



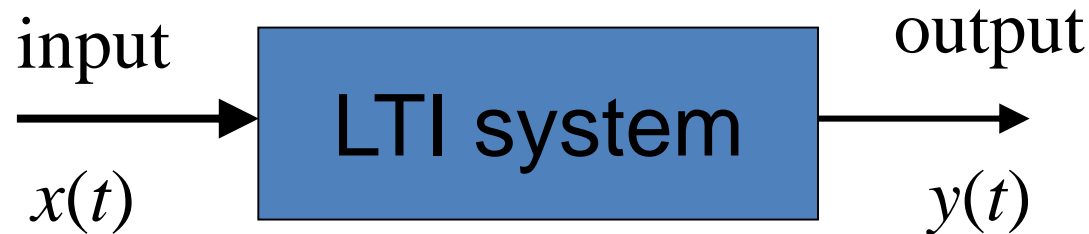
Industrial Control Systems

Chapter Four: Transfer Function, Stability and Block Diagram Modelling

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Transfer function



Definition: The transfer function of a linear time-invariant system is defined as the ratio of the Laplace transform of the output variable to the Laplace transform of the input variable when all **initial conditions are zero**.

$$G(s) = \frac{Y(s)}{X(s)}$$



Transfer function

Consider the linear time-invariant system described by the following differential equation:

$$\begin{aligned} a_0 \frac{d^n}{dt^n} y + a_1 \frac{d^{n-1}}{dt^{n-1}} y + \dots + a_{n-1} \frac{dy}{dt} + a_n y \\ = b_0 \frac{d^m}{dt^m} x + b_1 \frac{d^{m-1}}{dt^{m-1}} x + \dots + b_{m-1} \frac{dx}{dt} + b_m x, \quad n \geq m \end{aligned}$$

By definition, the transfer function is

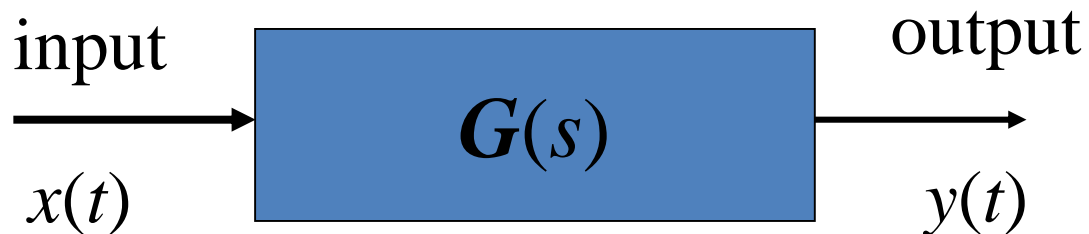
$$\frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n} := G(s)$$



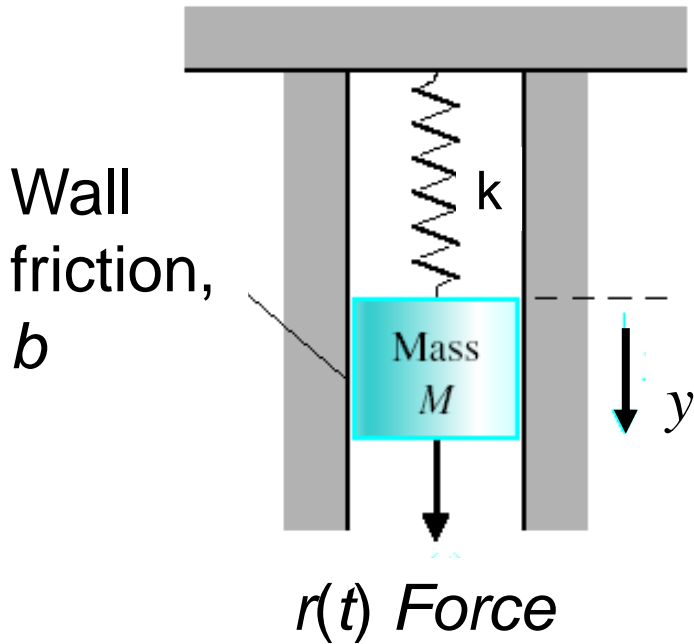
Transfer function

The advantage of transfer function: It represents system dynamics by algebraic equations and clearly shows the input-output relationship:

$$Y(s) = G(s)X(s)$$



Example. Spring-mass-damper system:



Let the input be the force $r(t)$ and the output be the displacement $y(t)$ of the mass. Find its transfer function.

Solution: The system differential equation is

$$M \frac{d^2 y(t)}{dt} + b \frac{dy(t)}{dt} + ky(t) = r(t)$$

From which we obtain its transfer function

$$\frac{Y(s)}{R(s)} = \frac{1}{Ms^2 + bs + k}$$



Transfer function

Transfer function helps us to check:

- The stability of the system.
- Time domain and frequency domain characteristics of the system.
- Response of the system for any given input.



Transfer function

Comments on transfer function:



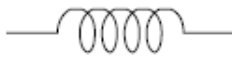
- Is limited to LTI systems.
- Is an operator to relate the output variable to the input variable of a linear differential equation.
- Is a property of a system itself, independent of the magnitude and nature of the input or driving function.
- Does not provide any information concerning the physical structure of the system. That is, the transfer functions of many physically different systems can be identical.

Transfer Function of Physical Systems (Electrical Systems)



Electrical Components

TABLE 2.3 Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
 Capacitor	$v(t) = \frac{1}{C} \int_0^1 i(\tau) d\tau$	$i(t) = C \frac{dv(t)}{dt}$	$v(t) = \frac{1}{C} q(t)$	$\frac{1}{Cs}$	Cs
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R} v(t)$	$v(t) = R \frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
 Inductor	$v(t) = L \frac{di(t)}{dt}$	$i(t) = \frac{1}{L} \int_0^1 v(\tau) d\tau$	$v(t) = L \frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ – V (volts), $i(t)$ – A (amps), $q(t)$ – Q (coulombs), C – F (farads), R – Ω (ohms), G – Ω (mhos), L – H (henries).



Transfer Function of RLC Circuit

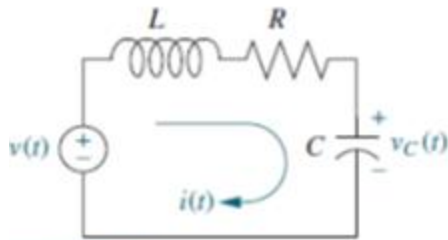


FIGURE 2.3 RLC network

PROBLEM: Find the transfer function relating the capacitor voltage, $V_C(s)$, to the input voltage, $V(s)$ in Figure 2.3.

Summing the voltages around the loop, assuming zero initial conditions, yields the integro-differential equation for this network as

$$L \frac{di(t)}{dt} + Ri(t) + \frac{1}{C} \int_0^t i(\tau) d\tau = v(t) \quad (2.61)$$

Changing variables from current to charge using $i(t) = dq(t)/dt$ yields

$$L \frac{d^2q(t)}{dt^2} + R \frac{dq(t)}{dt} + \frac{1}{C} q(t) = v(t) \quad (2.62)$$

From the voltage-charge relationship for a capacitor in Table 2.3,

$$q(t) = Cv_C(t) \quad (2.63)$$

Substituting Eq. (2.63) into Eq. (2.62) yields

$$LC \frac{d^2v_C(t)}{dt^2} + RC \frac{dv_C(t)}{dt} + v_C(t) = v(t) \quad (2.64)$$



Transfer Function of RLC Circuit

Taking the Laplace transform assuming zero initial conditions, rearranging terms, and simplifying yields

$$(LCs^2 + RCs + 1)V_C(s) = V(s) \quad (2.65)$$

Solving for the transfer function, $V_C(s)/V(s)$, we obtain

$$\frac{V_C(s)}{V(s)} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \quad (2.66)$$

as shown in Figure 2.4.

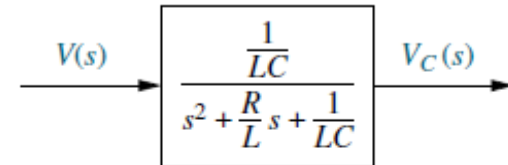


FIGURE 2.4 Block diagram of series *RLC* electrical network



Transfer Function of Electrical Circuits

For the capacitor,

$$V(s) = \frac{1}{Cs} I(s) \quad (2.67)$$

For the resistor,

$$V(s) = RI(s) \quad (2.68)$$

For the inductor,

$$V(s) = LsI(s) \quad (2.69)$$

Now define the following transfer function:

$$\frac{V(s)}{I(s)} = Z(s) \quad (2.70)$$



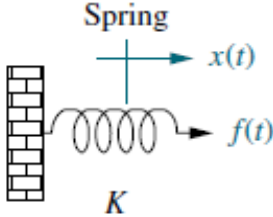
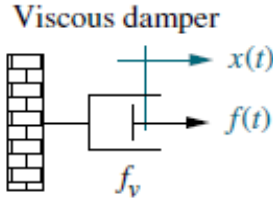
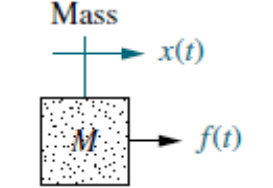
Part-I

TRANSFER FUNCTION OF TRANSLATIONAL MECHANICAL SYSTEMS



Transfer Function of Translational Mechanical Systems

TABLE 2.4 Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass

Component	Force-velocity	Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
 <p>Spring K</p>	$f(t) = K \int_0^t v(\tau) d\tau$	$f(t) = Kx(t)$	K
 <p>Viscous damper f_v</p>	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
 <p>Mass M</p>	$f(t) = M \frac{dv(t)}{dt}$	$f(t) = M \frac{d^2x(t)}{dt^2}$	Ms^2

Note: The following set of symbols and units is used throughout this book: $f(t) = \text{N}$ (newtons), $x(t) = \text{m}$ (meters), $v(t) = \text{m/s}$ (meters/second), $K = \text{N/m}$ (newtons/meter), $f_v = \text{N-s/m}$ (newton-seconds/meter), $M = \text{kg}$ (kilograms = newton-seconds²/meter).



Transfer Function of Translational Mechanical Systems

Transfer Function—One Equation of Motion

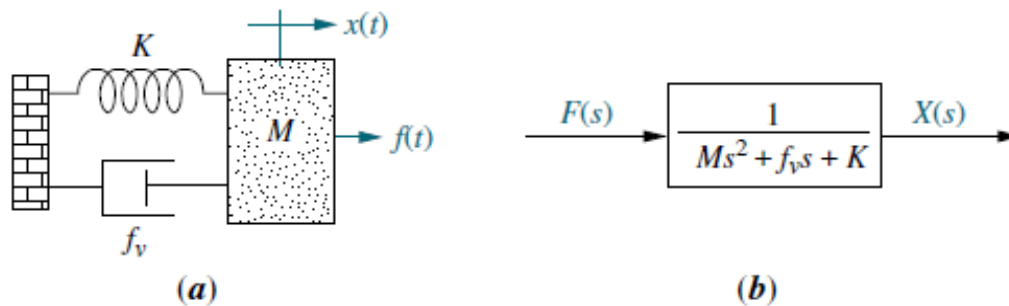


FIGURE 2.15 a. Mass, spring, and damper system; b. block diagram

PROBLEM: Find the transfer function, $X(s)/F(s)$, for the system of Figure 2.15(a).

$$M \frac{d^2x(t)}{dt^2} + f_v \frac{dx(t)}{dt} + Kx(t) = f(t) \quad (2.108)$$

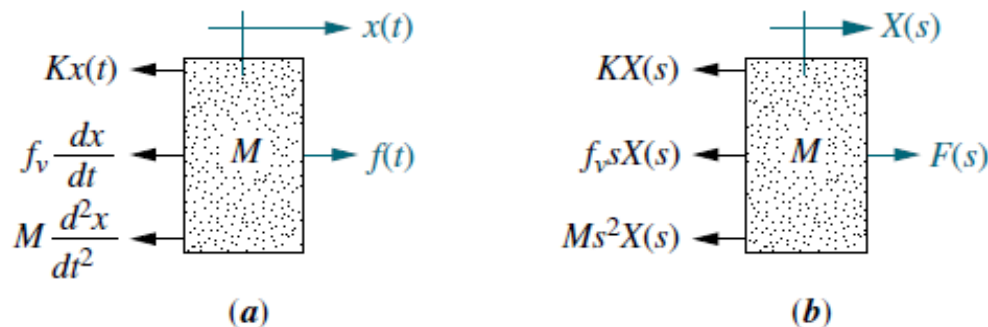


FIGURE 2.16 a. Free-body diagram of mass, spring, and damper system; b. transformed free-body diagram



Transfer Function of Translational Mechanical Systems

Taking the Laplace transform, assuming zero initial conditions,

$$Ms^2X(s) + f_v sX(s) + KX(s) = F(s) \quad (2.109)$$

or

$$(Ms^2 + f_v s + K)X(s) = F(s) \quad (2.110)$$

Solving for the transfer function yields

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + f_v s + K} \quad (2.111)$$

which is represented in Figure 2.15(b).



