Industrial Control Systems

Chapter Three:
Laplace Transform

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Linear Time-invariant Systems

Linearity: A system S is linear if it satisfies both

- *Homogeneity:* If $y = Sx$, and a is a constant then

$$ay = S(ax).$$

- *Superposition:* If $y_1 = Sx_1$ and $y_2 = Sx_2$, then

$$y_1 + y_2 = S(x_1 + x_2).$$

Combined Homogeneity and Superposition:
If $y_1 = Sx_1$ and $y_2 = Sx_2$, and a and b are constants,

$$ay_1 + by_2 = S(ax_1 + bx_2)$$

If H is time invariant, delaying the input and output both by a time τ should produce the same response

$$h_\tau(t) = h(t - \tau).$$

Linear Time-invariance Systems

How they can be both !

- A system can have dynamics (i.e., it's a dynamic system) but still be time-invariant if the laws that define its dynamics are constant.

- For example, a simple RLC circuit with constant resistance, inductance, and capacitance is a dynamic system because its output is described by a differential equation involving past inputs. It is also time-invariant because the values of R, L, and C are constant, meaning its response to a specific input will be the same no matter when the input is applied.
- Conversely, a system that is time-variant has dynamics that change over time, such as a circuit where the resistance changes with temperature.

Laplace Transforms

- Important analytical method for solving *linear* ordinary differential equations.

  Application to nonlinear ODEs? Must linearize first.

- Laplace transforms play a key role in important process control concepts and techniques.

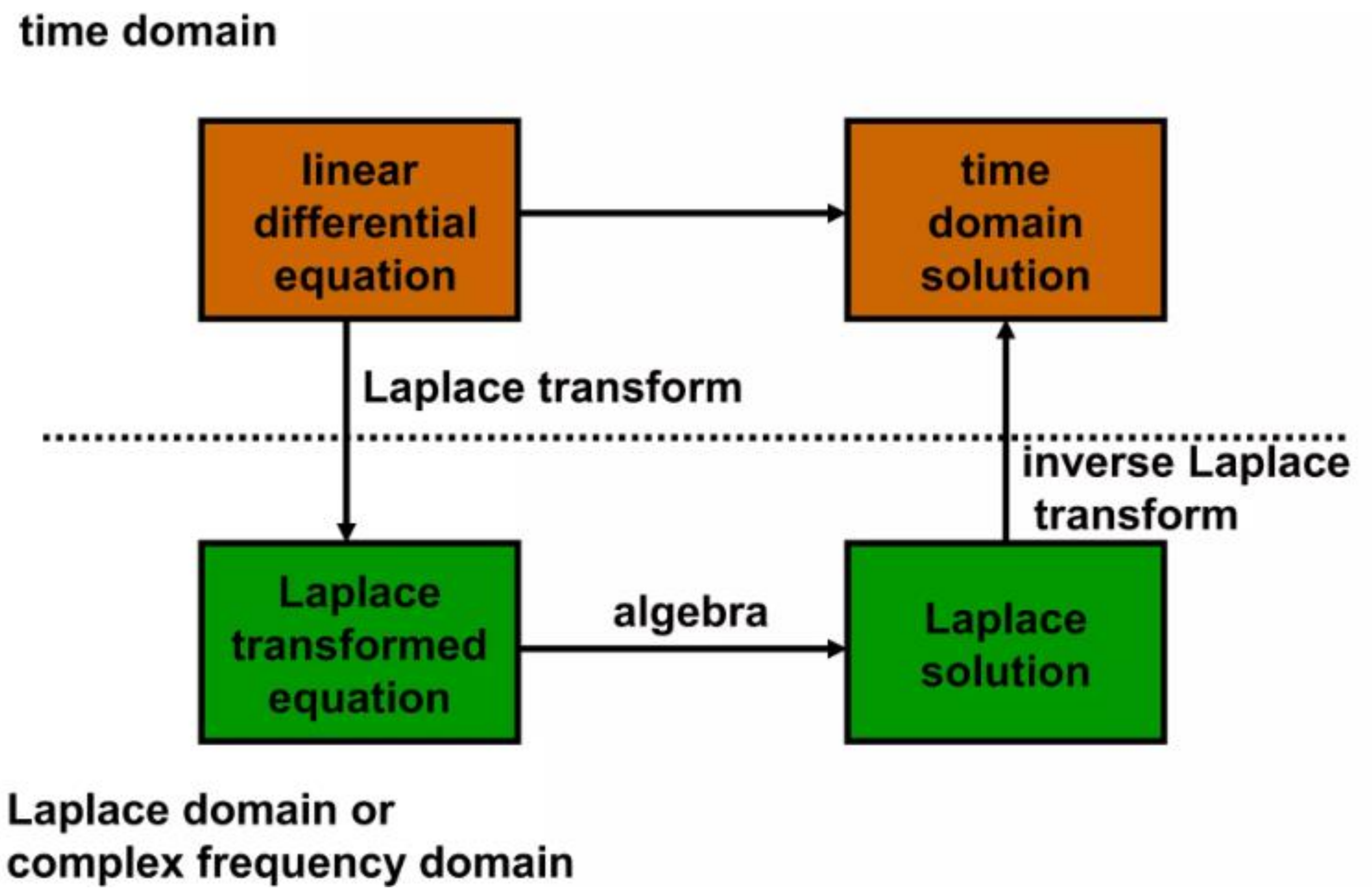
  - Examples:
    - Transfer functions
    - Frequency response
    - Control system design
    - Stability analysis

