement (no sensor required)
- does not destabilize the system

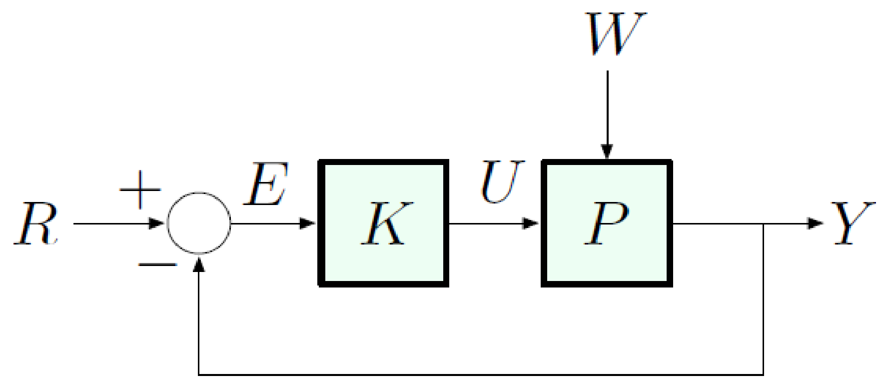
e.g., if both K and P are stable (all poles in OLHP),

$$\frac{Y}{R} = KP$$

is also stable:

$$\{\text{poles of } KP\} = \{\text{poles of } K\} \cup \{\text{poles of } P\}$$

Feedback Control

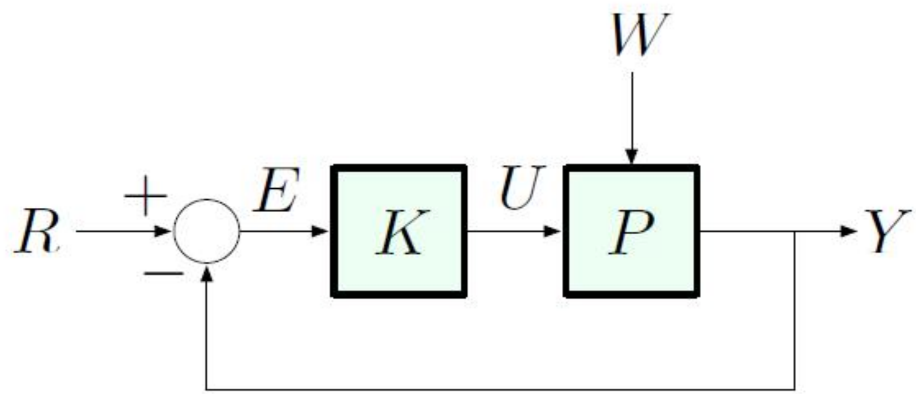


- more difficult/expensive to implement (requires a sensor and an information path from controller to actuator)
- may destabilize the system:

$$\frac{Y}{R} = \frac{KP}{1 + KP}$$

  has new poles, which may be unstable
- but: feedback control is the *only way* to stabilize an unstable plant (this was the Wright brothers' key insight)

Benefits of Feedback Control



Feedback control:

- reduces steady-state error to disturbances
- reduces steady-state sensitivity to model uncertainty (parameter variations)
- improves time response

Summary

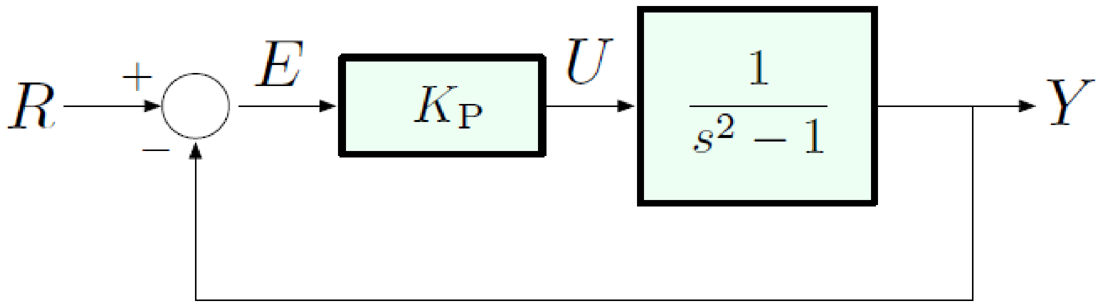
Feedback control:

- reduces steady-state error to disturbances
- reduces steady-state sensitivity to model uncertainty (parameter variations)
- improves time response

Word of caution: high-gain feedback only works well for 1st-order systems; for higher-order systems, it will typically cause underdamping and instability.

Thus, we need a more sophisticated design than just static gain. We will take this up in the next lecture with *Proportional-Integral-Derivative* (PID) control.

Proportional Feedback



K_P – "proportional gain" (P-gain) $\quad U = K_P E$

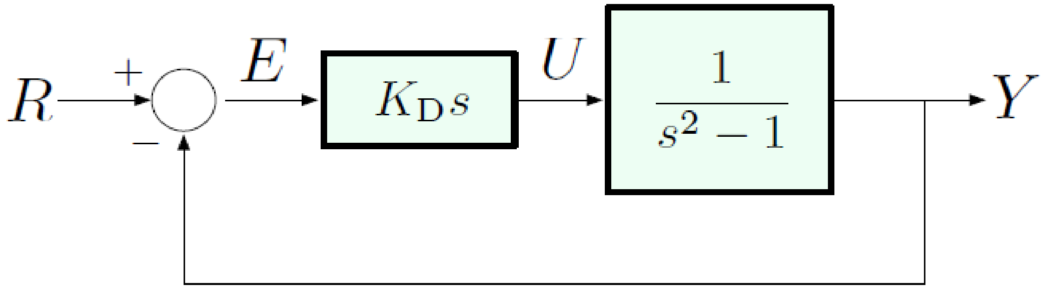
Let's try to find a value of K_P that would stabilize the system:

$$\frac{Y}{R} = \frac{\frac{K_P}{s^2-1}}{1+\frac{K_P}{s^2-1}} = \frac{K_P}{s^2-1+K_P}$$

— the polynomial in the denominator has zero coefficient of s
$\Longrightarrow$ necessary condition for stability is not satisfied.

The feedback system is *not stable for* ***any*** *value of* K_P*!!*

Back to Analysis: Derivative Feedback



$$\frac{Y}{R} = \frac{\frac{K_D s}{s^2-1}}{1 + \frac{K_D s}{s^2-1}} = \frac{K_D s}{s^2 + K_D s - 1}$$

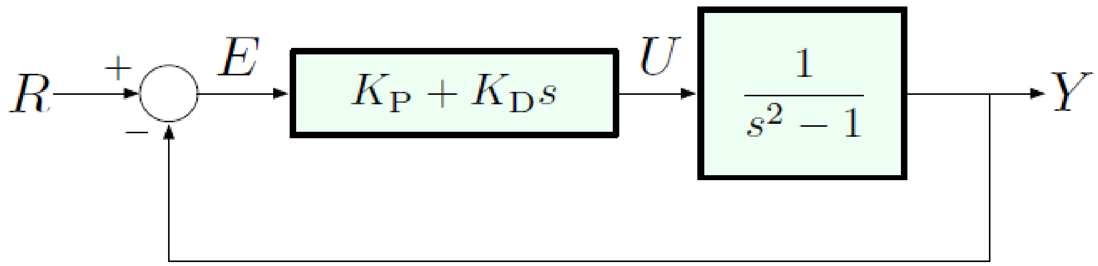
— still not good: the denominator has a negative coefficient
$\Longrightarrow$ not stable; also we have picked up a zero at the origin.

But:

- P-control affected the coefficient of s^0 (constant term)
- D-control affected the coefficient of s

— let's combine them!!

Proportional-Derivative (PD) Control



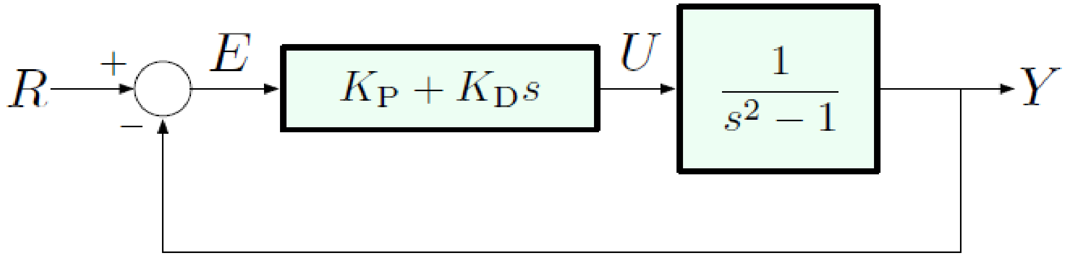
$$\frac{Y}{R} = \frac{\dfrac{K_P + K_D s}{s^2-1}}{1 + \dfrac{K_P + K_D s}{s^2-1}} = \frac{K_P + K_D s}{s^2 + K_D s + K_P - 1}$$

— now, if we set $K_D > 0$ and $K_P > 1$, then the transfer function will be stable.

Even more: by choosing K_P and K_D, we can *arbitrarily* assign coefficients of the denominator, and therefore the poles of the transfer function:

PD control gives us arbitrary pole placement!!

Proportional-Derivative (PD) Control



$$\frac{Y}{R} = \frac{K_{\mathrm{P}} + K_{\mathrm{D}} s}{s^2 + K_{\mathrm{D}} s + K_{\mathrm{P}} - 1}$$

By choosing $K_{\mathrm{P}}, K_{\mathrm{D}}$, we can achieve arbitrary pole placement!!

Also note that the addition of P-gain moves the zero:

$$K_{\mathrm{D}} s + K_P = 0 \qquad \text{LHP zero at } -\frac{K_{\mathrm{P}}}{K_{\mathrm{D}}}$$

But what's missing? $\qquad \text{DC gain} = \left.\frac{Y}{R}\right|_{s=0} = \frac{K_{\mathrm{P}}}{K_{\mathrm{P}} - 1} \neq 1$

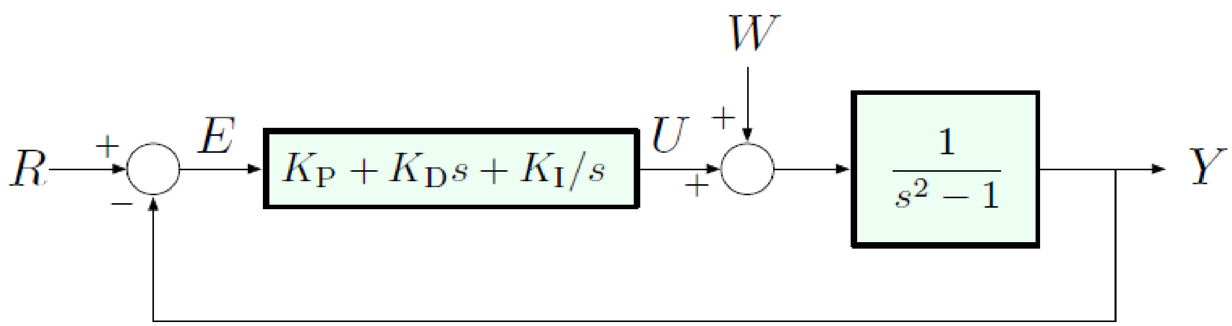
— can't have perfect tracking of constant reference.

Proportional-Integral-Derivative (PID) Control

Let us try

$$U = \left(K_{\mathrm{P}} + K_{\mathrm{D}} s + \frac{K_{\mathrm{I}}}{s} \right) E \qquad \text{– the classic three-term controller}$$

In fact, let's also throw in a constant disturbance:

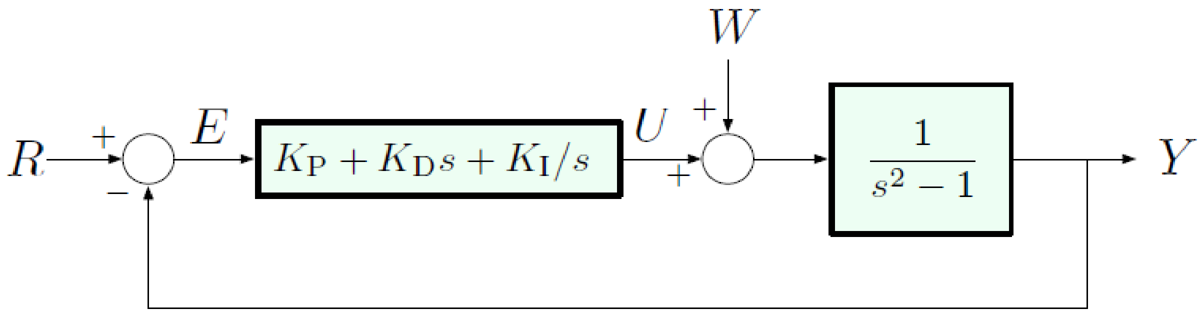


We will see that, with PID control, the goals of

- tracking a constant reference r
- rejecting a constant disturbance w

can be accomplished in one shot.