Transfer Function of Translational Mechanical Systems

we obtain for the spring,

$$F(s) = KX(s) \quad (2.112)$$

for the viscous damper,

$$F(s) = f_v s X(s) \quad (2.113)$$

and for the mass,

$$F(s) = Ms^2 X(s) \quad (2.114)$$

If we define impedance for mechanical components as

$$Z_M(s) = \frac{F(s)}{X(s)} \quad (2.115)$$



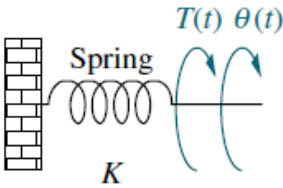
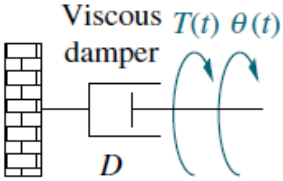
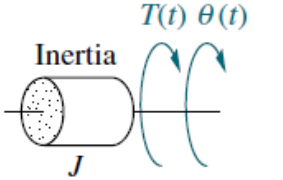
Part-II

TRANSFER FUNCTION OF ROTATIONAL MECHANICAL SYSTEMS



Transfer Function of Rotational Mechanical Systems

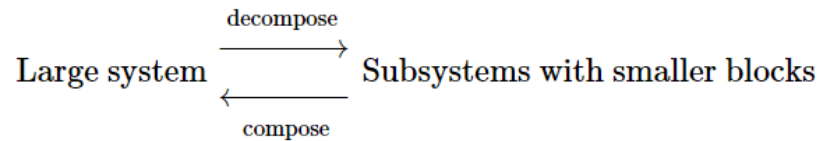
TABLE 2.5 Torque-angular velocity, torque-angular displacement, and impedance rotational relationships for springs, viscous dampers, and inertia

Component	Torque-angular velocity	Torque-angular displacement	Impedance $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K \int_0^t \omega(\tau) d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D \frac{d\theta(t)}{dt}$	Ds
	$T(t) = J \frac{d\omega(t)}{dt}$	$T(t) = J \frac{d^2\theta(t)}{dt^2}$	Js^2

Note: The following set of symbols and units is used throughout this book: $T(t)$ – N-m (newton-meters), $\theta(t)$ – rad(radians), $\omega(t)$ – rad/s(radians/second), K – N-m/rad(newton- meters/radian), D – N-m-s/rad (newton- meters-seconds/radian). J – kg-m²(kilograms-meters² – newton-meters-seconds²/radian).

System Modelling Diagrams

In this lecture, we will introduce **block diagram** with which we can visualize the algebra represented by differential equations for a given system, i.e., we can easily represent and analyze this system by means of block diagrams.



Usually, a system will be composed of subsystems with smaller blocks and these smaller blocks come from some given *library*. They are used as building blocks for more complicated systems. (Think of Lego bricks.)

Building Blocks

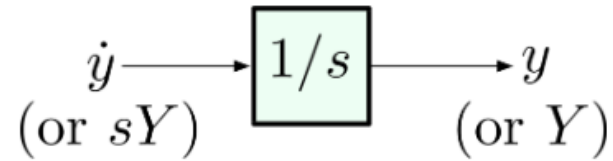


Figure 1: Integrator

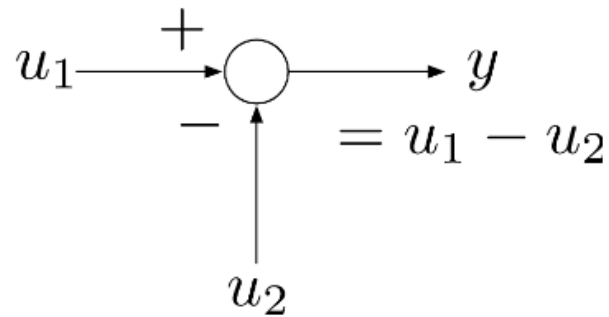


Figure 2: Summer

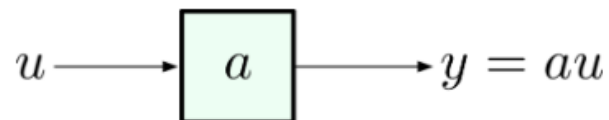


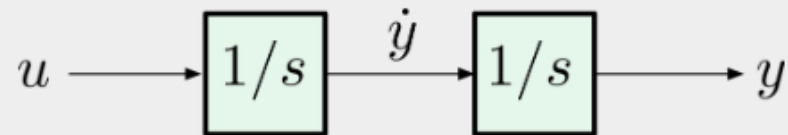
Figure 3: Gain

Building Blocks

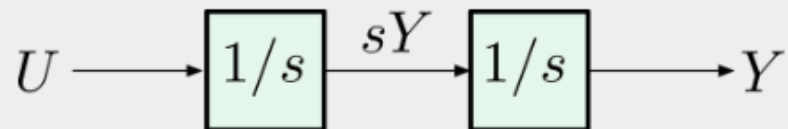
Example 1: Draw an all-integrator diagram for system dynamics

$$\ddot{y}(t) = u(t) \text{ or equivalently } s^2 Y(s) = U(s).$$

Solution: Recall the “chain” method we talked about before, leave the system output on the right and trace back based on the **degree** of the highest order term of the differential equation. In this case it is **2**, we need **two** integrators.



or equivalently in s -domain,



Building Blocks

Example 2: By introducing two extra $\dot{y}(t)$ and $y(t)$ terms to the left hand side of system dynamics of Example 1, we have a new system dynamics

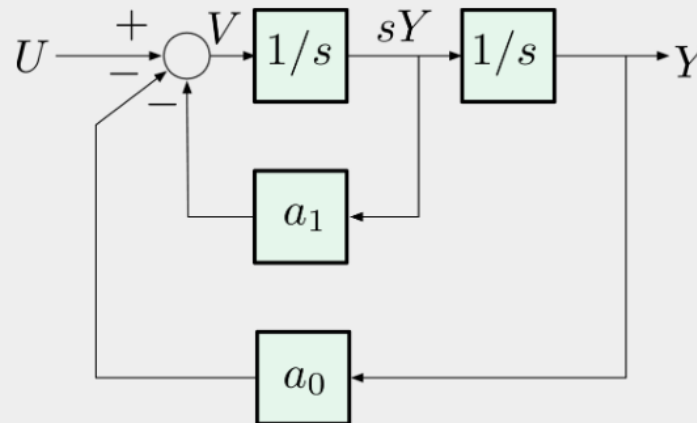
$$\ddot{y}(t) + a_1\dot{y}(t) + a_0y(t) = u(t) \iff s^2Y(s) + a_1sY(s) + a_0Y(s) = U(s),$$

or equivalently $Y(s) = \frac{1}{s^2 + a_1s + a_0}U(s).$

Draw an all-integrator diagram for the new system.

Solution: Keep the highest derivative on one side and everything else on the other,

$$\ddot{y} = \underbrace{-a_1\dot{y} - a_0y + u}_{=v}$$



Compare the above new diagram with Example 1, the chain of integrators stays the same but we included two **feedback** loops and one **summing junction** because of the two extra terms we introduced.

Series and Parallel Structure



Figure 10: Series connection

$$\frac{Y(s)}{U(s)} = G_1(s)G_2(s).$$

Series and Parallel Structure

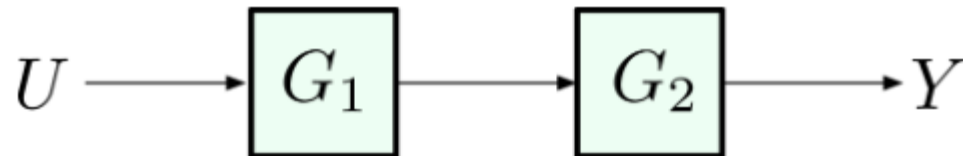


Figure 10: Series connection

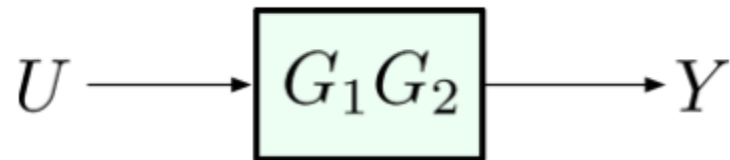


Figure 11: Series connection (reduced)

$$\frac{Y(s)}{U(s)} = G_1(s)G_2(s).$$

Series and Parallel Structure

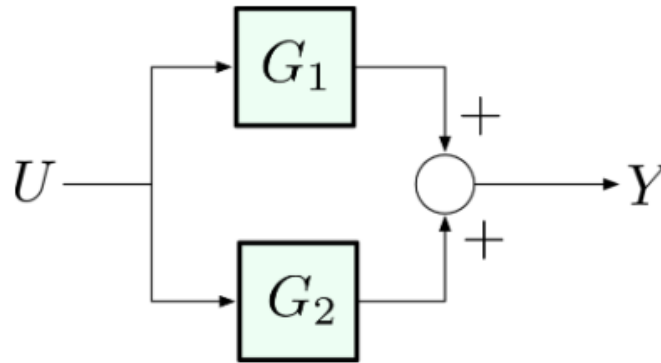


Figure 12: Parallel connection

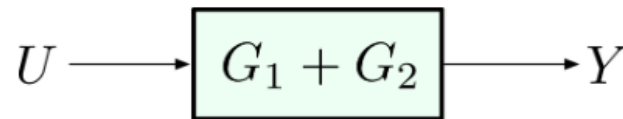


Figure 13: Parallel connection (reduced)

system output $Y(s) = G_1(s)U(s) + G_2(s)U(s)$, i.e., the sum of two branches due to input $U(s)$. We have the transfer function

$$\frac{Y(s)}{U(s)} = G_1(s) + G_2(s).$$

Negative Feedback and Unity Feedback

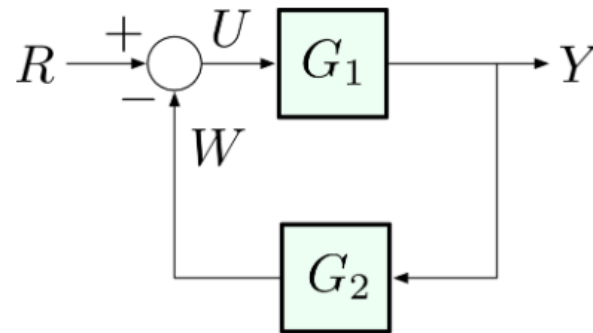


Figure 14: Negative feedback

First we can compute the system transfer function from **reference** $R(s)$ to output $Y(s)$,

$$\begin{aligned} U &= R - W, \\ Y &= G_1 U \\ &= G_1(R - W) \\ (2) \quad &= G_1 R - G_1 G_2 Y. \end{aligned}$$

Solving for $Y(s)$ from Equation (2),

$$Y(s) = \frac{G_1(s)}{1 + G_1(s)G_2(s)} R(s).$$

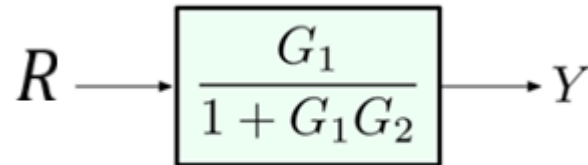
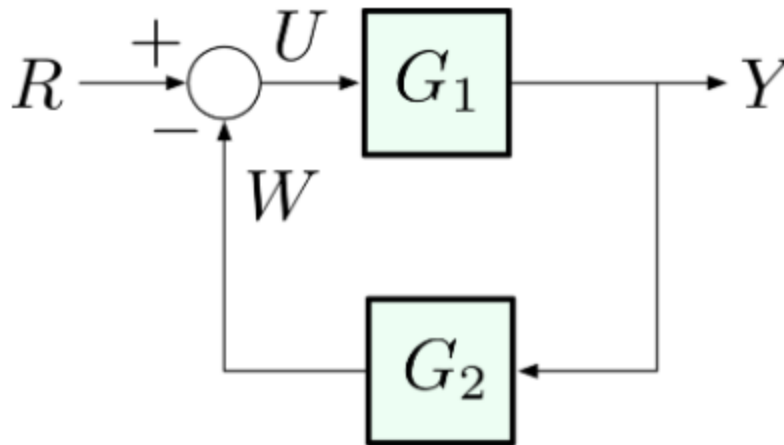


Figure 15: Negative feedback (reduced)

Negative Feedback and Unity Feedback

It reads in natural language

$$\text{negative feedback loop gain} = \frac{\text{forward gain}}{1 + \text{loop gain}}.$$



Negative Feedback and Unity Feedback

One special case of negative feedback is when $G_2(s) = 1$ or rather we move $G_2(s)$ block from feedback path to forward path.