Definition of Laplace Transform

$$\mathbf{L}[f(t)] = F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

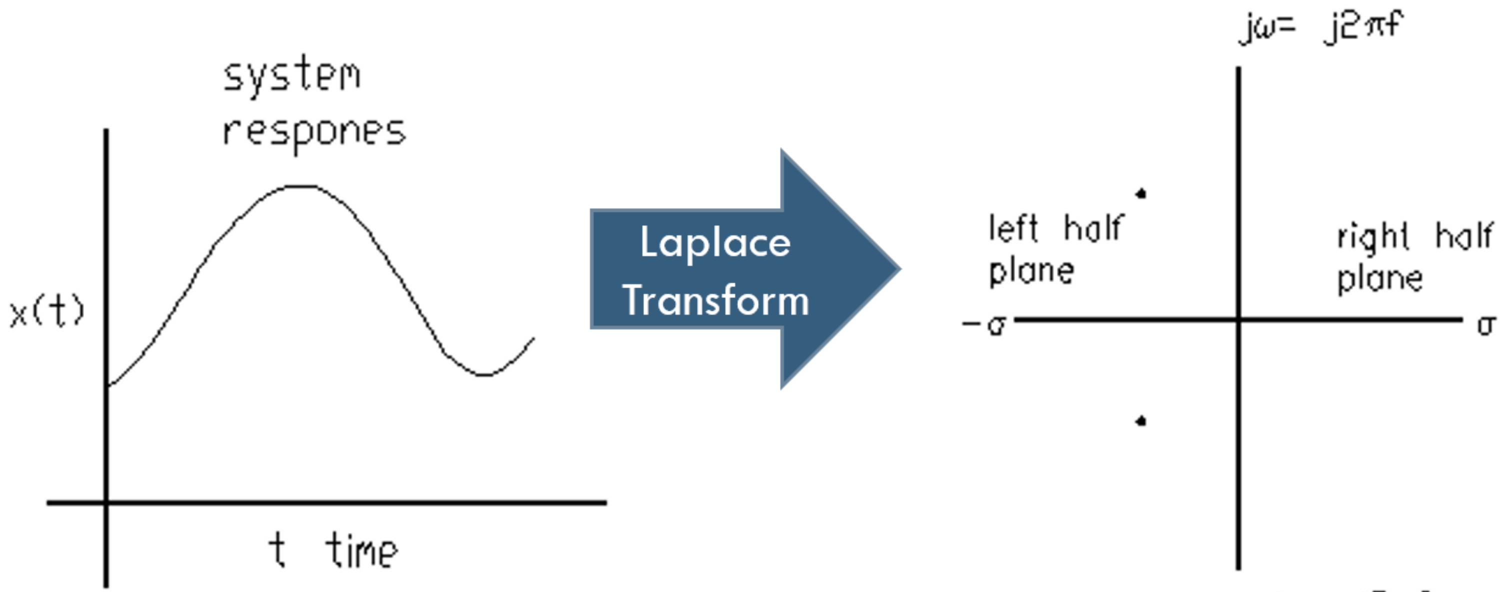
- Convert time-domain functions and operations into frequency-domain
  - $f(t) \rightarrow F(s) \quad (t \in R, s \in C)$
  - Linear differential equations (LDE) $\rightarrow$ algebraic expression in Complex plane
- Graphical solution for key LDE characteristics
- Discrete systems use the analogous z-transform

Definition of Laplace Transform



Laplace Transform

Laplace transform converts time domain problems into functions of a complex variable, s, that is related to the frequency response of the system