Proportional-Integral-Derivative (PID) Control



$$Y = \frac{1}{s^2-1}(U+W), \qquad U = \left(K_P + K_D s + \frac{K_I}{s}\right)(R-Y)$$

$$\text{so } Y = \frac{K_P + K_D s + \frac{K_I}{s}}{s^2-1}(R-Y) + \frac{1}{s^2-1}W$$

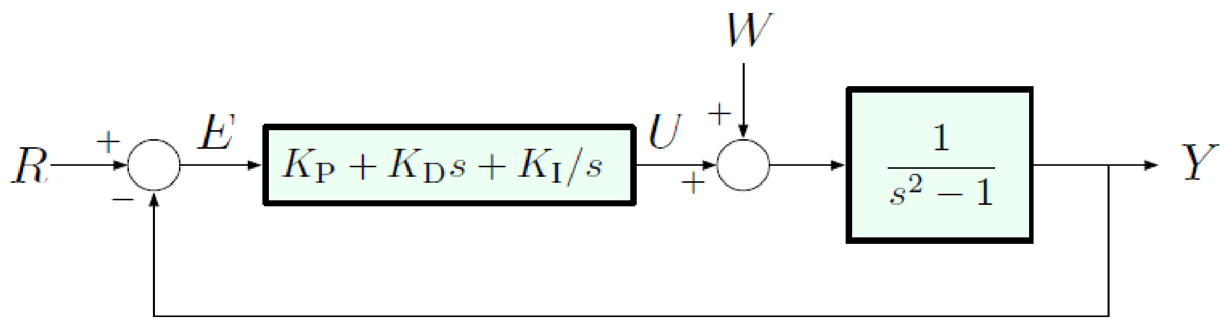
Simplify:

$$(s^2-1)Y = \left(K_P + K_D s + \frac{K_I}{s}\right)(R-Y) + W$$

$$\left(s^2 - 1 + K_P + K_D s + \frac{K_I}{s}\right)Y = \left(K_P + K_D s + \frac{K_I}{s}\right)R + W$$

$$(s^3 - s + K_P s + K_D s^2 + K_I)Y = (K_P s + K_D s^2 + K_I)R + Ws$$

Proportional-Integral-Derivative (PID) Control

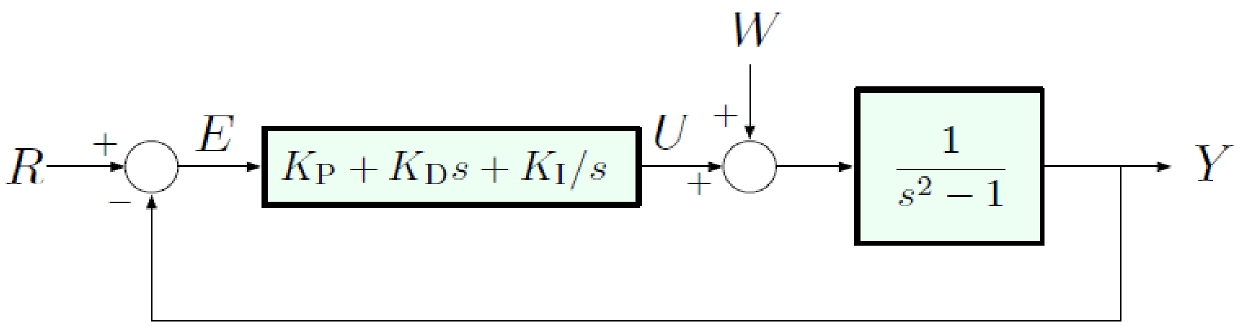


$$(s^3 - s + K_P s + K_D s^2 + K_I)Y = (K_P s + K_D s^2 + K_I)R + Ws$$

Therefore,

$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

Proportional-Integral-Derivative (PID) Control

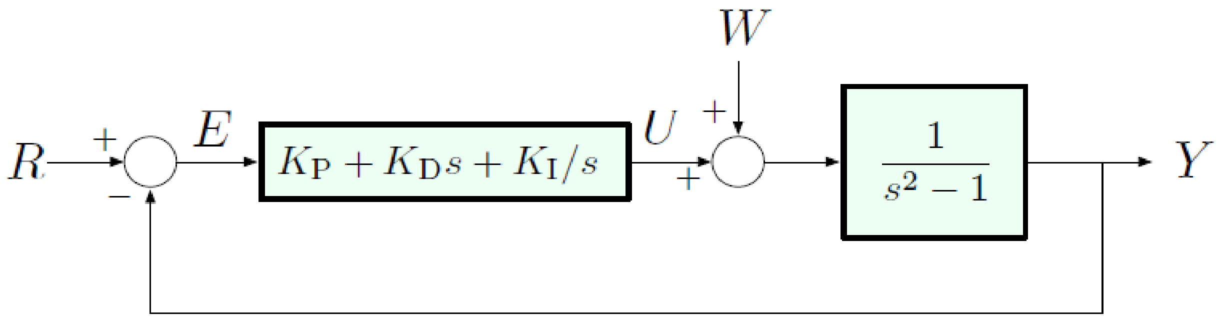


$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

Stability:

- need $K_D > 0$, $K_P > 1$, $K_I > 0$ (necessary condition) and $K_D(K_P - 1) > K_I$ (Routh–Hurwitz for 3rd-order)
- can still assign coefficients arbitrarily by choosing K_P, K_I, K_D

Proportional-Integral-Derivative (PID) Control



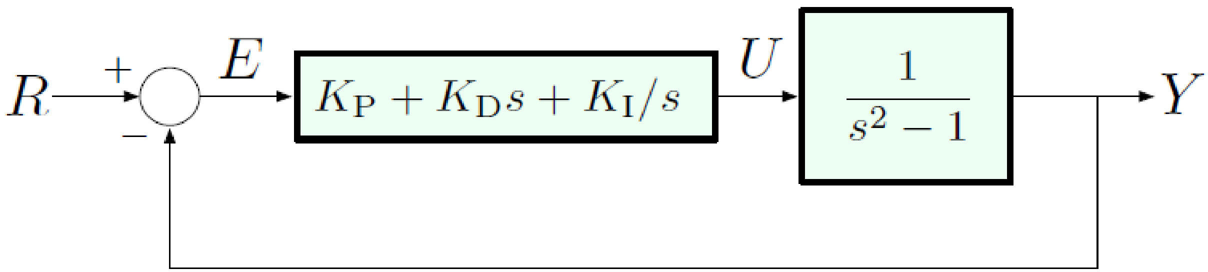
$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

Reference tracking:

$$\text{DC gain}(R \to Y) = \left. \frac{K_D s^2 + K_P s + K_I}{s^3 + (K_P - 1)s + K_D s^2 + K_I} \right|_{s=0} = 1$$

— so, with the addition of I-feedback, we remove earlier limitation and achieve *perfect tracking*!

Proportional-Integral-Derivative (PID) Control



$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

Disturbance rejection:

$$\text{DC gain}(W \to Y) = \left. \frac{s}{s^3 + (K_P - 1)s + K_D s^2 + K_I} \right|_{s=0} = 0$$

— so, integral gain also gives *complete attenuation* of *constant* disturbances!!

PID Control: Summary & Further Comments

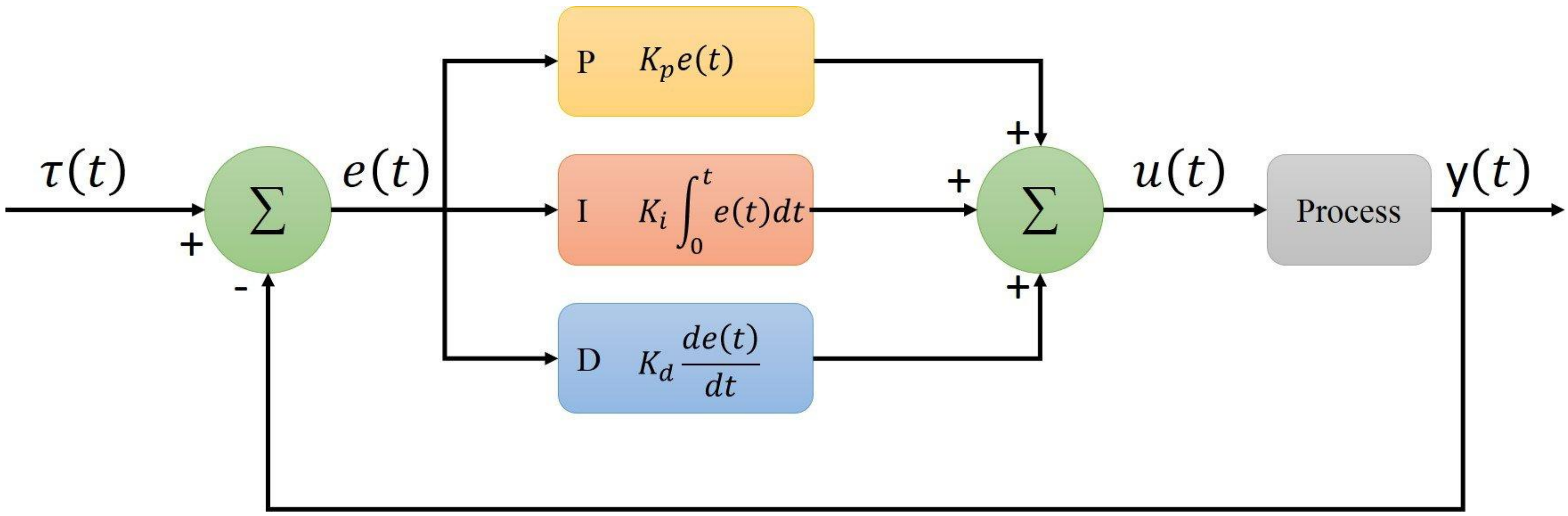
P-gain simplest to implement, but not always sufficient for stabilization

D-gain helps achieve stability, improves time response (more control over pole locations)

- arbitrary pole placement only valid for 2nd-order response; in general, we still have control over two *dominant poles*
- cannot be implemented directly, so need approximate implementation; D-gain also amplifies noise

I-gain essential for perfect steady-state tracking of constant reference and rejection of constant disturbance

PID Controller



PID Controller

- The output of the controller is proportional to the error signal (P-term), the integral signal (I-term), and the derivative of the error signal (D-term) through the gains K_p, K_i, and K_d, respectively.

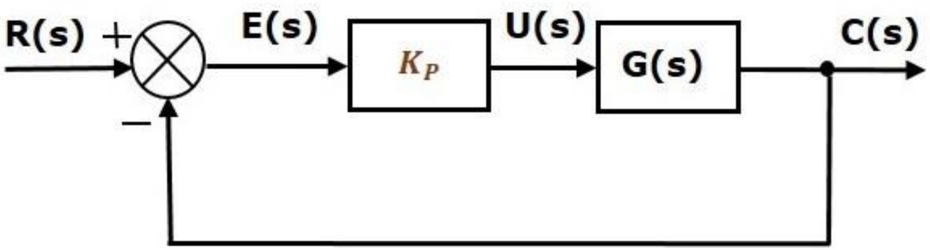
- In practice, variations of the above control law are also implemented, such as a PI controller, which has only the P and I terms, or a PD controller, which has only the P and D terms.

P - Controller

- With the proportional mode, the size of the controller output is proportional to the size of the error.

- This means that the correction elements of the control system (eg: valve), will receive a signal which is proportional to the size of the correction required.

- A gain element with transfer function Kp in series with the forward-path element G(s)



$$U(s) = K_P E(s)$$

$$\frac{U(s)}{E(s)} = K_P$$

D- Controller

- With the derivative mode of control the controller output is proportional to the rate of change with time of the error signal. This can be represented by the equation

$$\mathit{controller}\ \text{output} = K_D \frac{de}{dt}$$

Kp is the constant of proportionality. The transfer function is obtained by taking Laplace transforms, thus

$$\mathit{controller}\ \text{output(s)} = K_D sE(s)$$

Hence the transfer function is $K_D s$

D- Controller

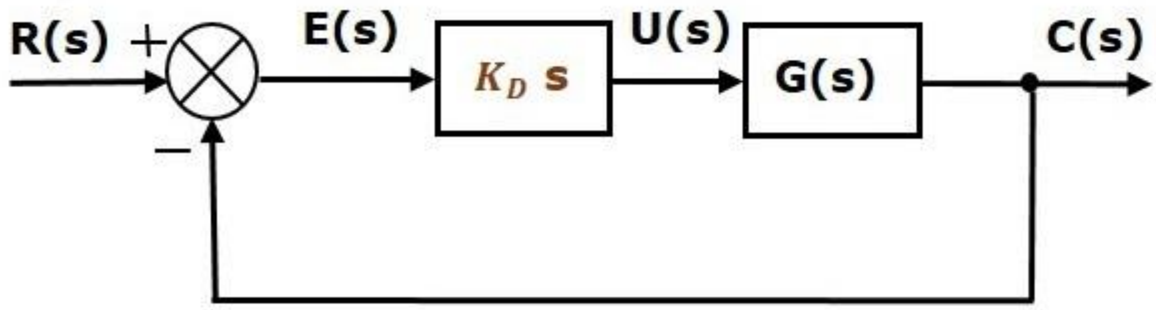
- With derivative control, as soon as the error signal begins to change, there can be quite a large controller output since it is proportional to the rate of change of the error signal and not its value.
    - Rapid initial responses to error signals thus occur.