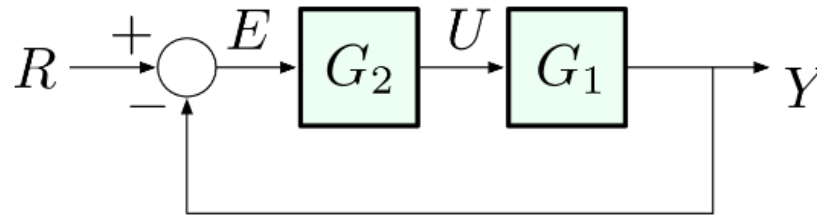


Figure 16: Unity feedback

This is called *unity feedback* –there is no component on the feedback path or feedback path is trivial 1.

- R = Reference
- U = Control input
- Y = Output
- E = Error
- G_1 = Plant (also denoted by P)
- G_2 = Controller or compensator (also denoted by C or K)

Negative Feedback and Unity Feedback

Derivation of the following three very important transfer functions will be left as an exercise. (Apply formula we derived for gain of negative feedback loop.)

- Reference R to output Y ,

$$\frac{Y}{R} = \frac{G_1 G_2}{1 + G_1 G_2}.$$

- Reference R to control input U ,

$$\frac{U}{R} = \frac{G_2}{1 + G_1 G_2}.$$

- Error E to output Y ,

$$\frac{Y}{E} = G_1 G_2. \quad (\text{no feedback path})$$

Block Diagram Reduction & Transformation

Now with the already discussed *series*, *parallel*, and *feedback* interconnections at our disposal, given a complicated diagram made up of some combination of those blocks, we can possibly write down an overall transfer function from one of the variables to another.

In general,

- Name all the variables in the diagram;
- Write down as many relationships between these variables as we can;
- Learn to recognize series, parallel, and feedback interconnections;
- Replace them by their equivalents;
- Repeat.

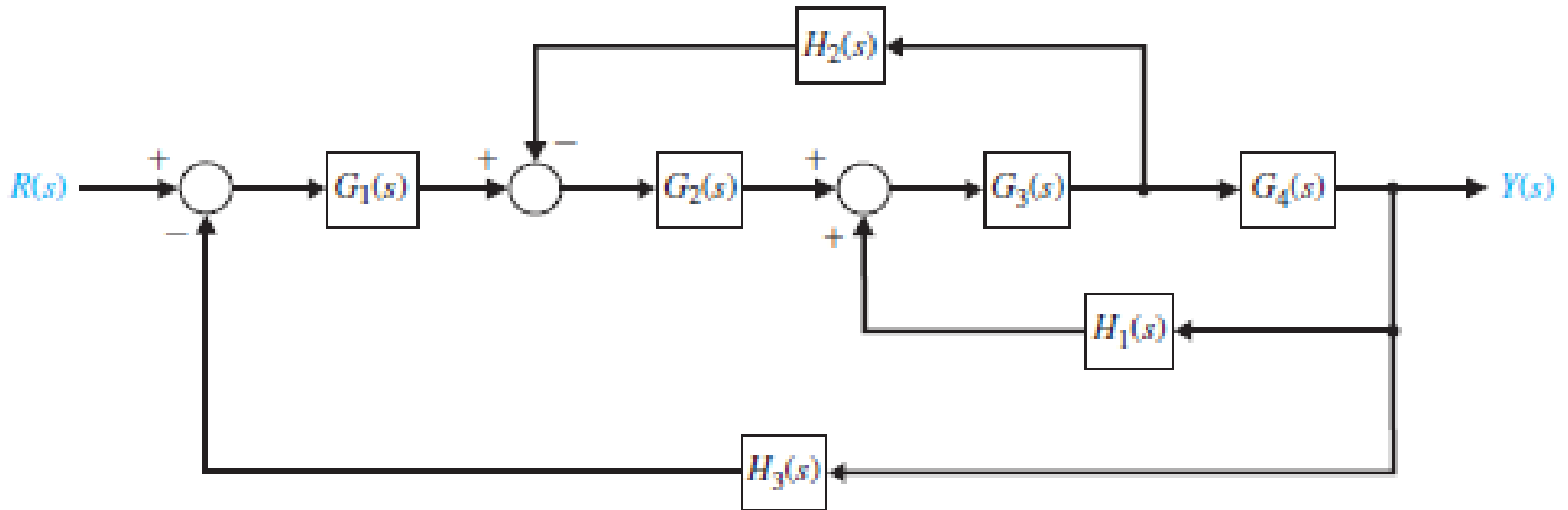
Block Diagram Reduction & Transformation

Table 2.5 Block Diagram Transformations

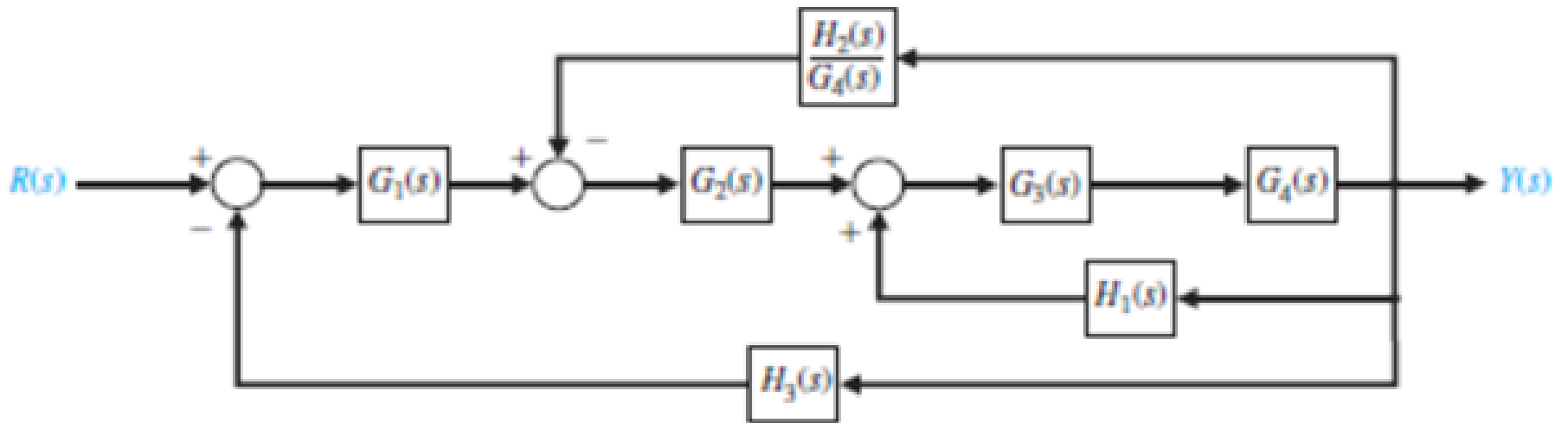
Transformation	Original Diagram	Equivalent Diagram
1. Combining blocks in cascade		<p>or</p>
2. Moving a summing point behind a block		
3. Moving a pickoff point ahead of a block		
4. Moving a pickoff point behind a block		
5. Moving a summing point ahead of a block		
6. Eliminating a feedback loop		

Block Diagram Reduction & Transformation

Reduce the following block diagram to find $Y(S) / R(S)$:

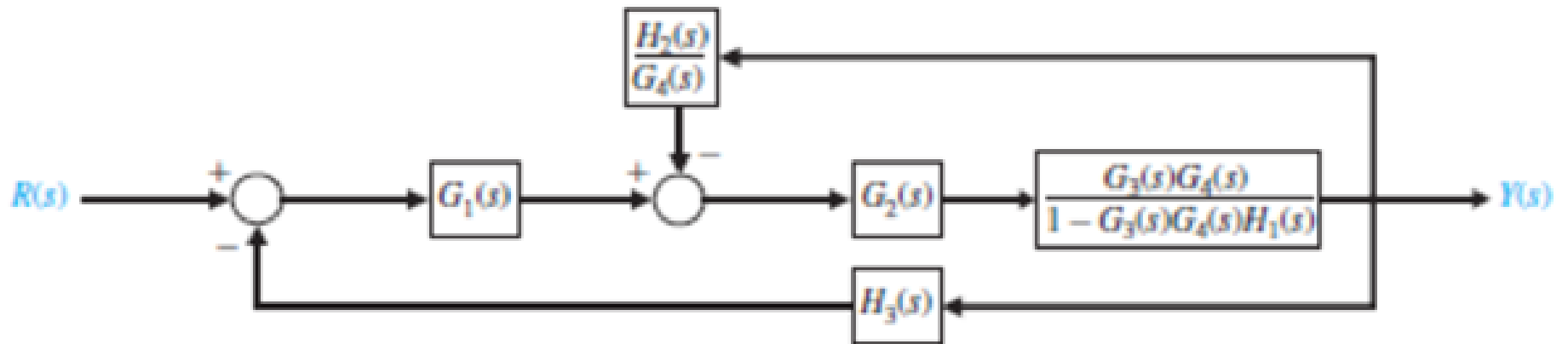


Block Diagram Reduction & Transformation



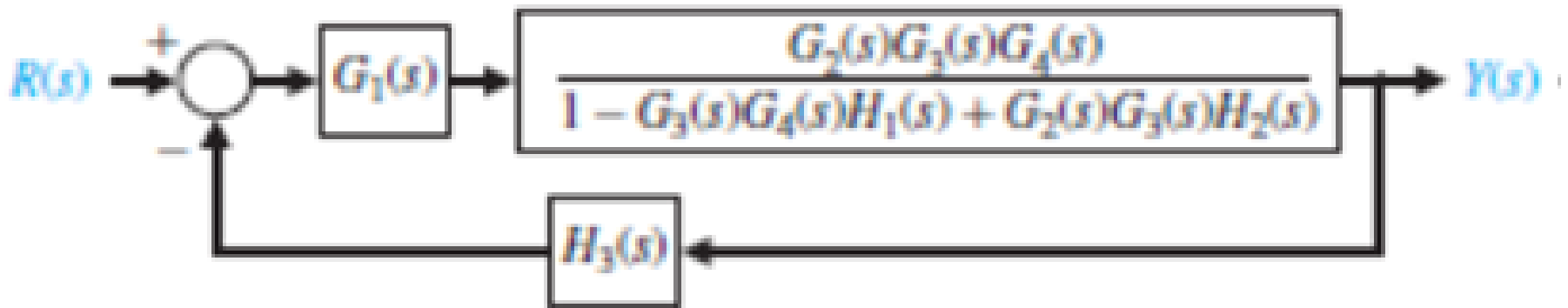
(a)