Time domain

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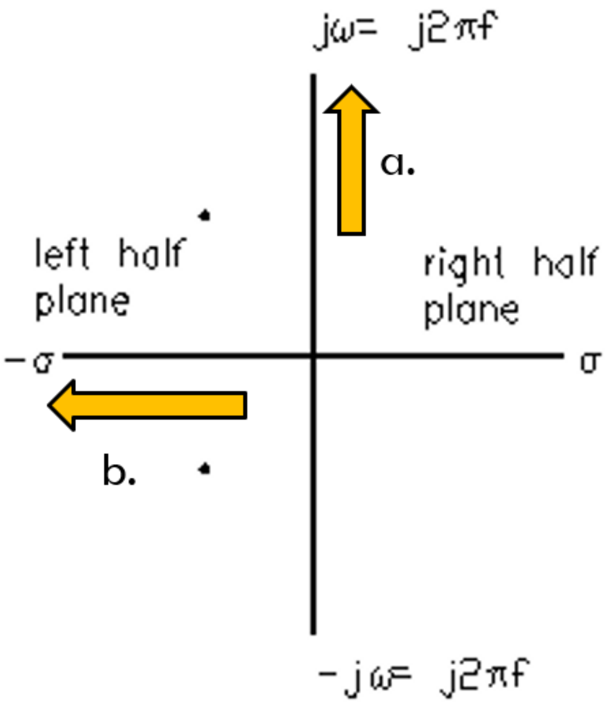
Complex frequency

Laplace Transform



a.) Higher frequency
b.) Faster time constants

Complex frequency combines transient response with sinusoidal steady-state response to get total response of system to input

$s = \sigma + j\omega$ ← Complex Frequency

σ = exponential decay/increase constant that is related to time constants of systems transient response. RC = L/R = σ in circuit analysis

$e^{\sigma t}$ Exponentially increasing function over time

$e^{-\sigma t}$ Exponentially decreasing function over time

Laplace Transform

The radian frequency $\omega = j2\pi f$ same frequency used in phasor analysis

Laplace related to sine response through Euler's Identity.
Euler's relates complex exponentials to sine and cosine time functions

$$e^{j\omega t} = \cos(\omega t) + j \cdot \sin(\omega t)$$
$$e^{-j\omega t} = \cos(\omega t) - j \cdot \sin(\omega t)$$

Adding and subtracting the above relationships gives the exponential forms of sine and cosine

Laplace Transform

Add the identities

$$e^{j\omega t} = \cos(\omega t) + j\cdot\sin(\omega t)$$
$$e^{-j\omega t} = \cos(\omega t) - j\cdot\sin(\omega t)$$

+

$$e^{j\omega t} + e^{-j\omega t} = 2\cdot\cos(\omega t)$$

$$\frac{e^{j\omega t} + e^{-j\omega t}}{2} = \cos(\omega t)$$

Exponential form of Cosine

Subtract the identities

$$e^{j\omega t} = \cos(\omega t) + j\cdot\sin(\omega t)$$
$$e^{-j\omega t} = \cos(\omega t) - j\cdot\sin(\omega t)$$

-

$$e^{j\omega t} - e^{-j\omega t} = 2j\cdot\sin(\omega t)$$

$$\frac{e^{j\omega t} - e^{-j\omega t}}{2j} = \sin(\omega t)$$

Exponential form of Sine

Since $e^{st} = e^{(\sigma + j\omega)t} = e^{\sigma t}\cdot e^{j\omega t}$

Laplace can give complete response: dc transient and steady-state sinusoidal

Laplace Transform

Time Domain Function	Laplace Domain Function
$\delta(t)$ Impulse	1
$u_s(t)$ Unit Step	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
e^{at}	$\frac{1}{s-a}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
t Linear ramp (slope 1)	$\frac{1}{s^2}$

Note: time functions multiplied by constants give Laplace function multiplied by constant

Examples:

$$5 \cdot u_s(t) \rightarrow \frac{5}{s}$$

$$3 \cdot \sin(4t) \rightarrow 3 \cdot \left(\frac{4}{s^2+16}\right) \quad \omega = 4$$

Laplace Transform for Elementary Functions

1) $x(t) = \delta(t)$ **Impulse signal**

Solution

$$\mathbf{L[\delta(t)]} = \int_0^\infty \delta(t)e^{-st}dt$$

$$= e^{-s(0)} = 1$$

$$\because \delta(t) = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{for } t \neq 0 \end{cases}$$

2) $x(t) = u(t)$ **Step signal**

Solution

$$\because u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

$$\mathbf{L[u(t)]} = \int_0^\infty u(t)e^{-st}dt = \int_0^\infty 1e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}$$

Laplace Transform for Elementary Functions

3) $L(1) = \frac{1}{s}$ **Constant**

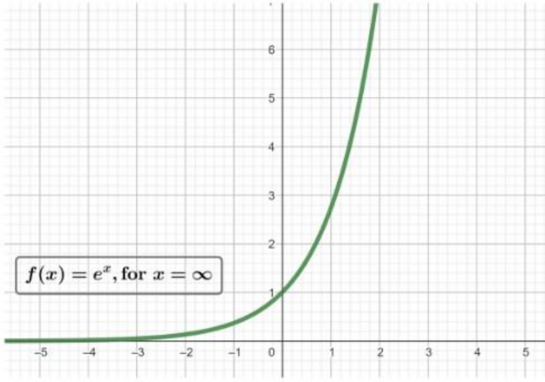
Solution

$$L(1) = \int_0^\infty e^{-st}.1\,dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}, (s > 0)$$