- The controller output is constant because the rate of change is constant and occurs immediately the deviation occurs.

- Derivative controllers do not, however, respond to steady-state error signals, since with a steady error the rate of change of error with time is zero.

- Because of this, derivative control is always combined with proportional; the proportional part gives a response to all error signals, including steady signals, while the derivative part response to the rate of change.

D- Controller



$$U(s) = K_D s E(s)$$

$$\frac{U(s)}{E(s)} = K_D s$$

PD- Controller

- Derivative control is never used alone because it is not capable of giving an output when there is a steady state error signal and so no correction is possible.

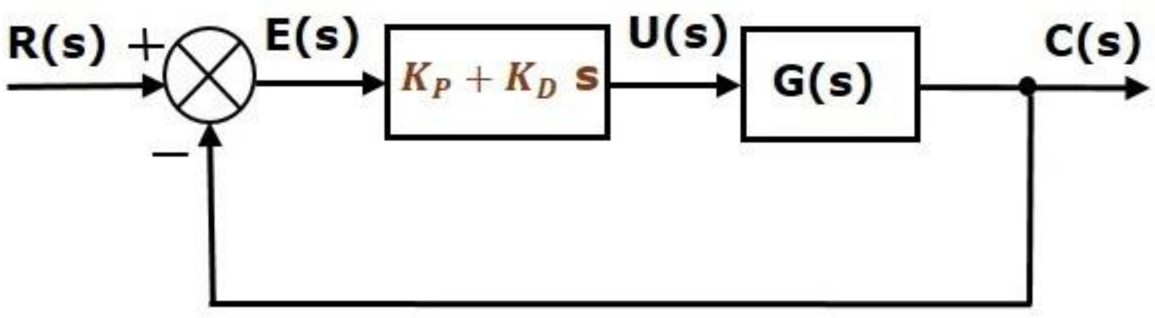
- It is thus invariably used in conjuction with proportional control so that this problem can be resolved.

- With proportional plus derivative control the controller output is given by

$$\textit{controller}\ \text{output} = K_p e + K_D \frac{de}{dt}$$

K_P is the proportionality constant and K_D the derivative constant, de/dt is the rate of change of error.

PD- Controller



$$U(s) = (K_P + K_D s)E(s)$$

$$\frac{U(s)}{E(s)} = K_P + K_D s$$

I - Controller

- The integral mode of control is one where the rate of change of th control output I is proportional to the input error signal e:

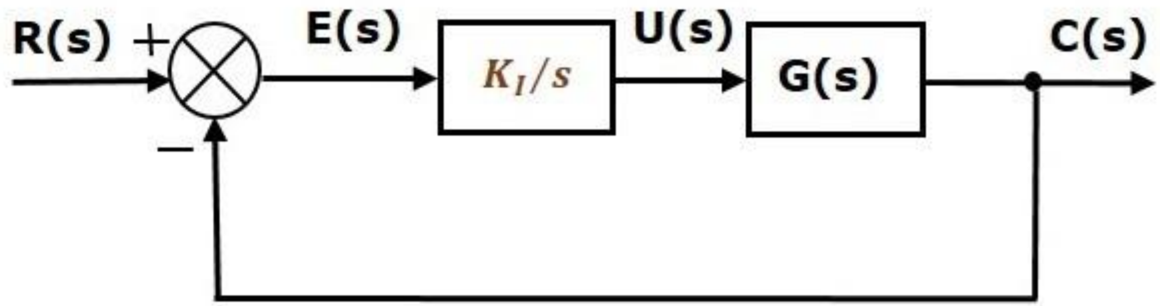
$$\frac{dI}{dt} = K_I e$$

K_I is the constant of proportionality and has units of $1/s$. integrating the above equations gives

$$\int_{Io}^{Iout} dI = \int_0^t K_I e \,\mathrm{d}t \qquad\qquad I_{out} - I_0 = \int_0^t K_I e \,\mathrm{d}t$$

I_0 is the controller output at zero time, I_{out} is the output at time t.

I - Controller



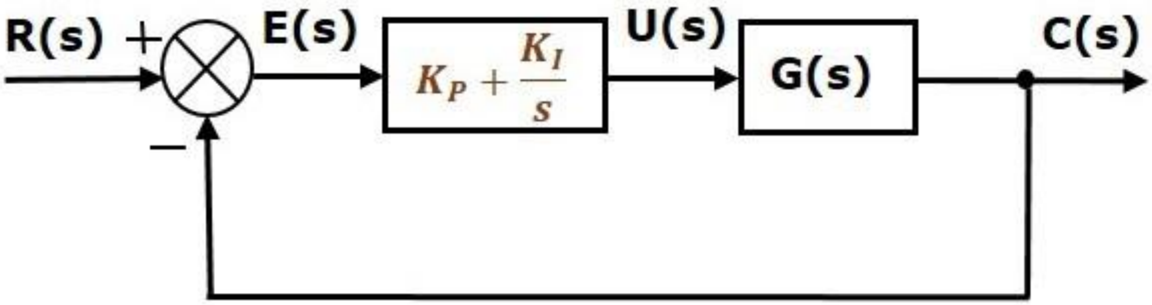
$$U(s) = \frac{K_I E(s)}{s}$$

$$\frac{U(s)}{E(s)} = \frac{K_I}{s}$$

PI - Controller

- The integral mode of control is not usually alone but is frequently used in conjuction with the proportional mode. When integral action is added to a proportional control system the controller output is given by

$$controller\ output = K_p e + K_I \int e\, dt$$

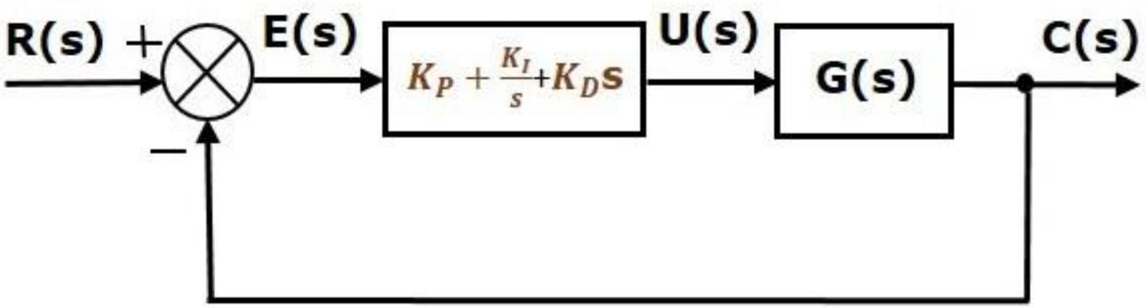


$$U(s) = \left(K_P + \frac{K_I}{s}\right) E(s)$$

$$\frac{U(s)}{E(s)} = K_P + \frac{K_I}{s}$$

PID - Controller

- Combining all three modes of control gives a controller known as a three-mode controller of PID controller.



$$u(t) = K_P e(t) + K_I \int e(t)dt + K_D \frac{\mathrm{d}e(t)}{\mathrm{d}t}$$

Apply Laplace transform on both sides -

$$U(s) = \left(K_P + \frac{K_I}{s} + K_D s\right) E(s)$$

$$\frac{U(s)}{E(s)} = K_P + \frac{K_I}{s} + K_D s$$

The Characteristics of P, I, and D controllers

A proportional controller (Kp) will have the effect of reducing the rise time and will reduce, but never eliminate, the steady-state error.

An integral control (Ki) will have the effect of eliminating the steady-state error, but it may make the transient response worse.

A derivative control (Kd) will have the effect of increasing the stability of the system, reducing the overshoot, and improving the transient response.

Proportional Control

By only employing proportional control, a steady state error occurs.

Proportional and Integral Control

The response becomes more oscillatory and needs longer to settle, the error disappears.

Proportional, Integral and Derivative Control

All design specifications can be reached.

The Characteristics of P, I, and D controllers

CL RESPONSE	RISE TIME	OVERSHOOT	SETTLING TIME	S-S ERROR
Kp ↑	Decrease	Increase	Small Change	Decrease
Ki	Decrease	Increase	Increase	Eliminate
Kd	Small Change	Decrease	Decrease	Small Change

Tips for Designing a PID Controller

1. Obtain an open-loop response and determine what needs to be improved
2. Add a proportional control to improve the rise time
3. Add a derivative control to improve the overshoot
4. Add an integral control to eliminate the steady-state error
5. Adjust each of Kp, Ki, and Kd until you obtain a desired overall response.

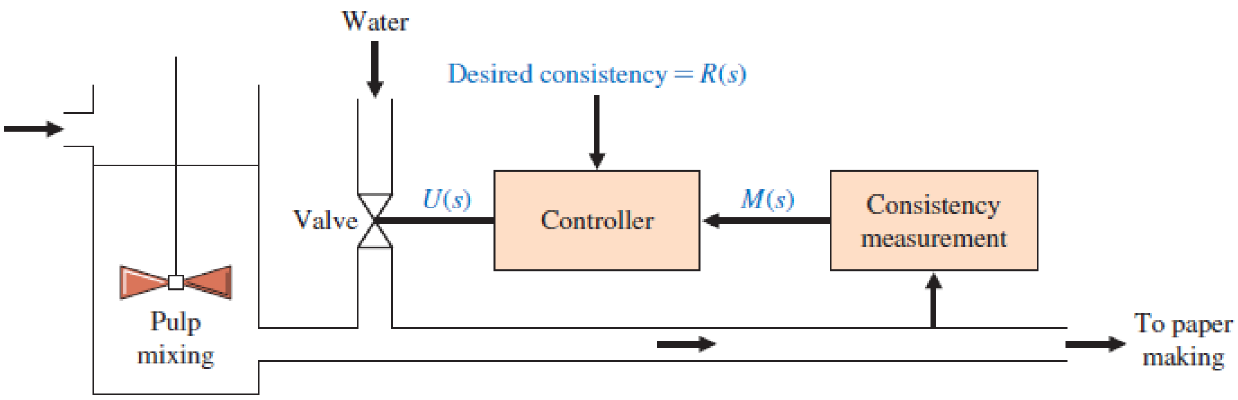
Lastly, please keep in mind that you do not need to implement all three controllers (proportional, derivative, and integral) into a single system, if not necessary. For example, if a PI controller gives a good enough response (like the above example), then you don't need to implement derivative controller to the system. Keep the controller as simple as possible.

Questions

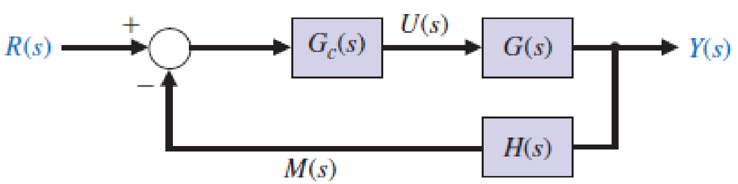
P4.11 One important objective of the paper-making process is to maintain uniform consistency of the stock output as it progresses to drying and rolling. A diagram of the thick stock consistency dilution control system is shown in Figure P4.11(a). The amount of water added determines the consistency. The block diagram of the system is shown in Figure P4.11(b). Let $H(s) = 1$ and

$$G_c(s) = \frac{K}{8s+1}, \qquad G(s) = \frac{1}{3s+1}.$$

Determine (a) the closed-loop transfer function $T(s) = Y(s)/R(s)$,
(c) the steady-state error for a step change in the desired consistency $R(s) = A/s$. (d) Calculate the value of K required for an allowable steady-state error of 2%.