Block Diagram Reduction & Transformation



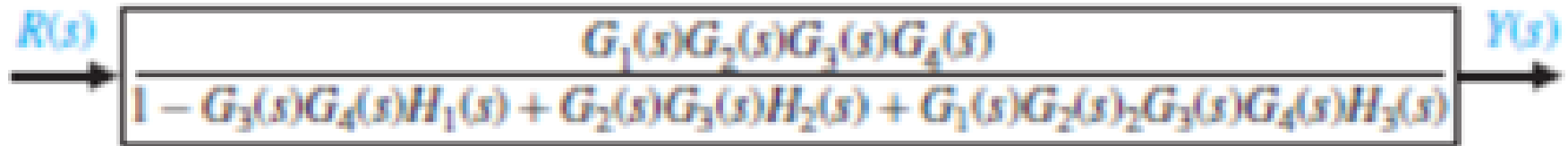
(b)

Block Diagram Reduction & Transformation



(c)

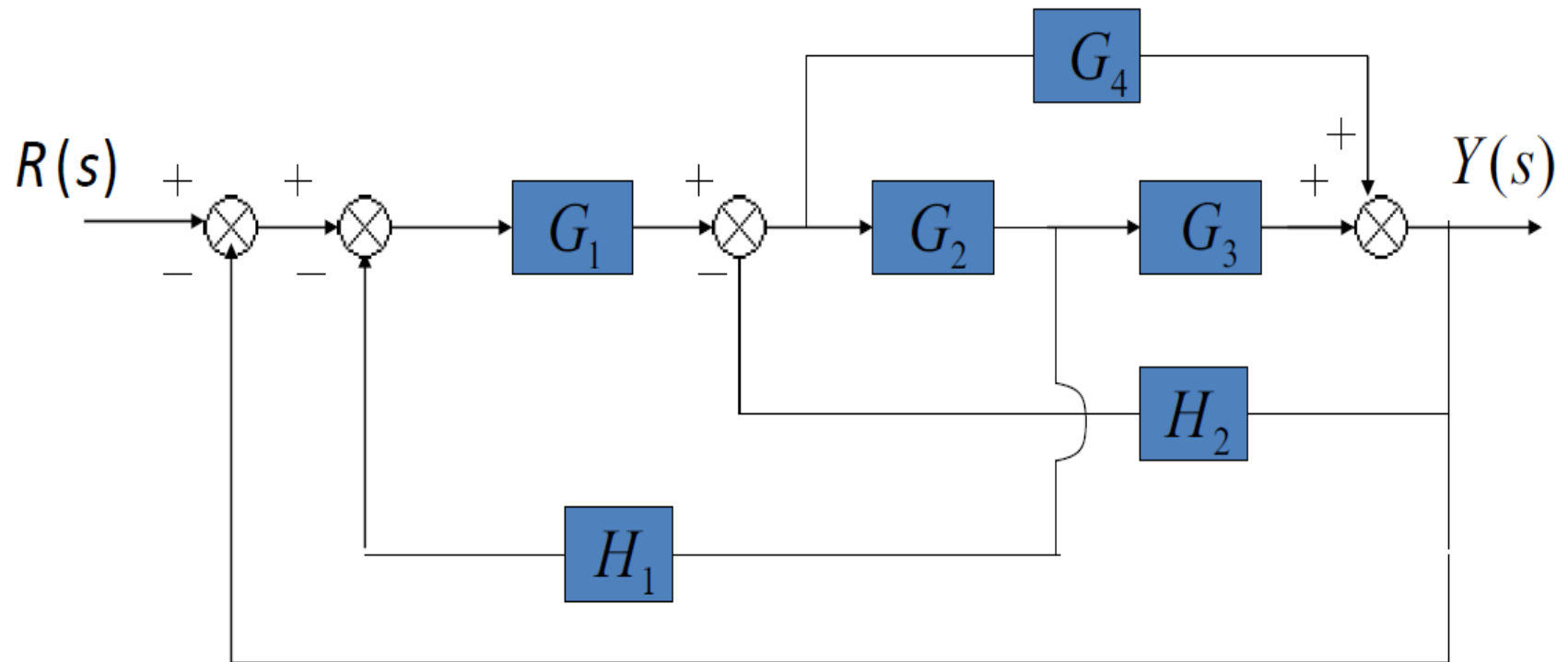
Block Diagram Reduction & Transformation



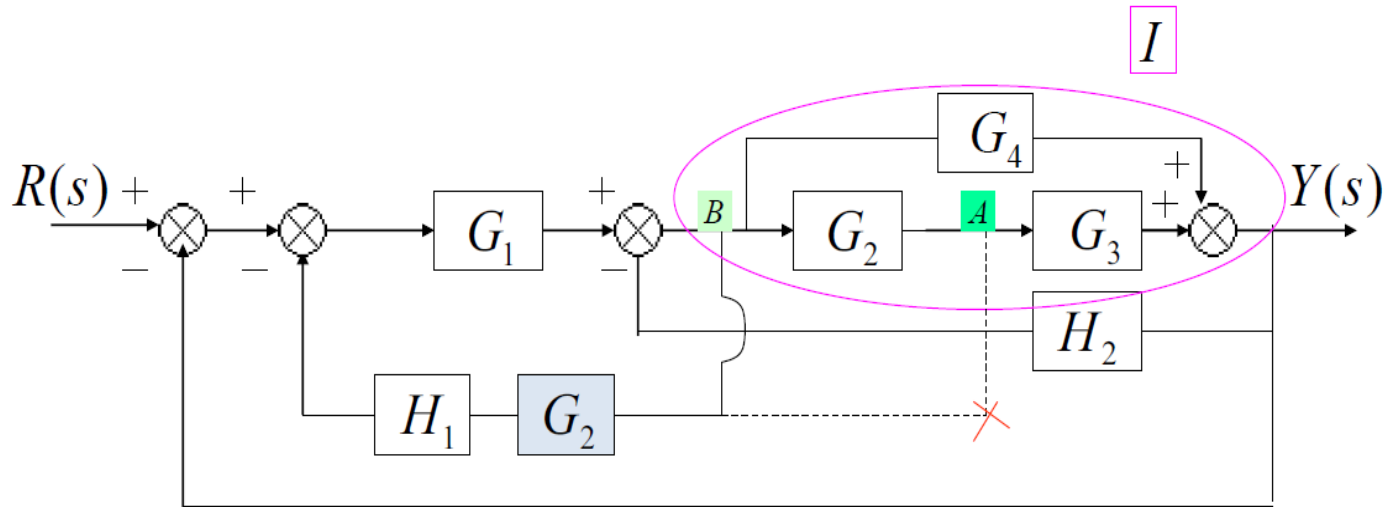
(d)

Block Diagram Reduction & Transformation

Reduce the following block diagram to find $Y(S) / R(S)$:



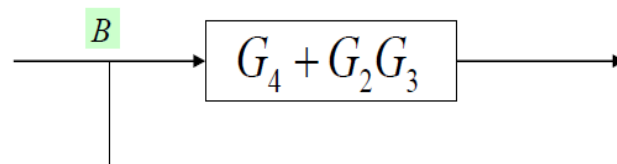
Block Diagram Reduction & Transformation



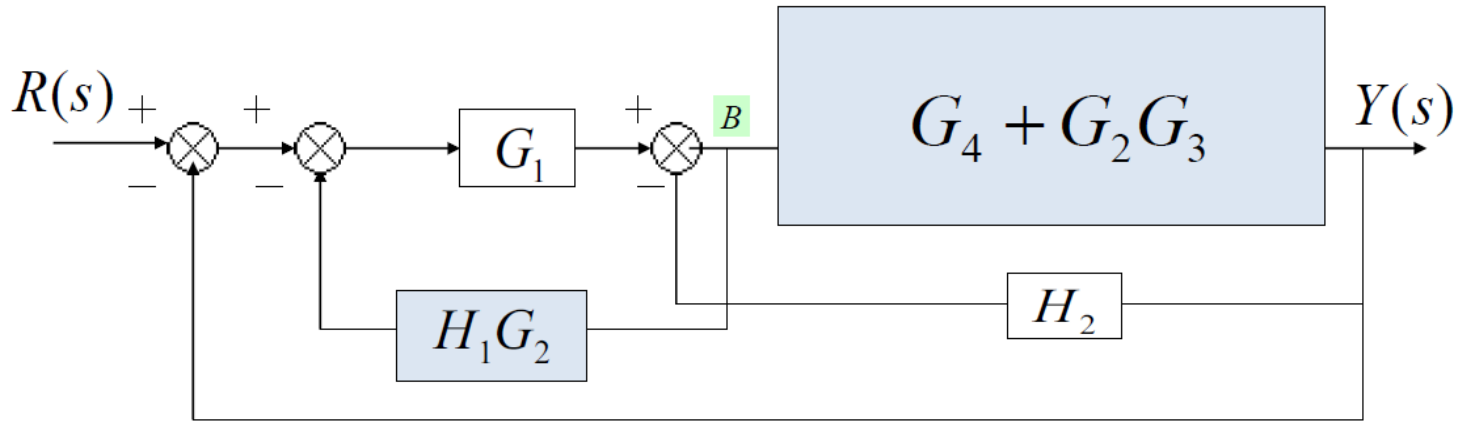
Solution:

1. Moving pickoff point A ahead of block G_2

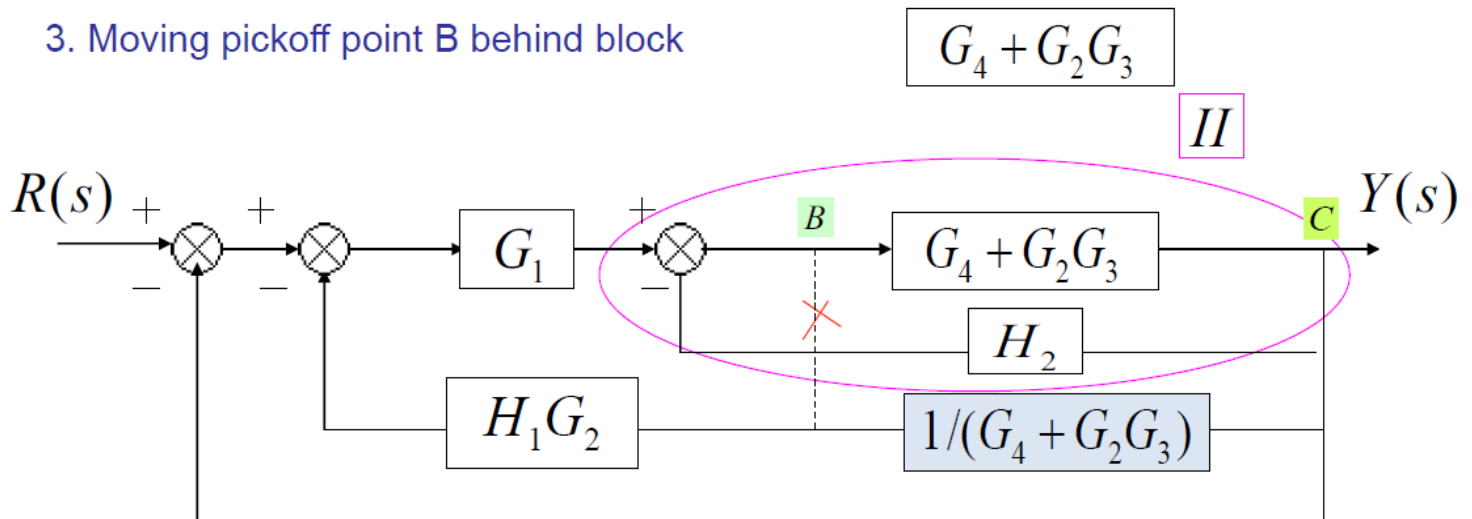
2. Eliminate loop I & simplify



Block Diagram Reduction & Transformation

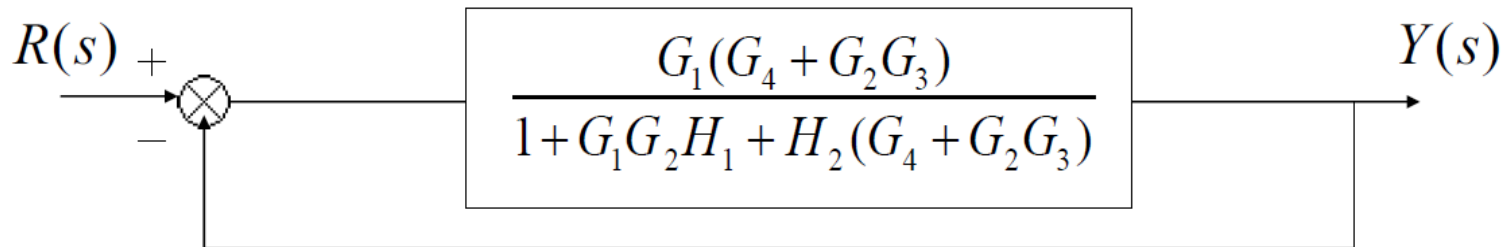
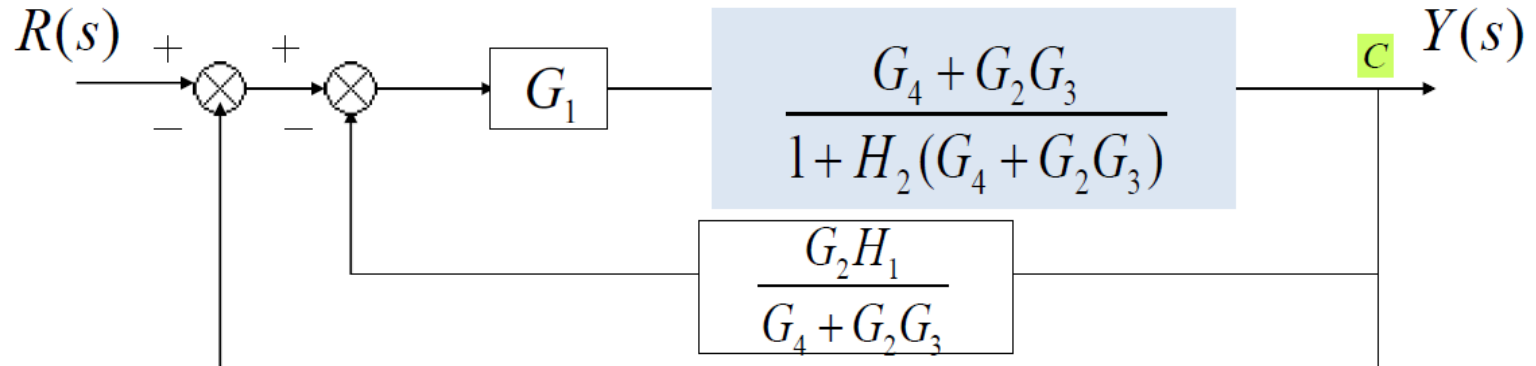


3. Moving pickoff point B behind block



Block Diagram Reduction & Transformation

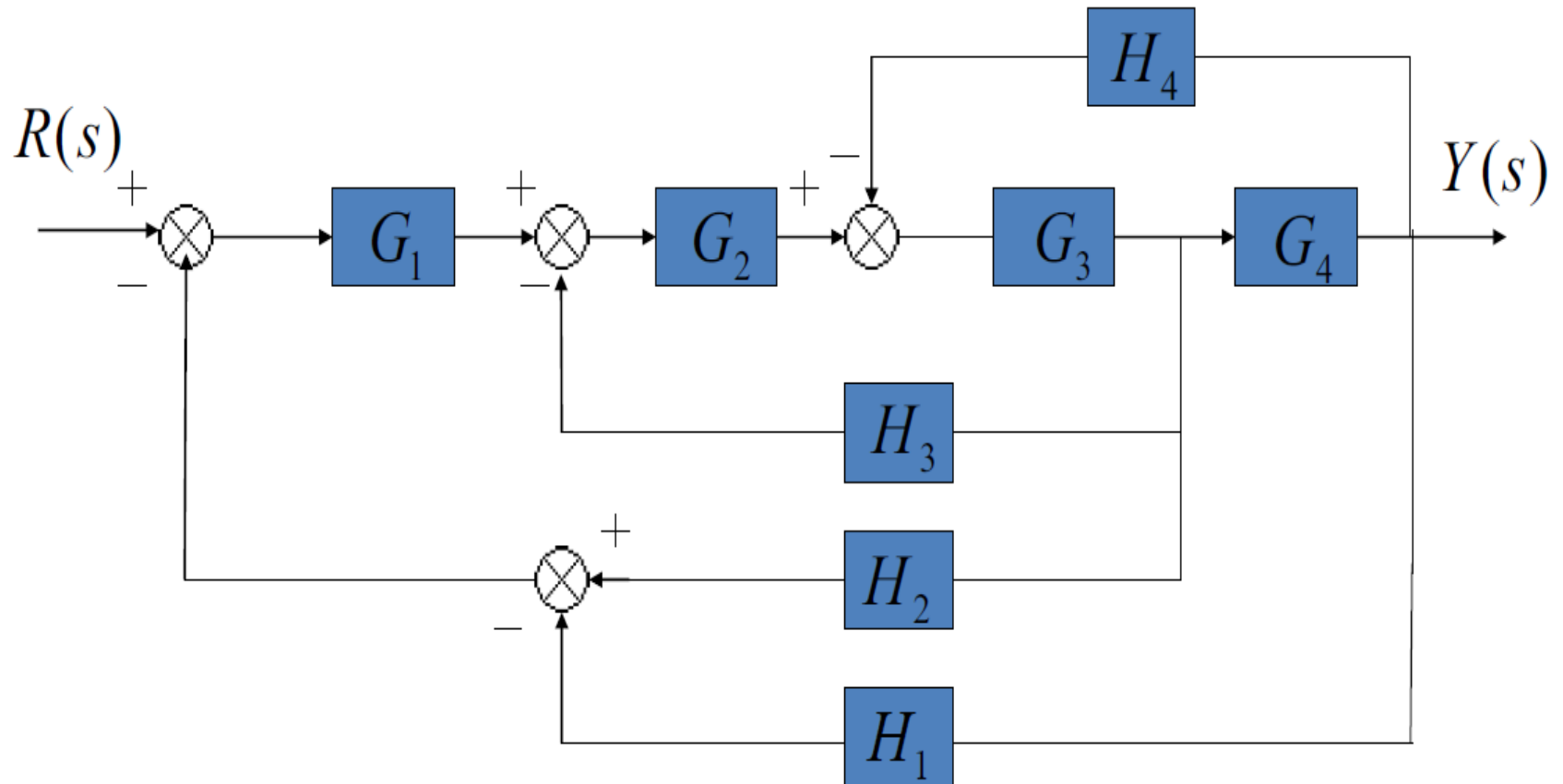
4. Eliminate loop III



$$T(s) = \frac{Y(s)}{R(s)} = \frac{G_1(G_4 + G_2G_3)}{1 + G_1G_2H_1 + H_2(G_4 + G_2G_3) + G_1(G_4 + G_2G_3)}$$

Block Diagram Reduction & Transformation

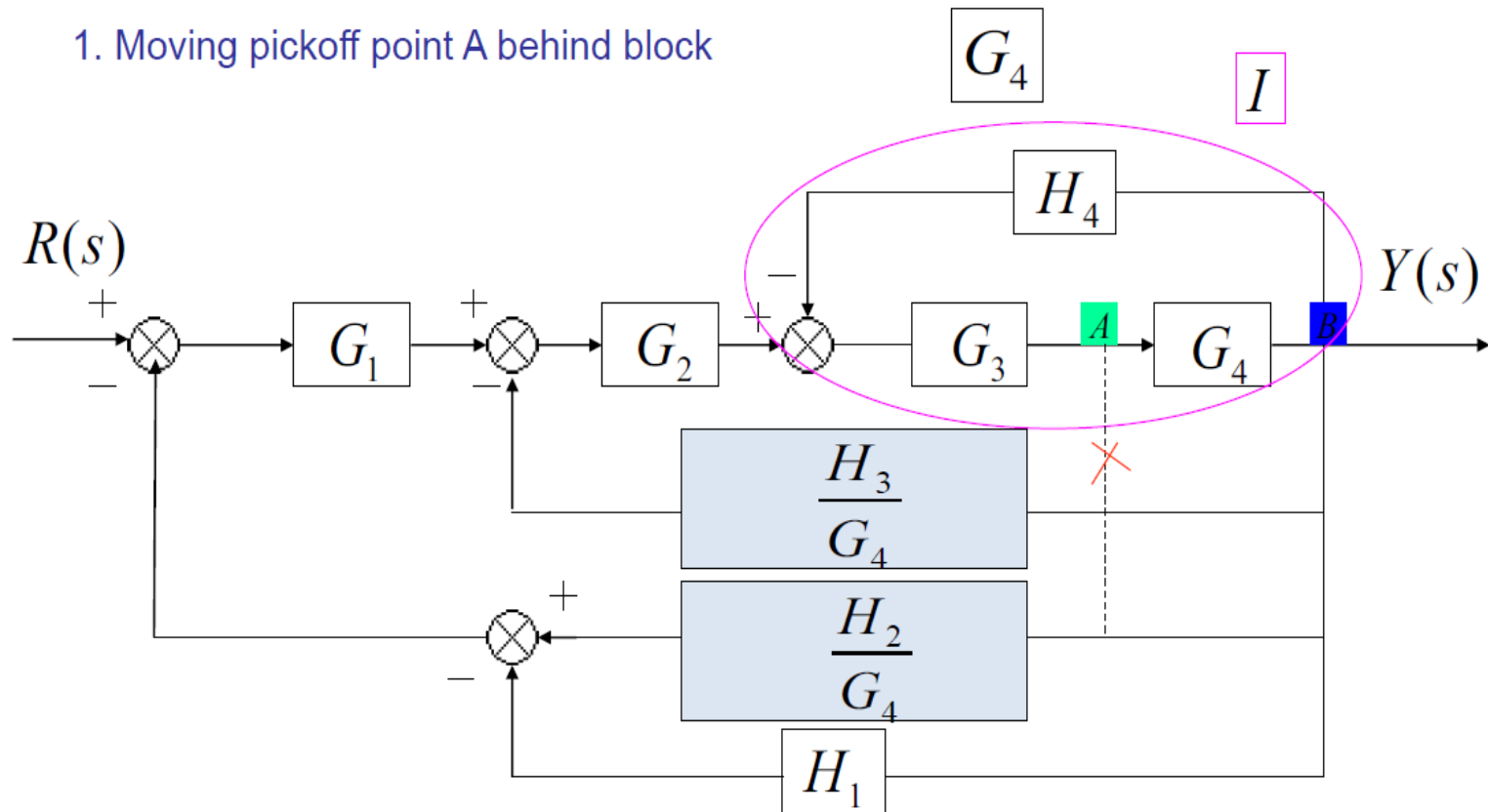
Reduce the following block diagram to find $Y(S) / R(S)$:



Block Diagram Reduction & Transformation

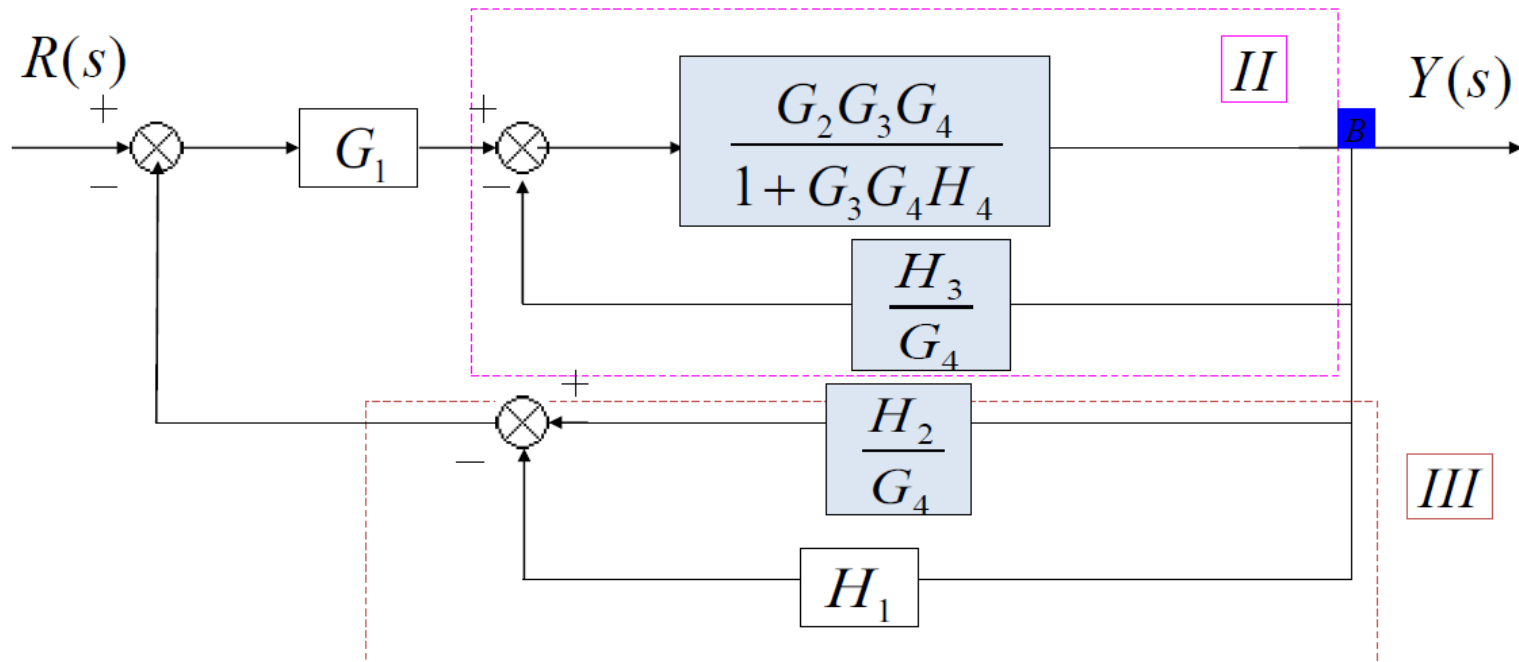
Solution:


1. Moving pickoff point A behind block




Block Diagram Reduction & Transformation

2. Eliminate loop I and Simplify



II  feedback

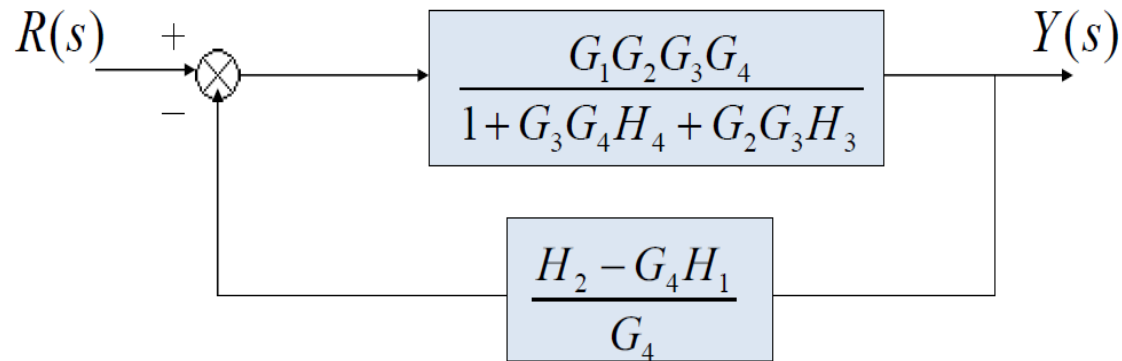
$$\frac{G_2 G_3 G_4}{1 + G_3 G_4 H_4 + G_2 G_3 H_3}$$

III  Not feedback

$$\frac{H_2 - G_4 H_1}{G_4}$$

Block Diagram Reduction & Transformation

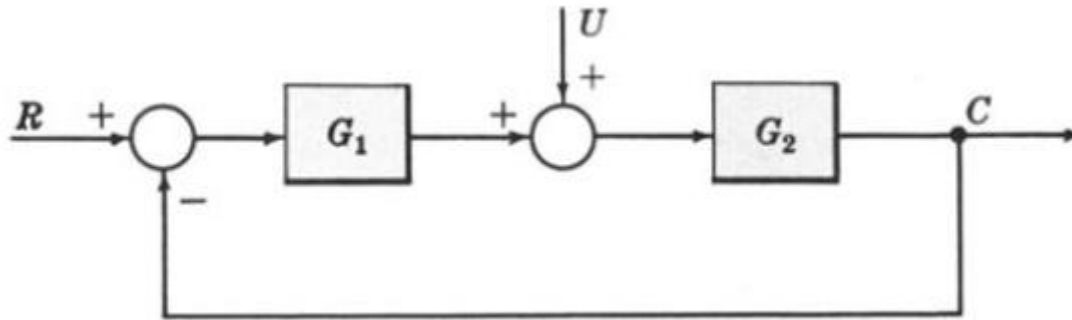
3. Eliminate loop II & III



$$T(s) = \frac{Y(s)}{R(s)} = \frac{G_1 G_2 G_3 G_4}{1 + G_2 G_3 H_3 + G_3 G_4 H_4 + G_1 G_2 G_3 H_2 - G_1 G_2 G_3 G_4 H_1}$$

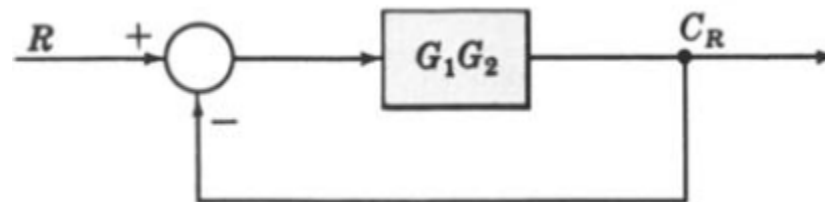
Multiple Input System

Determine the output C due to inputs R and U using the Superposition Method.



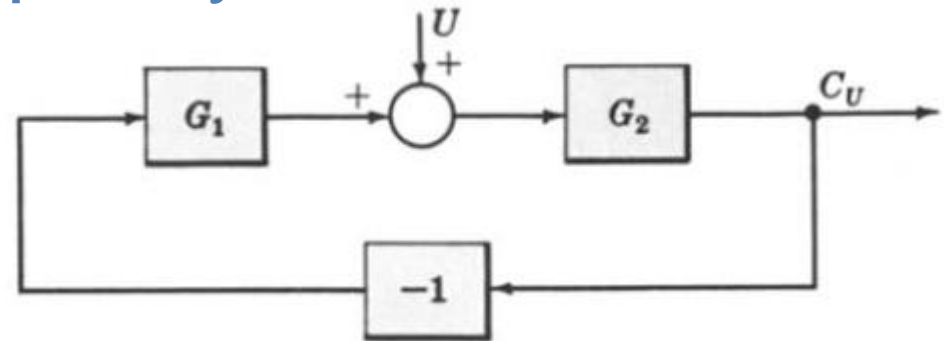
Step 1: Put $U \equiv 0$.

Step 2: The system reduces to



Step 3: the output C_R due to input R is $C_R = [G_1G_2/(1 + G_1G_2)]R$.

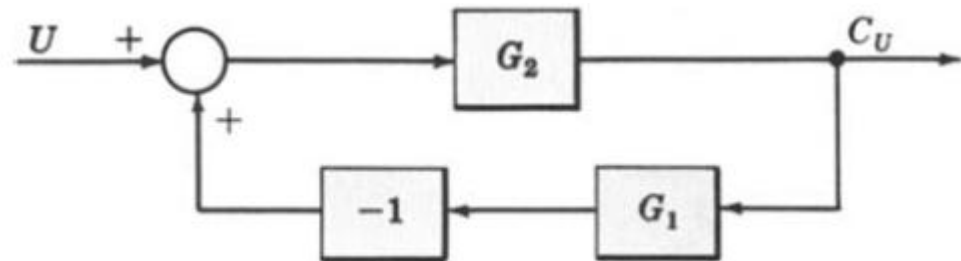
Multiple Input System



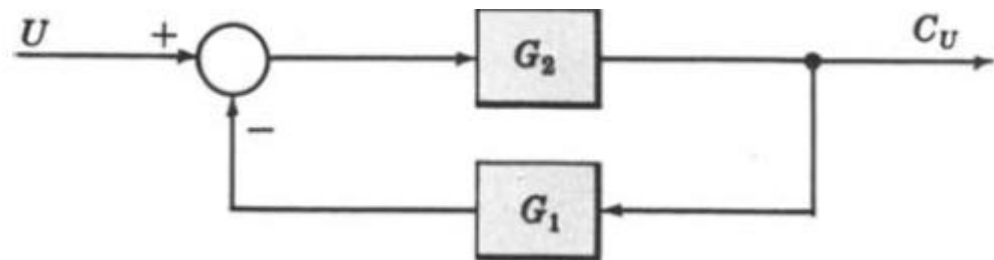
Step 4a: Put $R = 0$.

Step 4b: Put -1 into a block, representing the negative feedback effect:

Rearrange the block diagram:



Let the -1 block be absorbed into the summing point:



Step 4c: the output C_U due to input U is $C_U = [G_2/(1 + G_1G_2)]U$.

Multiple Input System.

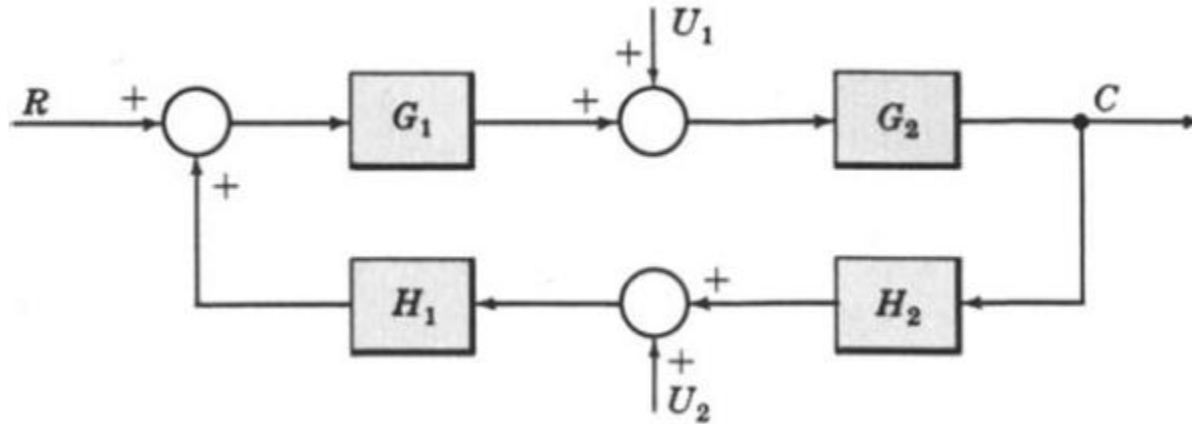
Step 5: The total output is $C = C_R + C_U$

$$= \left[\frac{G_1 G_2}{1 + G_1 G_2} \right] R + \left[\frac{G_2}{1 + G_1 G_2} \right] U$$

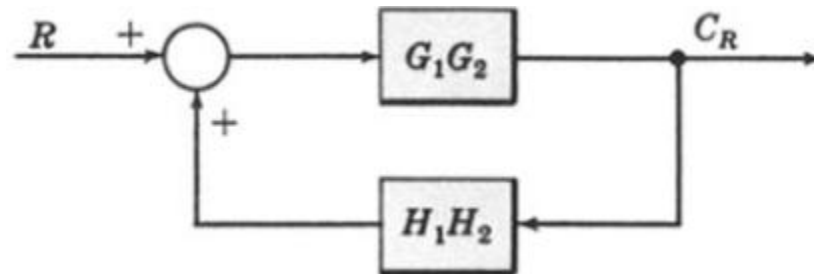
$$= \left[\frac{G_2}{1 + G_1 G_2} \right] [G_1 R + U]$$

Multiple Input System

Determine the output C due to inputs R , U_1 , and U_2 using the Superposition Method.



Let $U_1 = U_2 = 0$.