4) $L(e^{at}) = \frac{1}{s - a}$ **Exponential signal**

Solution

$$L(e^{at}) = \int_0^\infty e^{-st}.e^{at}dt = \int_0^\infty e^{-(s-a)t}dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = \frac{1}{s - a} \text{ if } s > a$$



Laplace Transform for Elementary Functions

5) $L[e^{-at}] = \frac{1}{s+a}$,s > -a **Exponential signal**

6) $L[\sinh at] = \frac{a}{s^2 - a^2}$

Solution

W.k.t $\sinh at = \frac{e^{at} - e^{-at}}{2}$

$$L(\sinh at) = L\left(\frac{e^{at} - e^{-at}}{2}\right) = \frac{1}{2}[L(e^{at}) - L(e^{-at})]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{a}{s^2 - a^2}, s > |a|$$

Laplace Transform for Elementary Functions

7) $L[\cosh at] = \dfrac{s}{s^2 - a^2}, s > |a|$ **Hint** $\cosh at = \dfrac{e^{at} + e^{-at}}{2}$

8) **x(t) = cos ω_0 t u(t)**

$$\mathcal{L}[\cos \omega_0 t\, u(t)] = \frac{1}{2}\mathcal{L}[e^{j\omega_0 t}u(t) + e^{-j\omega_0 t}u(t)]$$

$$= \frac{1}{2}\left[\frac{1}{s - j\omega_0} + \frac{1}{s + j\omega_0}\right] = \frac{s}{s^2 + \omega_0^2}$$

9) **x(t) = sin ω_0 t u(t)**

Laplace Transform for Elementary Functions

8) $L[\sin at] = \frac{a}{s^2+a^2}$ and $L[\cos at] = \frac{s}{s^2+a^2}, s > 0$

Solution

Using Euler's Formula

$$e^{iat} = \cos at + i \sin at \quad \text{----> (1)}$$

$$L[\cos at + i \sin at] = L[e^{iat}] = \frac{1}{s-ia} * \frac{(s+ia)}{(s+ia)}$$

$$= \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2} \quad \text{----> (2)}$$

Compare (1) and (2)

Real part $L[\cos at] = \frac{s}{s^2+a^2}$ Imaginary part $L[\sin at] = \frac{a}{s^2+a^2}$

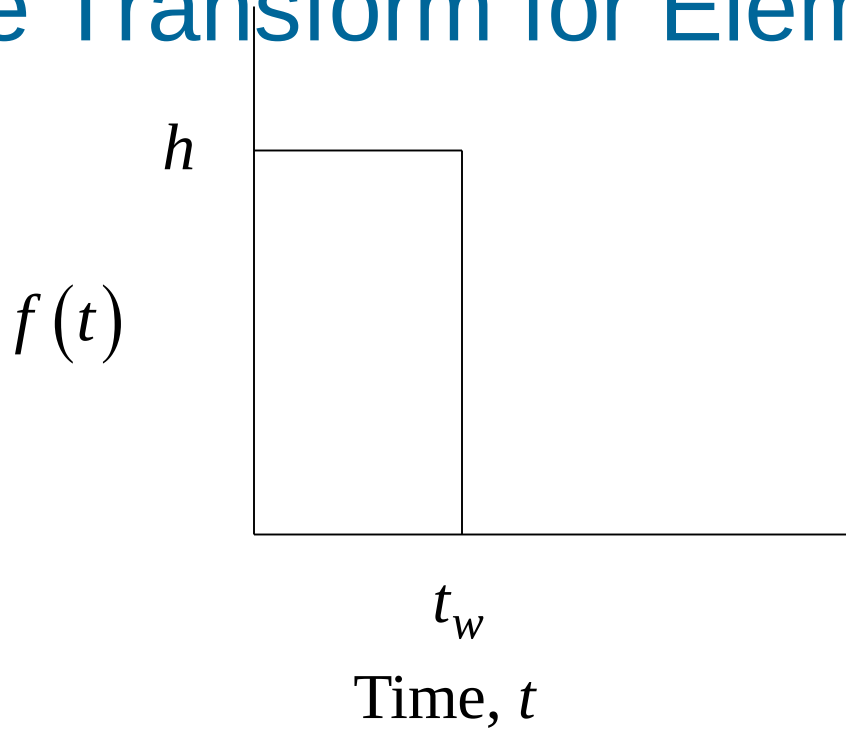
Laplace Transform for Elementary Functions