(a)



(b)

Questions

P4.11 (a) The closed-loop transfer function is

$$T(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)} = \frac{K}{(8s+1)(3s+1)+K} = \frac{K}{24s^2 + 11s + 1 + K} .$$

(c) Define $E(s) = R(s) - Y(s)$. Then

$$E(s) = \frac{R(s)}{1 + G_c(s)G(s)} = \left[\frac{24s^2 + 11s + 1}{24s^2 + 11s + K + 1}\right] R(s) .$$

With

$$R(s) = \frac{A}{s} ,$$

we have

$$e_{ss} = \lim_{s \to 0} sE(s) = \frac{A}{1+K} .$$

(d) We want $|e(t)| \leq 0.02A$ as $t \to \infty$. So,

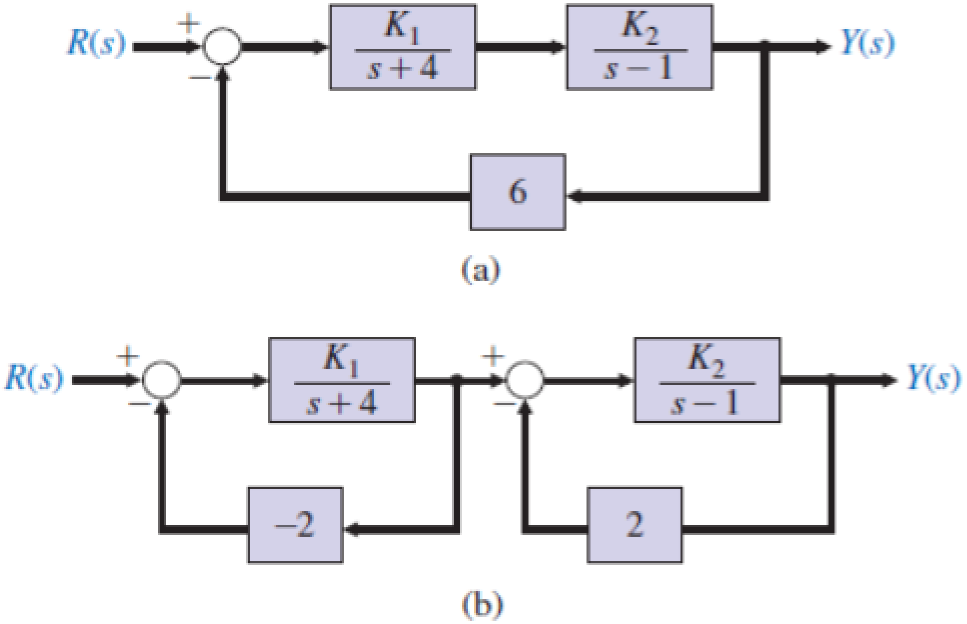
$$0.02A \geq \frac{A}{K+1}$$

implies

$$K \geq 49 .$$

Questions

P4.12 Two feedback systems are shown in Figures P4.12(a) and (b). (a) Evaluate the closed-loop transfer functions T_1 and T_2 for each system.



Questions

(a) The two transfer functions are

$$T_1(s) = \frac{K_1 K_2}{s^2 + 3s - 4 + 6K_1 K_2}$$

and

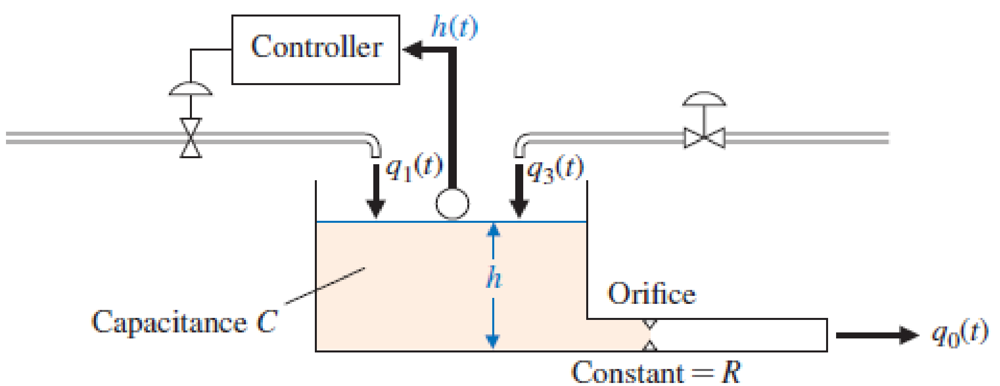
$$T_2(s) = \frac{K_1 K_2}{s^2 + (3 - 2K_1 + 2K_2)s - 4 + 8K_2 + 2K_1 - 4K_1 K_2} .$$

When $K_1 = K_2 = 1$, we find that

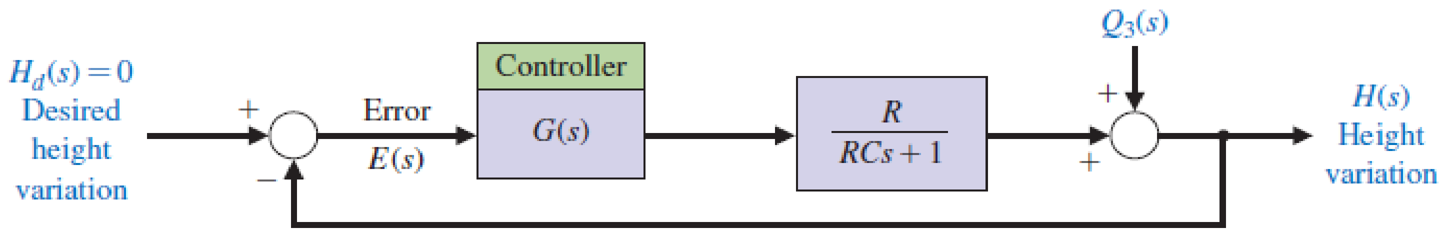
$$T_1(s) = T_2(s) = \frac{1}{s^2 + 3s + 2} .$$

Questions

AP4.1 A tank level regulator control is shown in Figure AP4.1(a). It is desired to regulate the level $H(s)$ in response to a disturbance change $Q_3(s)$. The block diagram shows small variable changes about the equilibrium conditions so that the desired $H_d(s) = 0$. Determine the equation for the error $E(s)$, and determine the steady-state error for a unit step disturbance when (a) $G(s) = K$ and (b) $G(s) = K/s$.



(a)



(b)

Questions

AP4.1 The plant transfer function is

$$G_p(s) = \frac{R}{RCs+1} \; .$$

The closed-loop output is given by

$$H(s) = \frac{1}{1+GG_p(s)} Q_3(s) + \frac{GG_p(s)}{1+GG_p(s)} H_d(s) \; .$$

Therefore, with $E(s) = H_d(s) - H(s)$, we have

$$E(s) = \frac{-1}{1+GG_p(s)} Q_3(s) \; ,$$

since $H_d(s) = 0$.

(a) When $G(s) = K$, we have

$$e_{ss} = \lim_{s \to 0} sE(s) = \frac{-1}{1+KR} \; .$$

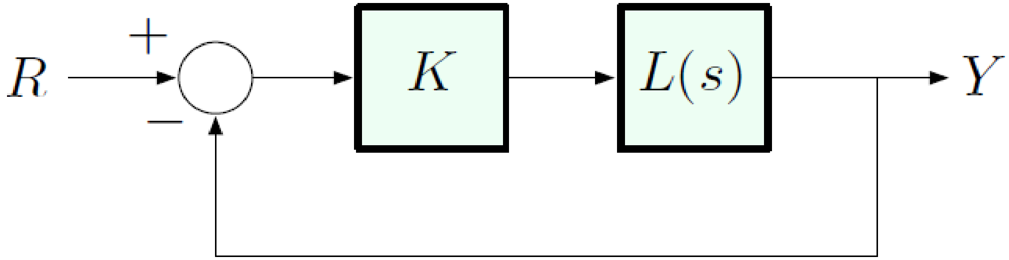
(b) When $G(s) = K/s$, we have

$$e_{ss} = \lim_{s \to 0} sE(s) = 0 \; .$$

The Root Locus Design Method

(invented by Walter R. Evans in 1948)

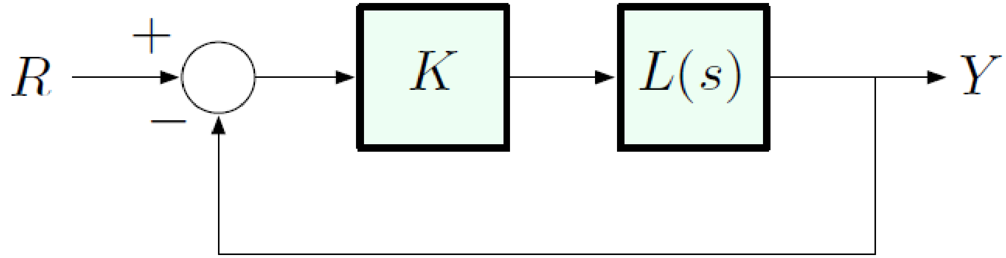
Consider this unity feedback configuration:



where

- K is a constant gain
- $L(s) = \dfrac{b(s)}{a(s)}$, where $a(s)$ and $b(s)$ are some polynomials

The Root Locus Design Method



Closed-loop transfer function: $\dfrac{Y}{R} = \dfrac{KL(s)}{1+KL(s)}, \quad L(s) = \dfrac{b(s)}{a(s)}$

Closed loop poles are solutions of:

$$1 + KL(s) = 0 \qquad \Leftrightarrow \qquad L(s) = -\frac{1}{K}$$

$$\Updownarrow$$

$$1 + \frac{Kb(s)}{a(s)} = 0$$

$$\Updownarrow$$

$$\underbrace{a(s) + Kb(s)}_{\substack{\text{characteristic} \\ \text{polynomial}}} = 0 \qquad \text{characteristic equation}$$

A Comment on Change of Notation

Note the change of notation:

$$\text{from } H(s) \text{ or } G(s) = \frac{q(s)}{p(s)} \quad \text{to } L(s) = \frac{b(s)}{a(s)}$$

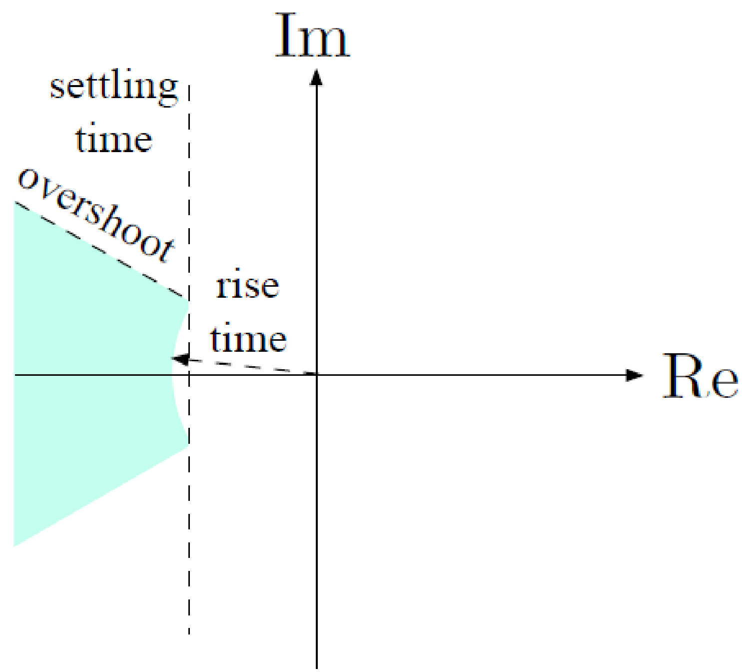
— the RL method is quite general, so $L(s)$ is not necessarily the *plant* transfer function, and K is not necessary *feedback gain* (could be *any parameter*).

E.g., $L(s)$ and K may be related to plant transfer function and feedback gain through some transformation.

As long as we can represent the poles of the closed-loop transfer function as roots of the equation $1 + KL(s) = 0$ for *some choice* of K and $L(s)$, we can apply the RL method.

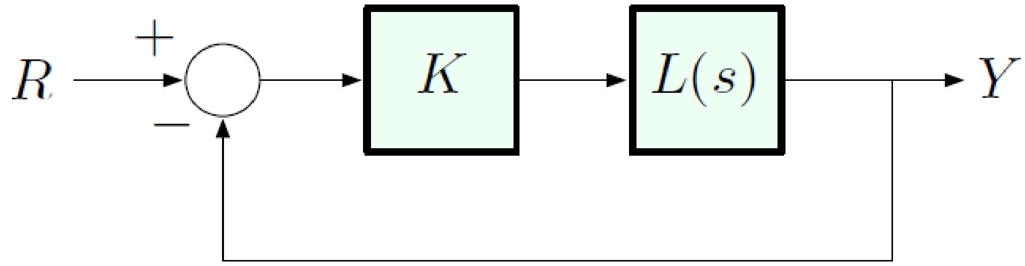
Towards Quantitative Characterization of Stability

Qualitative description of stability: Routh test gives us a range of K to guarantee stability.



For what values of K do we best satisfy given design specs?

Root Locus and Quantitative Stability



Closed-loop transfer function: $\dfrac{Y}{R} = \dfrac{KL(s)}{1 + KL(s)}, \quad L(s) = \dfrac{b(s)}{a(s)}$

For what values of K do we best satisfy given design specs?

Specs are encoded in pole locations, so:

The *root locus* for $1 + KL(s)$ is the set of all closed-loop poles, i.e., the roots of

$$1 + KL(s) = 0,$$

as K varies from 0 to ∞.