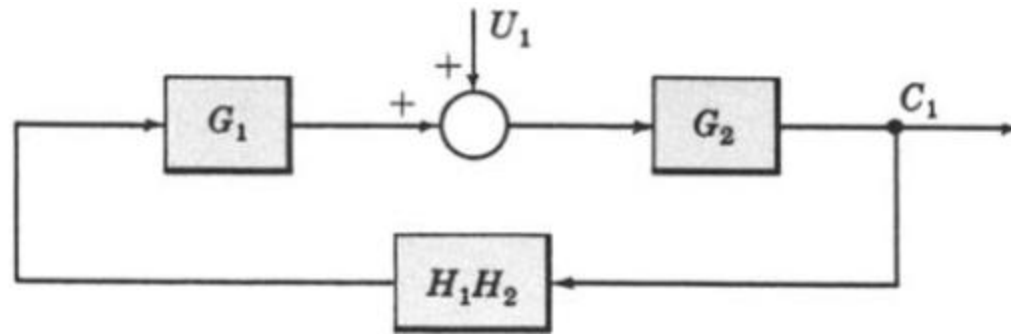


$$C_R = [G_1G_2/(1 - G_1G_2H_1H_2)]R$$

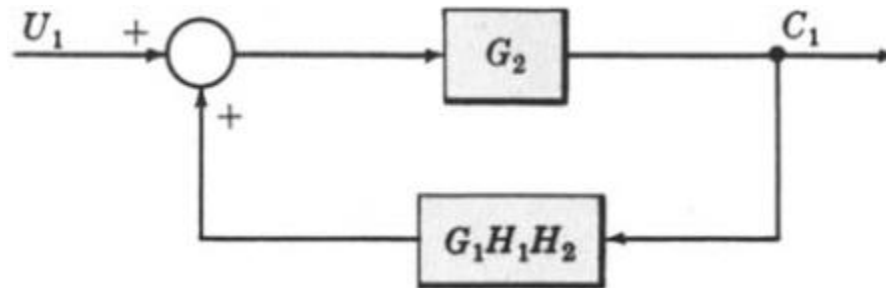
where C_R is the output due to R acting alone.

Multiple Input System

Now let $R = U_2 = 0$.



Rearranging the blocks, we get

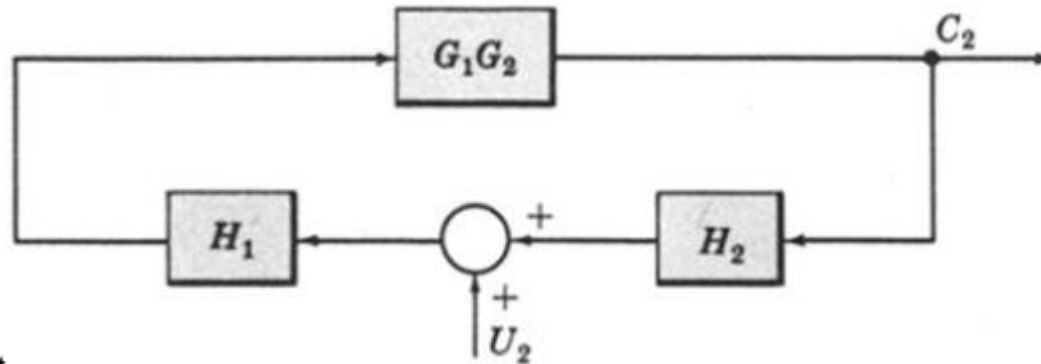


$$C_1 = [G_2 / (1 - G_1G_2H_1H_2)]U_1$$

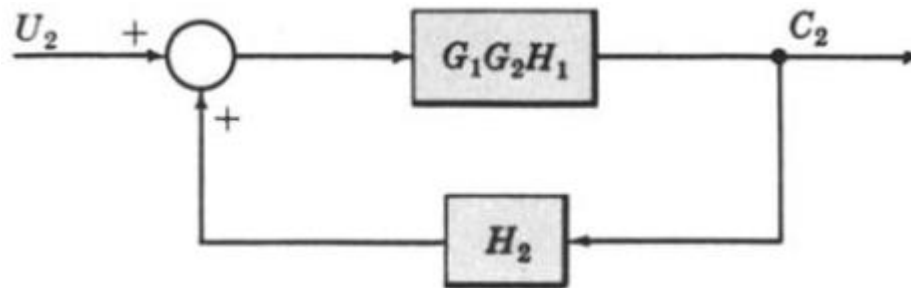
where C_1 is the response due to U_1 acting alone.

Multiple Input System

Finally, let $R = U_1 = 0$.



Rearranging the blocks, we get



$$C_2 = [G_1G_2H_1/(1 - G_1G_2H_1H_2)]U_2$$

where C_2 is the response due to U_2 acting alone.

By superposition, the total output is

$$C = C_R + C_1 + C_2 = \frac{G_1G_2R + G_2U_1 + G_1G_2H_1U_2}{1 - G_1G_2H_1H_2}$$

Stability of Control System

$$\frac{Y(s)}{X(s)} = \frac{b_0s^m + b_1s^{m-1} + \dots + b_{m-1}s + b_m}{a_0s^n + a_1s^{n-1} + \dots + a_{n-1}s + a_n}$$

- Roots of the denominator polynomial of a transfer function are called 'poles'.
- And the roots of numerator polynomials of a transfer function are called 'zeros'.

Stability of Control System

- Poles of the system are represented by 'x' and zeros of the system are represented by 'o'.
- System order is always equal to number of poles of the transfer function.
- Following transfer function represents n^{th} order plant.

$$\frac{Y(s)}{X(s)} = \frac{b_0s^m + b_1s^{m-1} + \dots + b_{m-1}s + b_m}{a_0s^n + a_1s^{n-1} + \dots + a_{n-1}s + a_n}$$

Stability of Control System

- Poles are also defined as “it is the frequency at which the *system becomes infinite*”. Hence the name pole, where the field is infinite.

$$\frac{Y(s)}{X(s)} = \frac{b_0s^m + b_1s^{m-1} + \dots + b_{m-1}s + b_m}{a_0s^n + a_1s^{n-1} + \dots + a_{n-1}s + a_n}$$

- And zero “*is the frequency at which the system becomes 0.*”

Stability of Control System

- Consider the Transfer function calculated in previous slides.

$$G(s) = \frac{X(s)}{Y(s)} = \frac{C}{As + B}$$

the denominator polynomial is $As + B = 0$

- The only pole of the system is

$$s = -\frac{B}{A}$$

Stability of Control System

- Consider the following transfer functions.
 - Determine
 - Whether the transfer function is proper or improper
 - Poles of the system
 - zeros of the system
 - Order of the system

$$\text{i) } G(s) = \frac{s + 3}{s(s + 2)}$$

$$\text{ii) } G(s) = \frac{s}{(s + 1)(s + 2)(s + 3)}$$

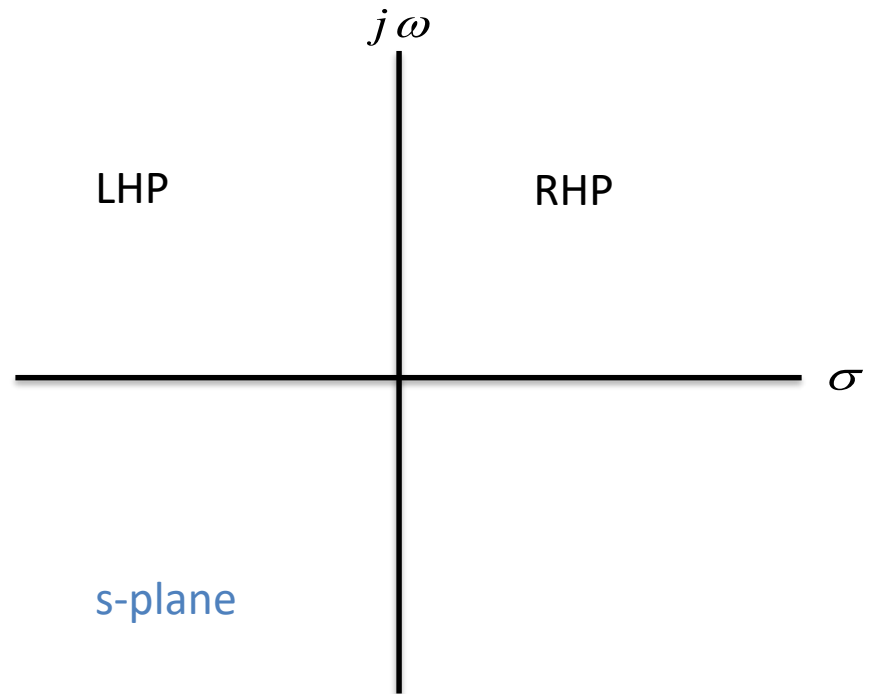
$$\text{iii) } G(s) = \frac{(s + 3)^2}{s(s^2 + 10)}$$

$$\text{iv) } G(s) = \frac{s^2(s + 1)}{s(s + 10)}$$

Stability of Control System

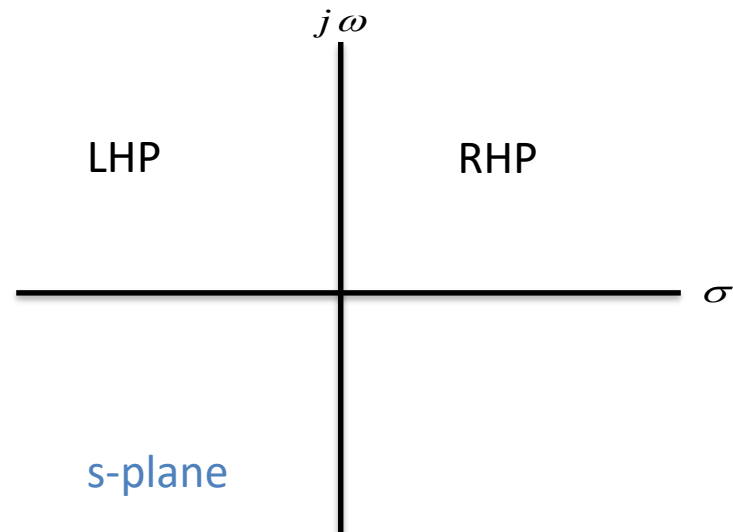
- The poles and zeros of the system are plotted in the s-plane to check the system's stability.

Recall $s = \sigma + j\omega$



Stability of Control System

- If all the poles of the system lie in left half plane the system is said to be **Stable**.
- If any of the poles lie in the right half plane, the system is said to be **unstable**.
- If pole(s) lie on the imaginary axis, the system is said to be **marginally stable**.



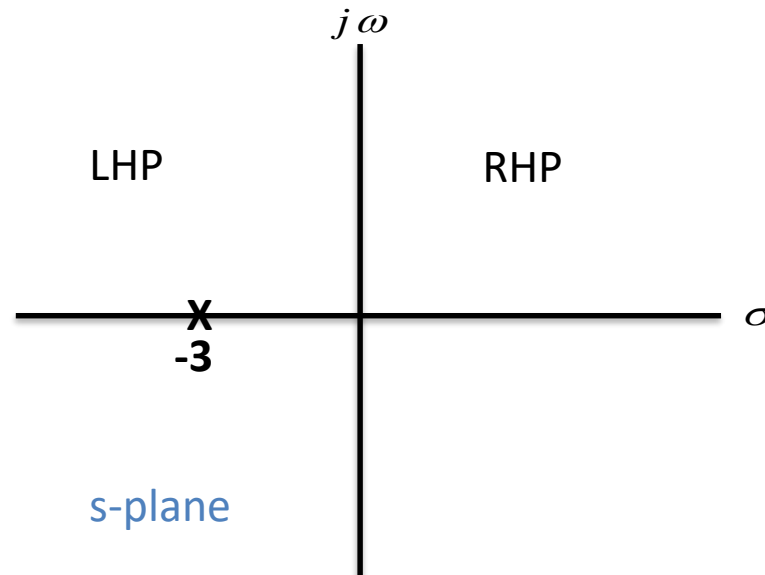
Stability of Control System

- For example

$$G(s) = \frac{C}{As + B}, \quad \text{if } A = 1, B = 3 \text{ and } C = 10$$

- Then the only pole of the system lie at

$$\text{pole} = -3$$



Stability of Control System

- Consider the following transfer functions.
 - Determine whether the transfer function is proper or improper
 - Calculate the Poles and zeros of the system
 - Determine the order of the system
 - Draw the pole-zero map
 - Determine the Stability of the system