9) Rectangular Pulse Function

It is defined by:

$$f(t)=\begin{cases}0 & \text{for } t<0\\ h & \text{for } 0\le t<t_w\\ 0 & \text{for } t\ge t_w\end{cases} \qquad (3\text{-}20)$$

Laplace Transform for Elementary Functions



The Laplace transform of the rectangular pulse is given by

$$F(s) = \frac{h}{s}\left(1 - e^{-t_w s}\right) \qquad (3\text{-}22)$$

Laplace Transform for Elementary Functions

Match the following time functions to correct Laplace domain function using the transform pairs.

a.) $10t$

b.) $t \cdot e^{-at}$

c.) e^{5t}

d.) e^{-2t}

e.) $3 \cdot \cos(t)$

1. $\frac{1}{s+2}$
2. $3 \cdot \left(\frac{s}{(s^2+1)} \right)$
3. $\frac{1}{s-5}$
4. $\frac{10}{s^2}$
5. $\frac{1}{(s+a)^2}$

Laplace Transform for Elementary Functions

Match the following time functions to correct Laplace domain function using the transform pairs.

a.) $10t$ (4)

b.) $t \cdot e^{-at}$ (5)

Laplace table 3.2 textbook

c.) e^{5t} (3)

d.) e^{-2t} (1)

e.) $3 \cdot \cos(t)$ (2)

(1) $\dfrac{1}{s+2}$

(2) $3 \cdot \left(\dfrac{s}{(s^2+1)} \right)$

(3) $\dfrac{1}{s-5}$

(4) $\dfrac{10}{s^2}$

(5) $\dfrac{1}{(s+a)^2}$

Laplace Transform for Elementary Functions

	f(t)	F(s)
1	$\delta(t)$	1
2	$\delta(t-a)$	e^{-as}
3	$u_0(t)$	$\frac{1}{s}$
4	$tu_0(t)$	$\frac{1}{s^2}$
5	$t^n u_0(t)$	$\frac{n!}{s^{n+1}}$
6	$e^{-at}u_0(t)$	$\frac{1}{s+a}$
7	$t^n e^{-at}u_0(t)$	$\frac{n!}{(s+a)^{n+1}}$
8	$\sin(\omega t)u_0(t)$	$\frac{\omega}{s^2+\omega^2}$
9	$\cos(\omega t)u_0(t)$	$\frac{s}{s^2+\omega^2}$
10	$e^{-at}\sin(\omega t)u_0(t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
11	$e^{-at}\cos(\omega t)u_0(t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

Laplace Transform Therorms

Laplace of an unknown function

$$\mathcal{L}(f_1(t)) = F_1(s)$$

Capitalize unknown function name

Replace t with s

Laplace Operator Symbol

Examples

$$\mathcal{L}(i_1(t)) = I_1(s)$$

$$\mathcal{L}(v_1(t)) = V_1(s)$$

Linearity of transform - can multiply by constant

If $\mathcal{L}(f_1(t)) = F_1(s)$ and $\mathcal{L}(f_2(t)) = F_2(s)$

Then $\mathcal{L}(a \cdot f_1(t) + b \cdot f_2(t)) = a \cdot F_1(s) + b \cdot F_2(s)$

Laplace Transform for Drivatives

Laplace Transform turns derivative into multiplication by s

If $\mathcal{L}(f_1(t)) = F_1(s)$

Then $\mathcal{L}\left(\frac{d}{dt} f_1(t)\right) = s \cdot F_1(s) - f_1(0)$

Subtract any non-zero initial conditions

For higher order derivatives

$$\mathcal{L}\left(\frac{d^2}{dt^2} f_1(t)\right) = s \cdot (s \cdot F_1(s) - f_1(0)) - \frac{d}{dt} f_1(0)$$

0 initial conditions reduces formula to

$$\mathcal{L}\left(\frac{d^2}{dt^2} f_1(t)\right) = s^2 \cdot F_1(s)$$

Laplace Transform for Integrals

Laplace turns integration into division by s

If $\mathscr{L}\left(f_1(t)\right) = F_1(s)$

Then $\mathscr{L}\left(\int \frac{d}{dt} f_1(t)\, dt\right) = \frac{1}{s} \cdot F_1(s)$

Examples from circuit analysis:

Capacitor voltage

$$\mathscr{L}\left(v_C(t)\right) = \mathscr{L}\left(\frac{1}{C} \cdot \int i_C(t)\, dt\right)$$

$$v_C(t) = \frac{1}{C} \cdot \int i_C(t)\, dt$$

$$V_C(s) = \frac{1}{C} \cdot \left(\frac{1}{s}\right) \cdot I_C(s) = \left[\frac{1}{C \cdot s}\right] \cdot I_C(s)$$