A Simple Example

$$L(s) = \frac{1}{s^2 + s} \qquad b(s) = 1,\ a(s) = s^2 + s$$

Characteristic equation: $\qquad a(s) + Kb(s) = 0$

$$s^2 + s + K = 0$$

Here, we can just use the quadratic formula:

$$s = -\frac{1 \pm \sqrt{1 - 4K}}{2} = -\frac{1}{2} \pm \frac{\sqrt{1 - 4K}}{2}$$

$$\text{Root locus} = \left\{ -\frac{1}{2} \pm \frac{\sqrt{1 - 4K}}{2} \ : 0 \le K < \infty \right\} \subset \mathbb{C}$$

TD Specs in Frequency Domain

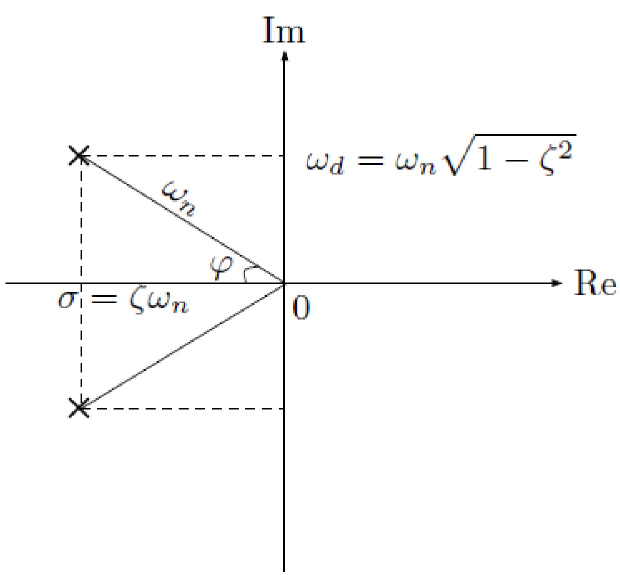
We want to *visualize* time-domain specs in terms of *admissible pole locations* for the 2nd-order system

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\sigma^2 + \omega_d^2}{(s+\sigma)^2 + \omega_d^2}$$

where $\sigma = \zeta\omega_n$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Step response: $y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$



$$\omega_n^2 = \sigma^2 + \omega_d^2$$

$$\zeta = \cos\varphi$$

Formulas for TD Specs

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\sigma^2 + \omega_d^2}{(s+\sigma)^2 + \omega_d^2}$$

$$t_r \approx \frac{1.8}{\omega_n}$$

$$t_p = \frac{\pi}{\omega_d}$$

$$M_p = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

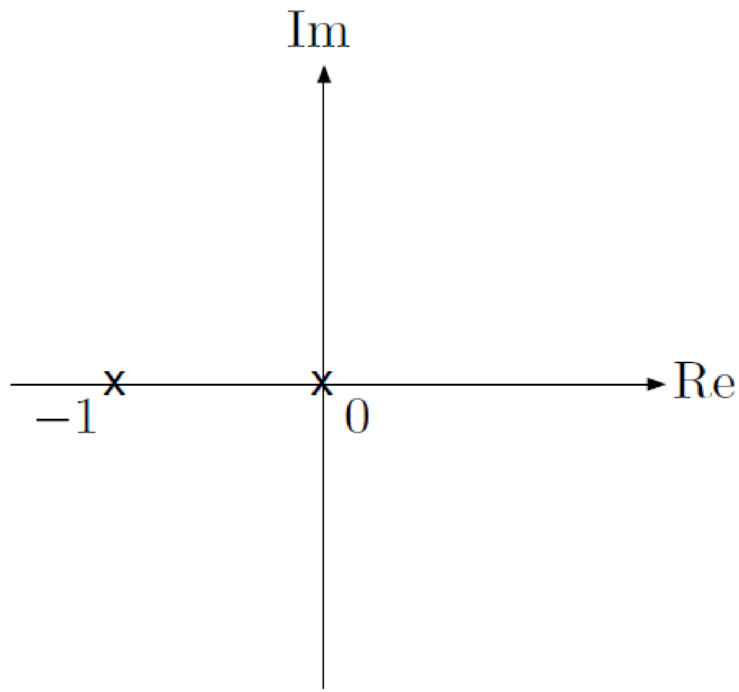
$$t_s \approx \frac{3}{\sigma}$$

Example, continued

$$\text{Root locus} = \left\{ -\frac{1}{2} \pm \frac{\sqrt{1-4K}}{2} \ : 0 \le K < \infty \right\} \subset \mathbb{C}$$

Let's plot it in the s-plane:

- start at $K = 0$ the roots are $-\frac{1}{2} \pm \frac{1}{2} \equiv -1, 0$
  note: these are poles of L (open-loop poles)

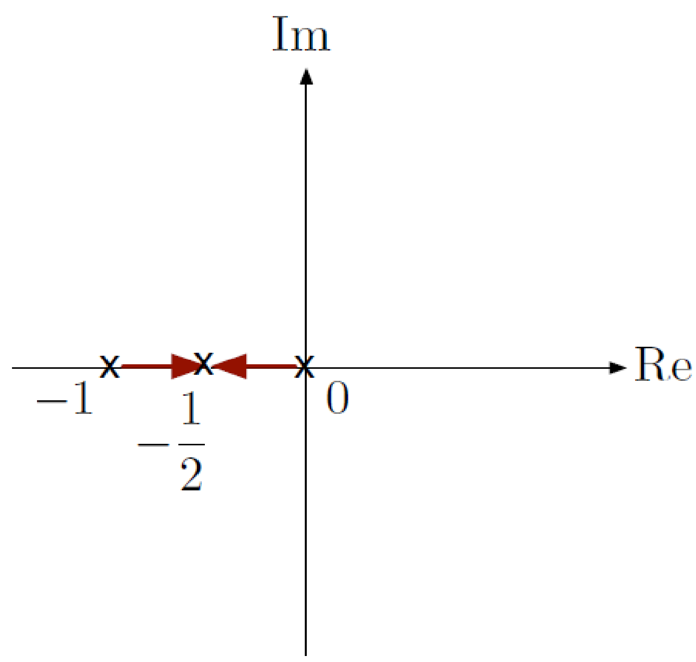


Example, continued

$$\text{Root locus: } \left\{ -\frac{1}{2} \pm \frac{\sqrt{1-4K}}{2} \ : 0 \le K < \infty \right\} \subset \mathbb{C}$$

- as K increases from 0, the poles start to move

$$\begin{aligned} 1 - 4K > 0 \quad &\Longrightarrow \text{ 2 real roots} \\ K = 1/4 \quad &\Longrightarrow \text{1 real root } s = -1/2 \end{aligned}$$

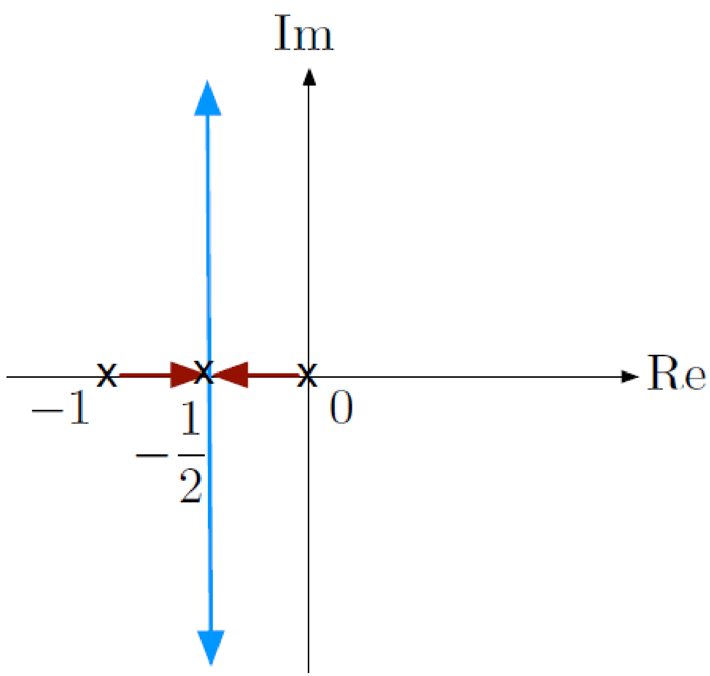


Example, continued

Root locus: $\left\{-\frac{1}{2} \pm \frac{\sqrt{1-4K}}{2} \;:\; 0 \le K < \infty\right\} \subset \mathbb{C}$

- as K increases from 0, the poles start to move

$$K > 1/4 \qquad \Longrightarrow \text{ 2 complex roots with } \mathrm{Re}(s) = -1/2$$



($s = -1/2$ is the *point of breakaway* from the real axis)

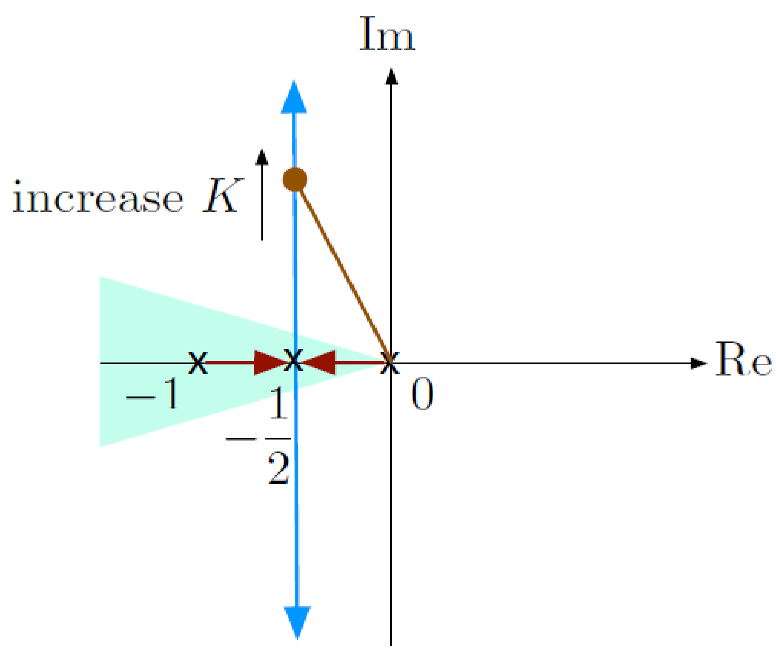
Example, continued

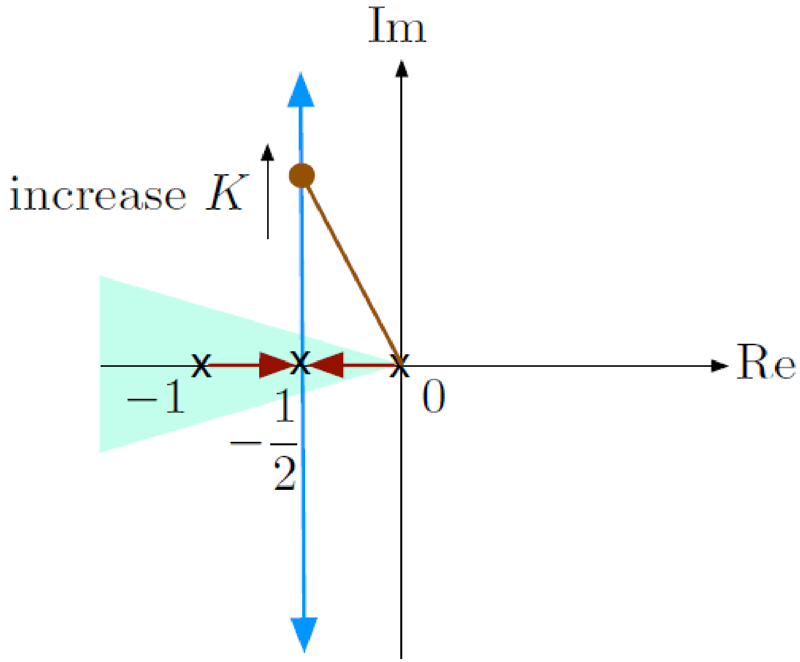
Compare this to admissible regions for given specs:

$t_s \approx \dfrac{3}{\sigma}$ — want σ large, can only have $\sigma = \dfrac{1}{2}$ $(t_s = 6)$

$t_r \approx \dfrac{1.8}{\omega_n}$ — want ω_n large $\Longrightarrow$ want K large

M_p — want to be inside the shaded region $\Longrightarrow$ want K small





Thus, the root locus helps us *visualize the trade-off* between all the specs in terms of K.

However, for order > 2, there will generally be no direct formula for the closed-loop poles as a function of K.

Our goal: develop simple rules for (approximately) sketching the root locus in the general case.

Equivalent Characterization of RL: Phase Condition

Recall our original definition: The *root locus* for $1 + KL(s)$ is the set of all closed-loop poles, i.e., the roots of

$$1 + KL(s) = 0,$$

as K varies from 0 to ∞.