$$\text{i) } G(s) = \frac{s + 3}{s(s + 2)}$$

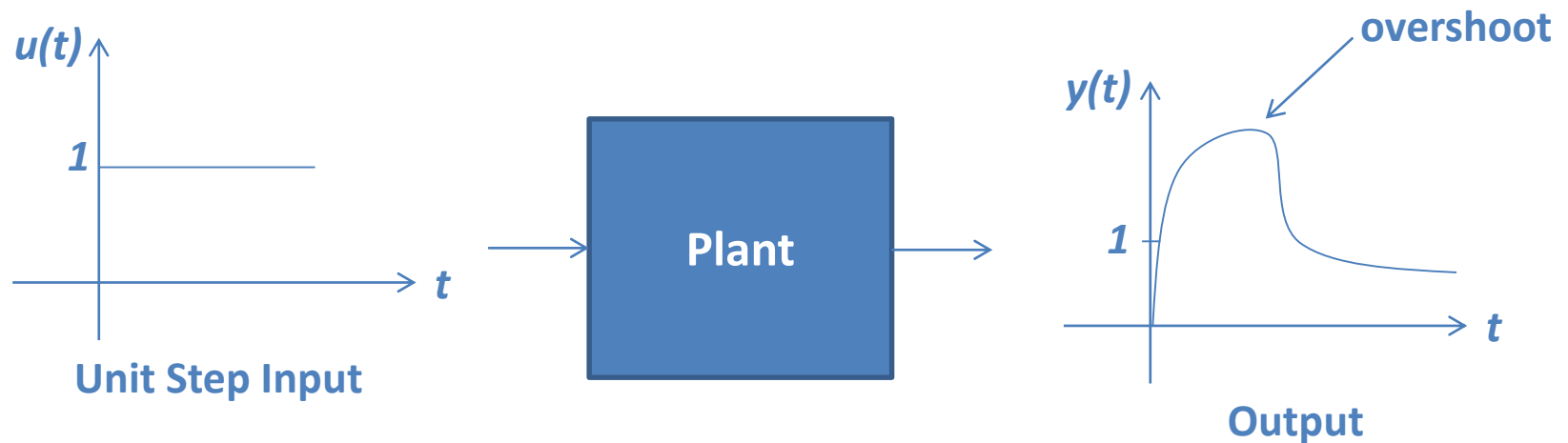
$$\text{ii) } G(s) = \frac{s}{(s + 1)(s + 2)(s + 3)}$$

$$\text{iii) } G(s) = \frac{(s + 3)^2}{s(s^2 + 10)}$$

$$\text{iv) } G(s) = \frac{s^2(s + 1)}{s(s + 10)}$$

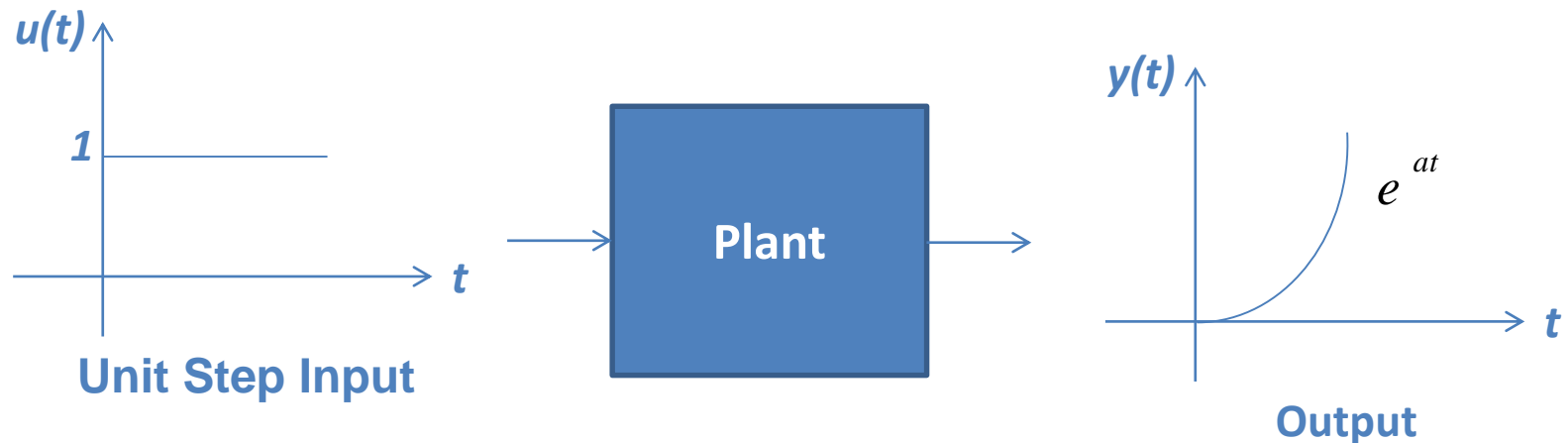
Stability of Control System

- The system is said to be stable if for any bounded input, the output of the system is also bounded (BIBO).
- Thu, for any bounded input, the output either remains constant or decreasese with time.



Stability of Control System

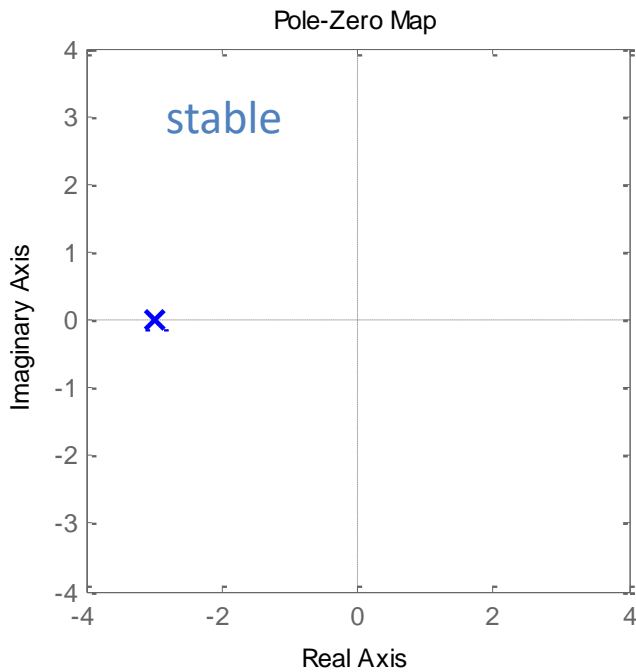
- If for any bounded input, the output is not bounded, the system is said to be unstable.



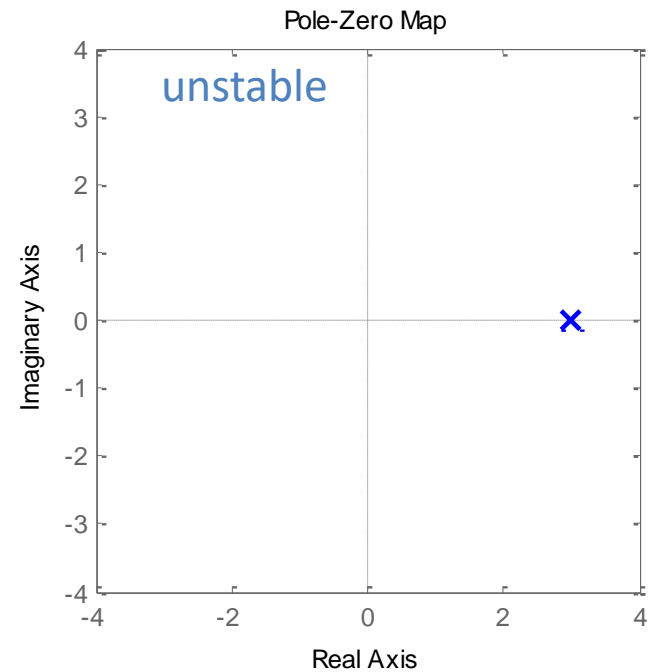
Stability of Control System

- For example

$$G_1(s) = \frac{Y(s)}{U(s)} = \frac{1}{s + 3}$$



$$G_2(s) = \frac{Y(s)}{U(s)} = \frac{1}{s - 3}$$



Stability of Control System

- For example

$$G_1(s) = \frac{Y(s)}{U(s)} = \frac{1}{s+3}$$

$$G_2(s) = \frac{Y(s)}{U(s)} = \frac{1}{s-3}$$

$$\ell^{-1}G_1(s) = \ell^{-1} \frac{Y(s)}{U(s)} = \ell^{-1} \frac{1}{s+3}$$

$$= y(t) = e^{-3t} u(t)$$

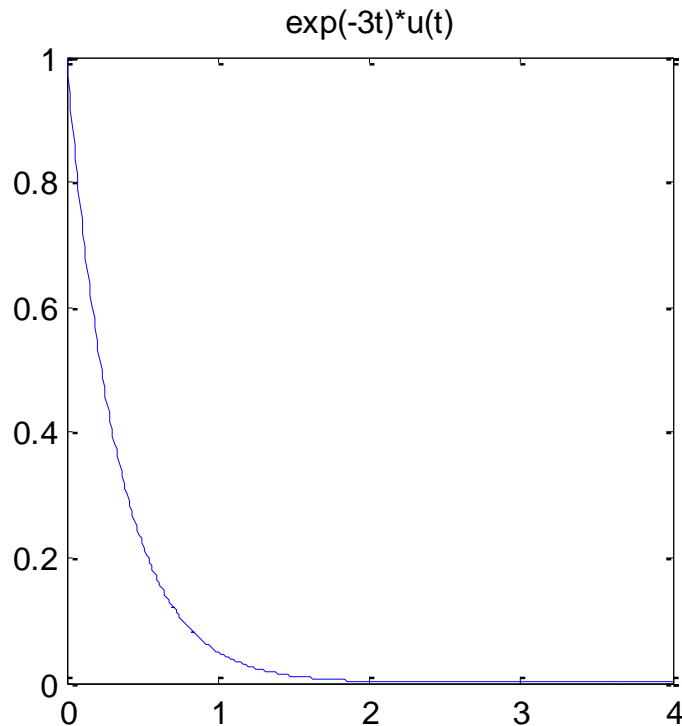
$$\ell^{-1}G_2(s) = \ell^{-1} \frac{Y(s)}{U(s)} = \ell^{-1} \frac{1}{s-3}$$

$$= y(t) = e^{3t} u(t)$$

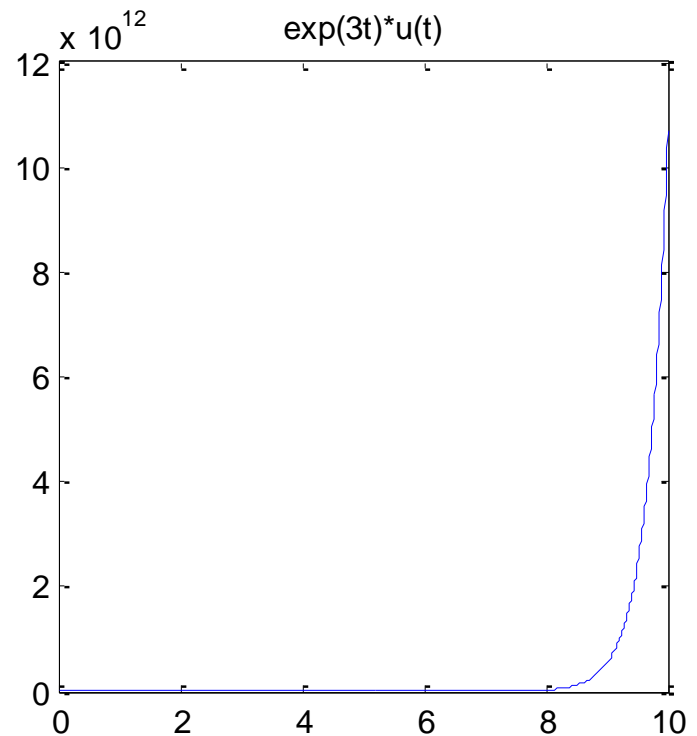
Stability of Control System

- For example

$$y(t) = e^{-3t} u(t)$$



$$y(t) = e^{3t} u(t)$$



Stability of Control System

- Whenever one or more poles are in RHP, the solution of the dynamic equations contains increasing exponential terms.
- Such as e^{3t} .
- This makes the system's response unbounded, and therefore, the overall response of the system is unstable.