Laplace Transform for s and t shifting

$$L[e^{at} f(t)](s) = F(s-a)$$

Example:

$$L[\cos bt] = \frac{s}{s^2 + b^2}$$

$$\Rightarrow \quad L[e^{at} \cos bt] = \frac{s-a}{[(s-a)^2 + b^2]}$$

$$L\big[H(t-a) f(t-a)\big](s) = e^{-as} F(s)$$

$$L^{-1}[e^{-as} F(s)](t) = H(t-a) f(t-a)$$

$$L^{-1}[FG] = f * g$$

Laplace Transform Initial and Final Values

Initial and Final Values

The initial-value and final-value properties allow us to find the initial value *f(0)* and *f(∞)* of *f(t)* directly from its Laplace transform *F(s)*.

$$f(0) = \lim_{s \to \infty} sF(s)$$

Initial-value theorem

$$f(\infty) = \lim_{s \to 0} sF(s)$$

Final-value theorem

The Inverse of Laplace Transform

Suppose F(s) has the general form of

$$F(s) = \frac{N(s)\text{......numerator polynomial}}{D(s)\text{...denominator polynomial}}$$

The finding the inverse Laplace transform of *F(s)* involves two steps:

1. Decompose F(s) into simple terms using partial fraction expansion.
2. Find the inverse of each term by matching entries in Laplace Transform Table.

The Inverse of Laplace Transform

Example 1

Find the inverse Laplace transform of

$$F(s) = \frac{3}{s} - \frac{5}{s+1} + \frac{6}{s^2+4}$$

Solution:

$$f(t) = L^{-1}\left(\frac{3}{s}\right) - L^{-1}\left(\frac{5}{s+1}\right) + L^{-1}\left(\frac{6}{s^2+4}\right)$$
$$= (3 - 5e^{-t} + 3\sin(2t)u(t), \quad t \geq 0$$

The Inverse of Laplace Transform

PARTIAL FRACTION EXPANSION

1) Distinct Real Roots of D(s)

$$F(s) = \frac{96(s+5)(s+12)}{s(s+8)(s+6)}$$

$s_1 = 0$, $s_2 = -8$

$s_3 = -6$

The Inverse of Laplace Transform

1) DISTINCT REAL ROOTS

$$F(s) = \frac{96(s+5)(s+12)}{s(s+8)(s+6)} \equiv \frac{K_1}{s} + \frac{K_2}{s+8} + \frac{K_3}{s+6}$$

- To find K_1: multiply both sides by s and evaluates both sides at s=0
- To find K_2: multiply both sides by s+8 and evaluates both sides at s=-8
- To find K_3: multiply both sides by s+6 and evaluates both sides at s=-6

The Inverse of Laplace Transform

FIND K_1

$$\left.\frac{96(s+5)(s+12)}{(s+8)(s+6)}\right|_{s=0} \equiv K_1 + \left.\frac{K_2 s}{s+8}\right|_{s=0} + \left.\frac{K_3 s}{s+6}\right|_{s=0}$$

$$\therefore K_1 = \frac{96(5)(12)}{(8)(6)} = 120$$

The Inverse of Laplace Transform

FIND K_2

$$\left.\frac{96(s+5)(s+12)}{s(s+6)}\right|_{s=-8} = \cancel{\left.\frac{K_1(s+8)}{s(s+6)}\right|_{s=-8}} + K_2 + \cancel{\left.\frac{K_3(s+8)}{s(s+6)}\right|_{s=-8}}$$

$$\therefore K_2 = \frac{96(-3)(4)}{(-8)(-2)} = -72$$

The Inverse of Laplace Transform

FIND K_3

$$\left.\frac{96(s+5)(s+12)}{s(s+8)}\right|_{s=-6} \equiv \left.\frac{K_1(s+6)}{s(s+8)}\right|_{s=-6} + \left.\frac{K_2(s+6)}{s(s+8)}\right|_{s=-6} + K_3$$

$$\therefore K_3 = \frac{96(-1)(6)}{(-6)(2)} = 48$$

The Inverse of Laplace Transform

INVERSE LAPLACE OF F(S)

$$F(s) = \frac{120}{s} - \frac{72}{s+8} + \frac{48}{s+6}$$

$$L^{-1}\left[\frac{120}{s} - \frac{72}{s+8} + \frac{48}{s+6}\right]$$

$$f(t) = \left(120 - 72e^{-8t} + 48e^{-6t}\right)u(t)$$

The Inverse of Laplace Transform

2) DISTINCT COMPLEX ROOTS

$$F(s) = \frac{100(s+3)}{(s+6)(s^2+6s+25)}$$

$S_1 = -6$ $S_2 = -3+j4$ $S_3 = -3-j4$

The Inverse of Laplace Transform

PARTIAL FRACTION EXPANSION

$$F(s) = \frac{100(s+3)}{(s+6)(s^2+6s+25)}$$

$$\equiv \frac{K_1}{s+6} + \frac{K_2}{s+3-j4} + \frac{K_3}{s+3+j4}$$

$$\equiv \frac{K_1}{s+6} + \frac{K_2}{s+3-j4} + \frac{K_2^*}{s+3+j4}$$

✓ Complex roots appears in conjugate pairs.