A point $s \in \mathbb{C}$ is on the RL if and only if

$$L(s) = \underbrace{-\frac{1}{K}}_{\text{negative and real}} \qquad \text{for some } K > 0$$

This gives us an equivalent characterization:

The phase condition: The root locus of $1 + KL(s)$ is the set of all $s \in \mathbb{C}$, such that $\angle L(s) = 180^\circ$, i.e., $L(s)$ is real and negative.

Rules for Sketching Root Loci

There are *six rules* for sketching root loci. These rules are mainly qualitative, and their purpose is to give intuition about impact of poles and zeros on performance.

These rules are:

- Rule A — number of branches
- Rule B — start points
- Rule C — end points

Rule A: Number of Branches

$$1 + K\frac{b(s)}{a(s)} = 1 + K\frac{s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n} = 0$$

$$\implies (s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n)$$
$$+ K(s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m) = 0$$

Since $\deg(a) = n \geq m = \deg(b)$, the characteristic polynomial $a(s) + Kb(s) = 0$ has degree n.

The characteristic polynomial has n solutions (roots), some of which may be repeated. As we vary K, these n solutions also vary to form n branches.

Rule A:

$$\#(\text{branches}) = \deg(a)$$

Rule B: Start Points

The locus starts from $K = 0$. What happens near $K = 0$?

If $a(s) + Kb(s) = 0$ and $K \sim 0$, then $a(s) \approx 0$.

Therefore:

- s is close to a root of $a(s) = 0$, or
- s is close to a pole of $L(s)$

Rule B: branches start at open-loop poles.

Rule C: End Points

What happens to the locus as $K \to \infty$?

$$a(s) + Kb(s) = 0$$
$$b(s) = -\frac{1}{K}a(s)$$

— as $K \to \infty$,

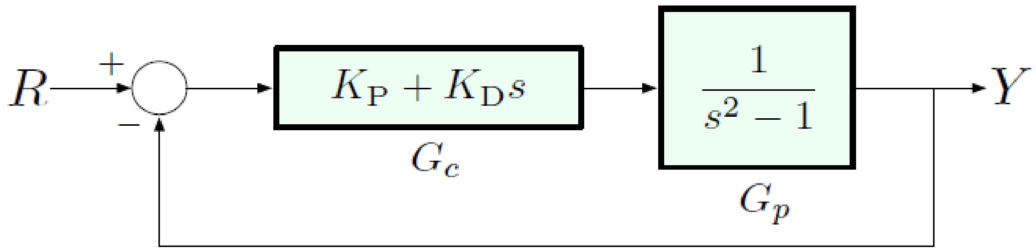
- branches end at the roots of $b(s) = 0$, or
- branches end at zeros of $L(s)$

Rule C: branches end at open-loop zeros.

Note: if $n > m$, we have n branches, but only m zeros. The remaining $n - m$ branches go off to infinity (end at "zeros at infinity").

Example

PD control of an unstable 2nd-order plant



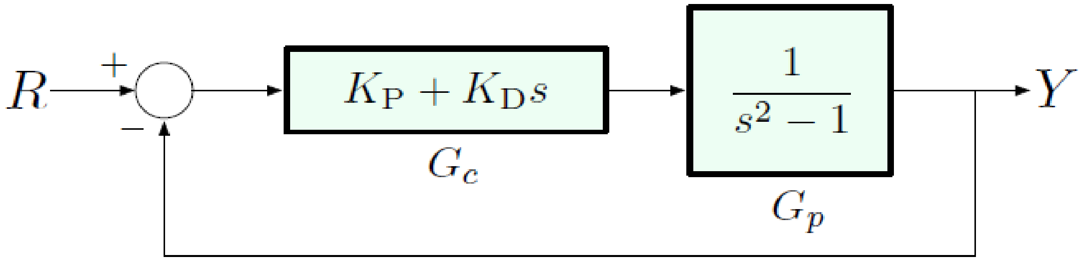
$$\frac{Y}{R} = \frac{G_c G_p}{1 + G_c G_p} \qquad \text{poles: } 1 + G_c(s)G_p(s) = 0$$

$$1 + (K_{\mathrm{P}} + K_{\mathrm{D}}s)\left(\frac{1}{s^2-1}\right) = 0$$

We will examine the impact of varying $K = K_{\mathrm{D}}$, assuming the ratio $K_{\mathrm{P}}/K_{\mathrm{D}}$ *fixed.*

Example

PD control of an unstable 2nd-order plant



We will examine the impact of varying $K = K_{\mathrm{D}}$, assuming the ratio $K_{\mathrm{P}}/K_{\mathrm{D}}$ *fixed*.

Let us write the characteristic equation in *Evans form*:

$$1 + \underbrace{K_{\mathrm{D}}}_{K} \left(s + \frac{K_{\mathrm{P}}}{K_{\mathrm{D}}} \right) \left(\frac{1}{s^2 - 1} \right) = 1 + K \underbrace{\frac{s + K_{\mathrm{P}}/K_{\mathrm{D}}}{s^2 - 1}}_{L(s)} = 0$$

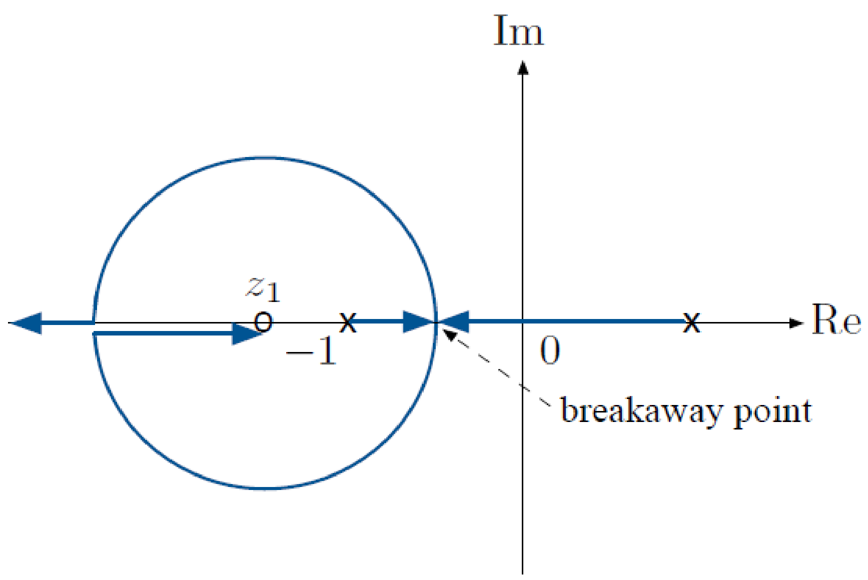
$$L(s) = \frac{s - z_1}{s^2 - 1} \qquad \text{zero at } s = z_1 = -K_{\mathrm{P}}/K_{\mathrm{D}} < 0$$

Example

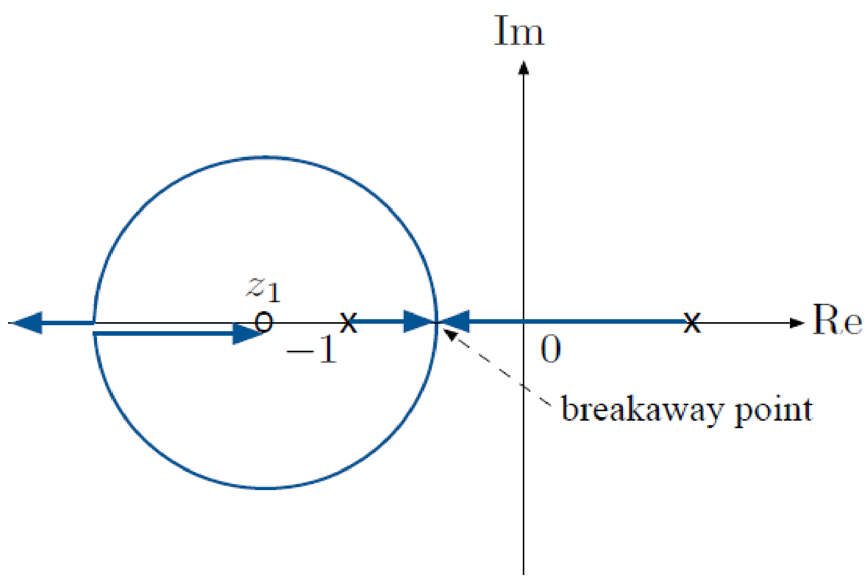
$$L(s) = \frac{s - z_1}{s^2 - 1}$$

- Rule A: $\begin{cases} m = 1 \\ n = 2 \end{cases} \implies 2$ branches
- Rule B: branches start at open-loop poles $\qquad s = \pm 1$
- Rule C: branches end at open-loop zeros $\qquad s = z_1, -\infty$
  (we will see why $-\infty$ later)

So the root locus will look something like this:



$$L(s) = \frac{s - z_1}{s^2 - 1}$$



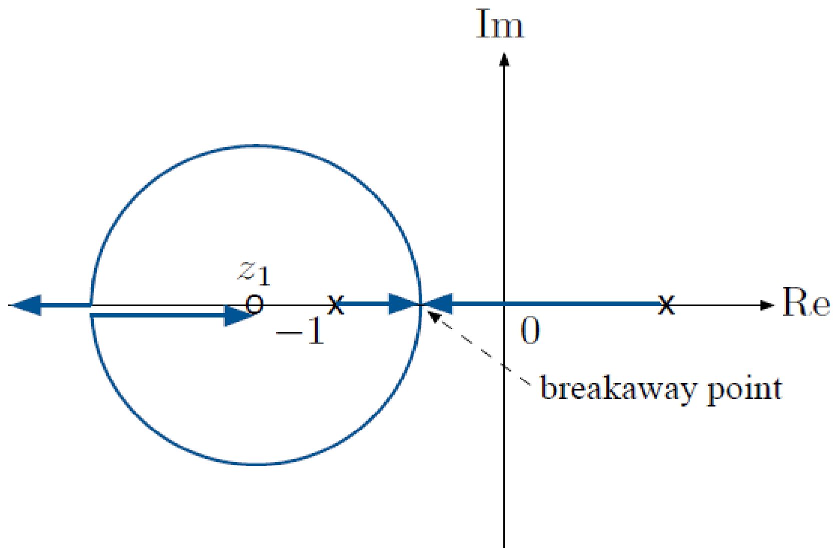
Why does one of the branches go off to $-\infty$?

$s^2 - 1 + K(s - z_1) = 0$

$s^2 + Ks - (Kz_1 + 1) = 0$

$s = -\frac{K}{2} \pm \sqrt{\frac{K^2}{4} + Kz_1 + 1}, \ z_1 < 0$ as $K \to \infty$, s will be < 0

$$L(s) = \frac{s - z_1}{s^2 - 1}$$



Is the point $s = 0$ on the root locus?

Let's see if there is any value $K > 0$, for which this is possible:

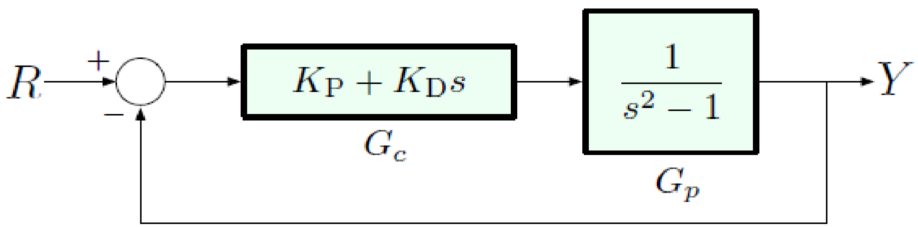
$$1 + KL(0) = 0$$

$$1 + Kz_1 = 0 \qquad K = -\frac{1}{z_1} > 0 \text{ does the job}$$

From Root Locus to Time Response Specs

For concreteness, let's see what happens when

$$K_{\mathrm{P}}/K_{\mathrm{D}} = -z_1 = 2 \qquad \text{and} \qquad K = K_{\mathrm{D}} = 5 \Longrightarrow K_{\mathrm{P}} = 10$$



$$G_c(s) = 10 + 5s$$

$$u = 10e + 5\dot{e}, \qquad e = r - y$$

Characteristic equation: $\quad 1 + 5\left(\frac{s+2}{s^2-1}\right) = 0$

$$s^2 + 5s + 9 = 0$$

Relate to 2nd-order response: $\quad \omega_n^2 = 9,\ 2\zeta\omega_n = 5 \Longrightarrow \zeta = 5/6$

Example 1: System with Two Real Poles

System Transfer Function: $G(s) = \dfrac{K}{s(s+2)}$

Goal: Sketch the root locus as K varies from 0 to ∞.

Step-by-Step Solution

1. **Locate Open-Loop Poles and Zeros:**
   - **Poles** (roots of the denominator when $K = 0$): $p_1 = 0$, $p_2 = -2$.
   - **Zeros** (roots of the numerator): None in the finite s-plane.
2. **Determine the Number of Branches:**
   - The number of branches is equal to the number of poles (P), which is 2. The number of finite zeros is $Z = 0$.