The Inverse of Laplace Transform

FIND K_1

$$K_1 = \left.\frac{100(s+3)}{s^2+6s+25}\right|_{s=-6}$$

$$= \frac{100(-3)}{25}$$

$$= -12$$

The Inverse of Laplace Transform

FIND K_2 AND K_2^*

$$K_2 = \left.\frac{100\,(s+3)}{(s+6)(s+3+j4)}\right|_{s=-3+j4}$$

$$= \frac{100\,(j4)}{(3+j4)(j8)}$$

$$= 6 - j8 = 10e^{-j53.13^\circ}$$

$$K_2^* = 6 + j8 = 10\,e^{j53.13}$$

Coefficients associated with conjugate pairs are themselves conjugates.

The Inverse of Laplace Transform

INVERSE LAPLACE OF F(S)

$$F(s) = \frac{100(s+3)}{(s+6)(s^2+6s+25)}$$

$$= \frac{-12}{s+6} + \frac{10\angle -53.13^\circ}{s+3-j4} + \frac{10\angle 53.13^\circ}{s+3+j4}$$

The Inverse of Laplace Transform

INVERSE LAPLACE OF F(S)

$$L^{-1}\left\{\frac{-12}{s+6}+\frac{10e^{-j53.13^\circ}}{s+3-j4}+\frac{10e^{j53.13^\circ}}{s+3+j4}\right\}$$

$$=\Big(-12e^{-6t}+10e^{-j53.13^\circ}e^{-(3-j4)t}+10e^{j53.13^\circ}e^{-(3+j4)t}\Big)u(t)$$

The Inverse of Laplace Transform

USEFUL TRANSFORM PAIRS

1) $$\frac{K}{s+a} \Leftrightarrow Ke^{-at}u(t)$$

2) $$\frac{K}{(s+a)^2} \Leftrightarrow Kte^{-at}u(t)$$

3) $$\frac{K}{s+\alpha-j\beta}+\frac{K}{s+\alpha+j\beta} \Leftrightarrow 2|K|e^{-\alpha t}\cos(\beta t+\theta)u(t)$$

4) $$\frac{K}{(s+\alpha-j\beta)^2}+\frac{K}{(s+\alpha+j\beta)^2} \Leftrightarrow 2t|K|e^{-\alpha t}\cos(\beta t+\theta)u(t)$$

Operational Laplace Transform

OPERATIONAL TRANSFORMS

- Indicate how mathematical operations performed on either f(t) or F(s) are converted into the opposite domain.
- The operations of primary interest are:
  1. Multiplying by a constant
  2. Addition/subtraction
  3. Differentiation
  4. Integration
  5. Translation in the time domain
  6. Translation in the frequency domain
  7. Scale changing

Operational Laplace Transform

OPERATION	f(t)	F(s)
Multiplication by a constant	$Kf(t)$	$KF(s)$
Addition/ Subtraction	$f_1(t)+f_2(t)-f_3(t)+\cdots$	$F_1(s)+F_2(s)-F_3(s)+\cdots$
First derivative (time)	$df(t)/dt$	$sF(s)-f(0^-)$
Second derivative (time)	$d^2(t)/dt^2$	$s^2F(s)-sf(0^-)-df(0^-)/dt$

Operational Laplace Transform

OPERATION	f(t)	F(s)
n th derivative (time)	$d^n(t)/dt^n$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}df(0^-)/dt - s^{n-3}df^2(0^-)/dt - \dots - df^{n-1}(0^-)/dt^{n-1}$
Time integral	$\int_0^t f(x)dx$	$F(s)/s$
Translation in time	$f(t-a)u(t-a)$, $a>0$	$e^{-as}F(s)$
Translation in frequency	$e^{-at}f(t)$	$F(s+a)$

Operational Laplace Transform

OPERATION	f(t)	F(s)
Scale changing	$f(at), a > 0$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
First derivative (s)	$tf(t)$	$-\frac{dF(s)}{ds}$
n th derivative	$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$
s integral	$\frac{f(t)}{t}$	$\int_s^{\infty} F(u)du$