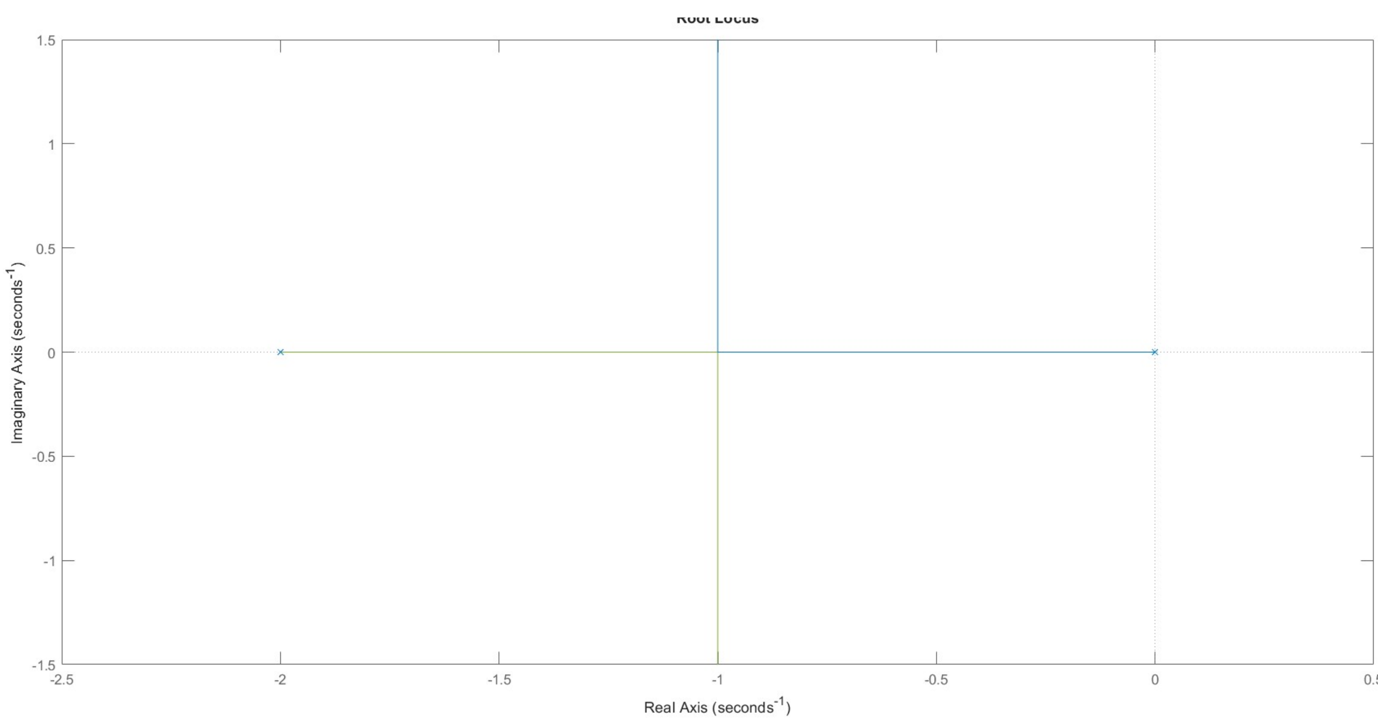
3. **Identify Root Locus on the Real Axis:**
   - A point on the real axis is part of the root locus if there is an **odd** number of poles and zeros to its right.
   - Region between $s = 0$ and $s = -2$: To the right of any point in this region, there is one pole at $s = 0$ (odd number). So, the locus exists here.
   - Region to the left of $s = -2$: To the right of any point here, there are two poles (even number). No locus here.
   - Region to the right of $s = 0$: To the right of any point here, there are no poles (even number). No locus here.
4. **Find Breakaway/Break-in Points:**
   - The branches on the real axis move towards each other as K increases and meet at a breakaway point.

- Form the characteristic equation: $1 + \frac{K}{s(s+2)} = 0 \Longrightarrow s^2 + 2s + K = 0.$
- Express K in terms of s: $K = -(s^2 + 2s).$
- Differentiate K with respect to s and set the derivative to zero to find the points where the locus might break away:
  $\frac{dK}{ds} = -(2s + 2) = 0.$
- Solving for s, we get $s = -1$.
- This point is within the valid real-axis locus segment (between 0 and -2), so $s = -1$ is the breakaway point.

5. **Determine Asymptotes (if any):**
   - Number of asymptotes: $P - Z = 2 - 0 = 2.$
   - Angles of asymptotes: $\theta_a = \frac{\pm 180^\circ(2m+1)}{P-Z} = \frac{\pm 180^\circ}{2} = \pm 90^\circ.$
   - Centroid of asymptotes: $\sigma_a = \frac{\sum \text{Poles} - \sum \text{Zeros}}{P-Z} = \frac{(0 + (-2))}{2} = -1.$

6. **Sketch the Root Locus:**

- The two branches start at $s = 0$ and $s = -2$ for $K = 0$.
- They move towards each other along the real axis, meet at the breakaway point $s = -1$.
- For $K > K_{\text{breakaway}}$, the roots become a complex-conjugate pair and move vertically along the asymptotes ($s = -1 \pm j\omega$) into the complex plane (forming a vertical line centered at $s = -1$).
- The system is always stable for all $K > 0$ as the locus remains in the left-half s-plane.



Example 2: System with Two Real Poles and One Zero

System Transfer Function: $G(s) = \dfrac{K(s+1)}{s(s+2)}$

Goal: Sketch the root locus as K varies from 0 to ∞.

Step-by-Step Solution

1. **Locate Open-Loop Poles and Zeros:**
   - **Poles**: $p_1 = 0$, $p_2 = -2$.
   - **Zeros**: $z_1 = -1$.
2. **Determine the Number of Branches:**
   - Number of branches: $P = 2$.
   - One branch will end at the finite zero $s = -1$ (as $K \to \infty$) and the other will go to infinity.

3. **Identify Root Locus on the Real Axis:**

- Region between $s = 0$ and $s = -1$: To the right, we have one pole at $s = 0$ (odd number). Locus exists here.
- Region between $s = -1$ and $s = -2$: To the right, we have one pole and one zero (even number). No locus here.
- Region to the left of $s = -2$: To the right, we have two poles and one zero (odd number). Locus exists here.
- Region to the right of $s = 0$: No locus here.

4. **Find Breakaway/Break-in Points:**

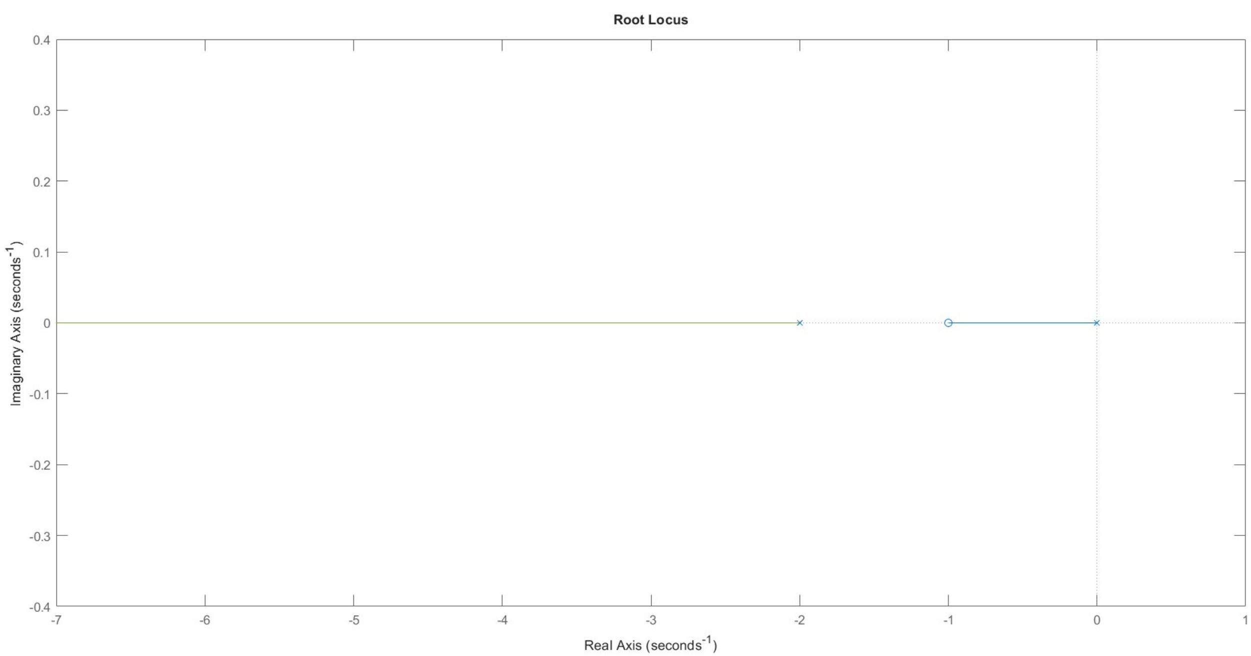
- Characteristic equation:
$$1 + \frac{K(s+1)}{s(s+2)} = 0 \implies s^2 + 2s + K(s+1) = 0 \implies s^2 + (2+K)s + K = 0.$$
- Express K in terms of s: $K = -\frac{s^2 + 2s}{s+1}$.
- Differentiate K with respect to s using the quotient rule, and set the derivative to zero:
$$\frac{dK}{ds} = -\frac{(2s+2)(s+1) - (s^2+2s)(1)}{(s+1)^2} = 0.$$
$$(2s+2)(s+1) - (s^2+2s) = 0$$
$$2s^2 + 2s + 2s + 2 - s^2 - 2s = 0$$
$$s^2 + 2s + 2 = 0.$$
- The roots of this quadratic equation are
$$s = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm j1.$$
- These are complex-conjugate points, not on the real axis, so they are **break-in points** where the complex loci meet the real axis (though in this specific second-order case, the locus doesn't reach the complex plane).

5. **Determine Asymptotes:**

- Number of asymptotes: $P - Z = 2 - 1 = 1$.
- Angle of asymptote: $\theta_a = \dfrac{\pm 180^\circ(2m+1)}{1} = 180^\circ$.
- Centroid: $\sigma_a = \dfrac{(0 + (-2)) - (-1)}{1} = -1$.
- The single asymptote is on the real axis at $s = -1$.

6. **Sketch the Root Locus:**

- One branch starts at $s = 0$ and moves left along the real axis towards $s = -1$. It terminates at the zero at $s = -1$ as $K \to \infty$.
- The other branch starts at $s = -2$ and moves left along the real axis. It goes to infinity along the 180° asymptote centered at $s = -1$.
- The entire root locus lies on the real axis and the system is always stable for all $K > 0$. Adding the zero pulled the locus further into the left-half plane compared to Example 1, allowing for a higher stable gain.



Transfer Function: $G(s)H(s) = \dfrac{K(s+1)}{s(s+2)(s+3)}$

1. **Poles and Zeros**: Poles at $s = 0, -2, -3$. Zero at $s = -1$.

2. **Real Axis Segments**: Locus exists between 0 and -1, and between -2 and -3.

3. **Asymptotes**:

    1. Number of asymptotes $(n - m) = 3 - 1 = 2$.

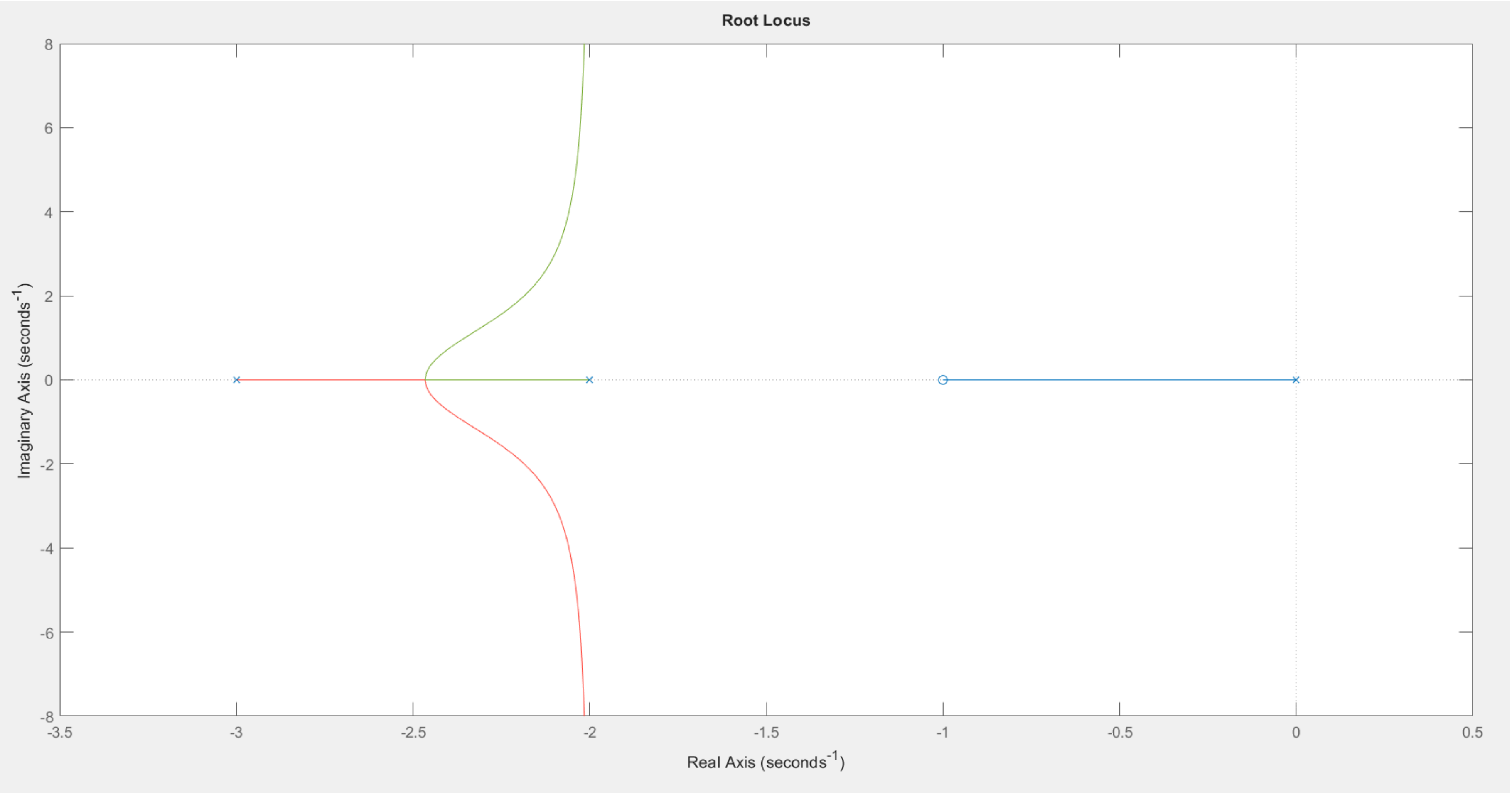
    2. **Angles**: $\dfrac{180^\circ(2k+1)}{2} = \pm 90^\circ$.

    3. **Centroid (σ_a)**: $\dfrac{(0 - 2 - 3) - (-1)}{2} = \dfrac{-4}{2} = -2$.

4. **Breakaway/Break-in Points**: Solve $\frac{dK}{ds} = 0$.

    1. $K = -\frac{s(s+2)(s+3)}{s+1} = -\frac{s^3+5s^2+6s}{s+1}$.
    2. Differentiating gives $s \approx -0.634$ (breakaway between 0 and -1) and $s \approx -2.366$ (break-in between -2 and -3).

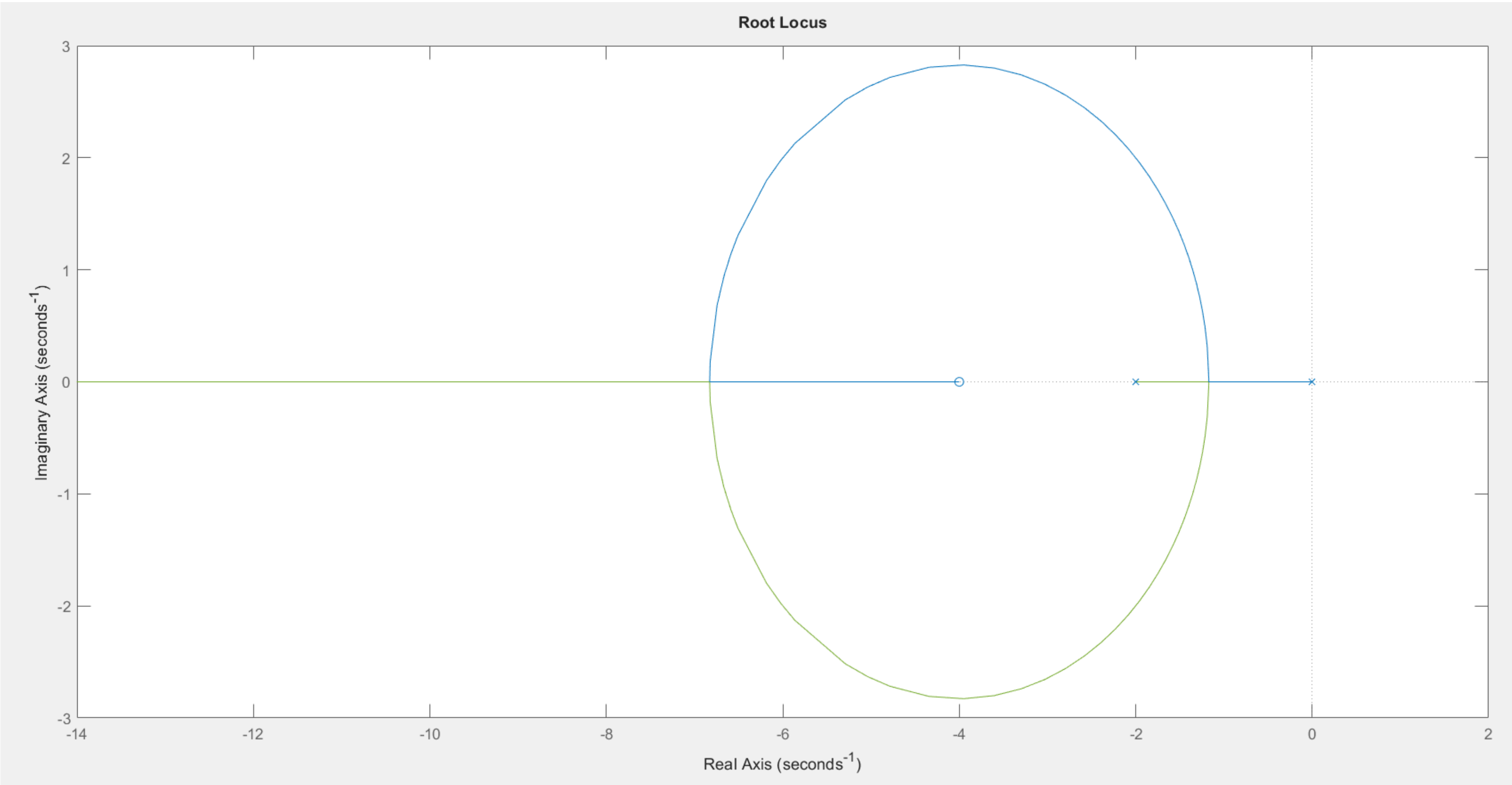
5. **Stability**: Since branches move toward the left-half plane or terminate at the zero at -1, the system is stable for all $K > 0$.



Root Locus
8
6
4
2
0
-2
-4
-6
-8
Imaginary Axis (seconds⁻¹)
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
Real Axis (seconds⁻¹)

Transfer Function: $G(s)H(s) = \frac{K(s+4)}{s(s+2)}$

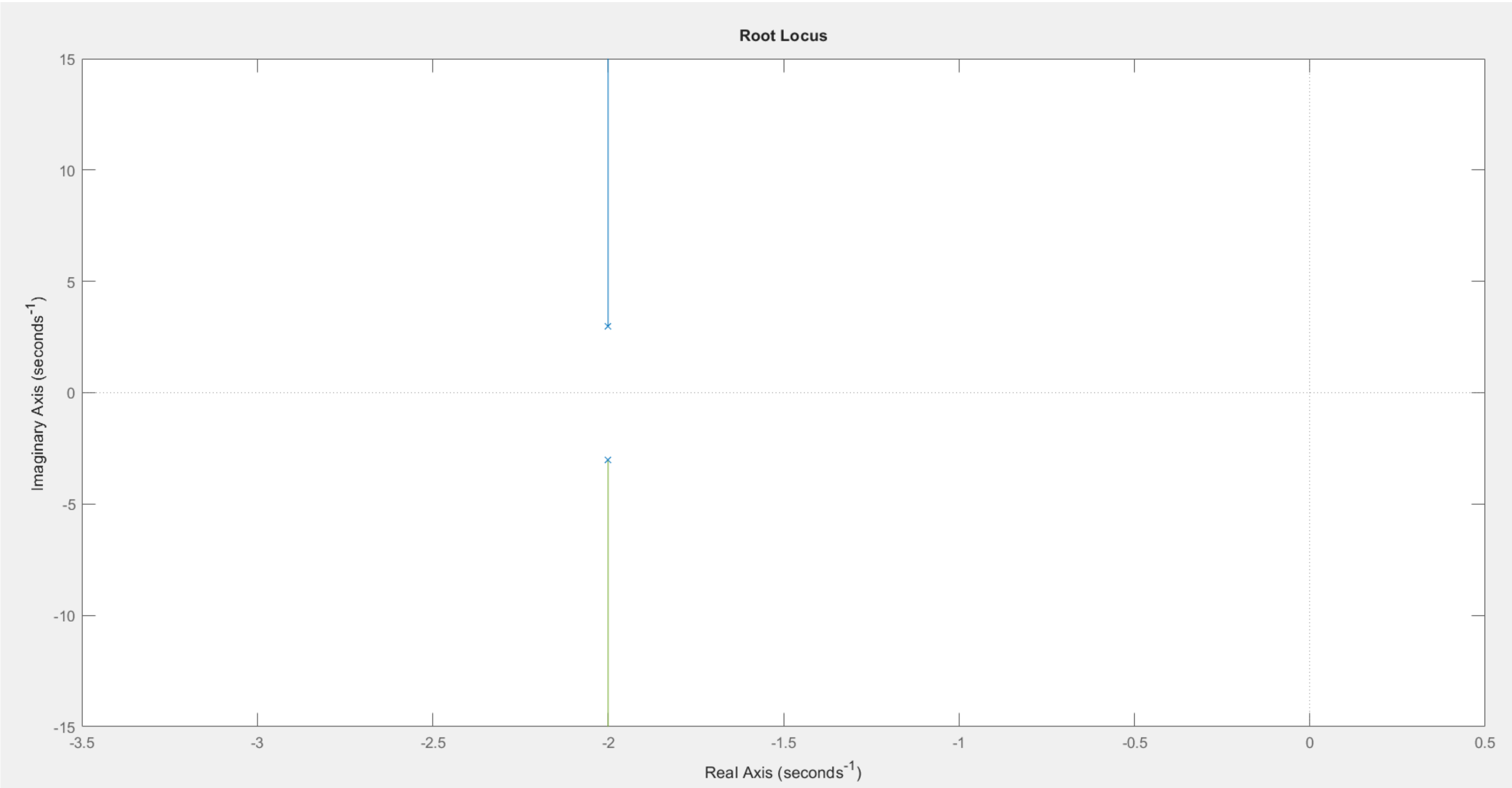
1. **Poles and Zeros**: Poles at $s = 0, -2$; Zero at $s = -4$.
2. **Real Axis Segments**: Locus exists between 0 and -2, and to the left of -4 (from -4 to $-\infty$).
3. **Breakaway/Break-in Points**:
    1. Characteristic Equation: $s^2 + 2s + K(s+4) = 0 \Rightarrow K = -\frac{s^2+2s}{s+4}$.
    2. $\frac{dK}{ds} = -\frac{(2s+2)(s+4) - (s^2+2s)(1)}{(s+4)^2} = 0.$
    3. Simplifying: $s^2 + 8s + 8 = 0$.
    4. **Breakaway**: $s \approx -1.17$ (between 0 and -2).
    5. **Break-in**: $s \approx -6.83$ (to the left of -4).
4. **Locus Shape**: The branches break away at -1.17, form a circular arc, and break back into the real axis at -6.83. One branch ends at the zero ($s = -4$) and the other goes to $-\infty$.


Root Locus
3
2
1
0
-1
-2
-3
Imaginary Axis (seconds^{-1})
-14
-12
-10
-8
-6
-4
-2
0
2
Real Axis (seconds^{-1})

Transfer Function: $G(s)H(s) = \dfrac{K}{s^2 + 4s + 13}$

1. **Poles**: $s = -2 \pm j3$ (Found via quadratic formula: $\dfrac{-4 \pm \sqrt{16 - 52}}{2}$).
2. **Real Axis Segment**: None (no real poles or zeros to provide an odd count on the axis).
3. **Asymptotes**:
   1. **Angles**: $\pm 90°$.
   2. **Centroid**: $\dfrac{(-2 + j3) + (-2 - j3)}{2} = -2$.

. **Locus Shape**: Two vertical lines starting from the complex poles and moving toward $\pm j\infty$ at $Re(s) = -2$.



Root Locus
15
10
5
0
-5
-10
-15
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})