Operational Laplace Transform

TRANSLATION IN TIME DOMAIN

If we start with any function:

$$f(t)u(t)$$

we can represent the same function translated in time by the constant a, as:

$$f(t-a)u(t-a)$$

In frequency domain:

$$f(t-a)u(t-a) = e^{-as}F(s)$$

Operational Laplace Transform

EX:

$$L[tu(t)] = {}^{1}\!/_{s^2}$$

$$L[(t-a)u(t-a)] = {}^{e^{-as}}\!/_{s^2}$$

Operational Laplace Transform

TRANSLATION IN FREQUENCY DOMAIN

Translation in the frequency domain is defined as:

$$L\left[e^{-at} f(t)\right] = F(s+a)$$

Operational Laplace Transform

EX:

$$L\left[\cos \omega t\right] = \frac{s}{s^2 + \omega^2}$$

$$L\left[e^{-at} \cos \omega t\right] = \frac{s + a}{(s + a)^2 + \omega^2}$$

Operational Laplace Transform

EX:

$$L[\cos t] = \frac{s}{s^2+1}$$

$$L[\cos \omega t] = \frac{1}{\omega}\frac{s/\omega}{(s/\omega)^2+1} = \frac{s}{s^2+\omega^2}$$

Table 3.1 Laplace Transforms for Various Time-Domain Functions[a]

$f(t)$		$F(s)$
1. $\delta(t)$	(unit impulse)	1
2. $S(t)$	(unit step)	$\frac{1}{s}$
3. t	(ramp)	$\frac{1}{s^2}$
4. t^{n-1}		$\frac{(n-1)!}{s^n}$
5. e^{-bt}		$\frac{1}{s+b}$
6. $\frac{1}{\tau} e^{-t/\tau}$		$\frac{1}{\tau s + 1}$
7. $\frac{t^{n-1}e^{-bt}}{(n-1)!}$	$(n > 0)$	$\frac{1}{(s+b)^n}$
8. $\frac{1}{\tau^n (n-1)!} t^{n-1} e^{-t/\tau}$		$\frac{1}{(\tau s + 1)^n}$
9. $\frac{1}{b_1 - b_2}(e^{-b_2 t} - e^{-b_1 t})$		$\frac{1}{(s+b_1)(s+b_2)}$

Table 3.1 Laplace Transforms for Various Time-Domain Functions[a]

$f(t)$	$F(s)$
10. $\frac{1}{\tau_1 - \tau_2}(e^{-t/\tau_1} - e^{-t/\tau_2})$	$\frac{1}{(\tau_1 s + 1)(\tau_2 s + 1)}$
11. $\frac{b_3 - b_1}{b_2 - b_1} e^{-b_1 t} + \frac{b_3 - b_2}{b_1 - b_2} e^{-b_2 t}$	$\frac{s + b_3}{(s + b_1)(s + b_2)}$
12. $\frac{1}{\tau_1} \frac{\tau_1 - \tau_3}{\tau_1 - \tau_2} e^{-t/\tau_1} + \frac{1}{\tau_2} \frac{\tau_2 - \tau_3}{\tau_2 - \tau_1} e^{-t/\tau_2}$	$\frac{\tau_3 s + 1}{(\tau_1 s + 1)(\tau_2 s + 1)}$
13. $1 - e^{-t/\tau}$	$\frac{1}{s(\tau s + 1)}$
14. $\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
15. $\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
16. $\sin(\omega t + \phi)$	$\frac{\omega \cos \phi + s \sin \phi}{s^2 + \omega^2}$
17. $e^{-bt} \sin \omega t$ (b, ω real)	$\frac{\omega}{(s + b)^2 + \omega^2}$
18. $e^{-bt} \cos \omega t$ (b, ω real)	$\frac{s + b}{(s + b)^2 + \omega^2}$
19. $\frac{1}{\tau\sqrt{1 - \zeta^2}} e^{-\zeta t/\tau} \sin(\sqrt{1 - \zeta^2}\, t/\tau)$ $(0 \leq \lvert\zeta\rvert < 1)$	$\frac{1}{\tau^2 s^2 + 2\zeta\tau s + 1}$

Table 3.1 Laplace Transforms for Various Time-Domain Functions[a] (continued)

$f(t)$	$F(s)$
20. $1 + \frac{1}{\tau_2 - \tau_1}(\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2})$ $(\tau_1 \neq \tau_2)$	$\frac{1}{s(\tau_1 s + 1)(\tau_2 s + 1)}$
21. $1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta t/\tau} \sin[\sqrt{1-\zeta^2}\, t/\tau + \psi]$ $\psi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}, \quad (0 \le \lvert\zeta\rvert < 1)$	$\frac{1}{s(\tau^2 s^2 + 2\zeta\tau s + 1)}$
22. $1 - e^{-\zeta t/\tau}[\cos(\sqrt{1-\zeta^2}\; t/\tau) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\sqrt{1-\zeta^2}\, t/\tau)]$ $(0 \le \lvert\zeta\rvert < 1)$	$\frac{1}{s(\tau^2 s^2 + 2\zeta\tau s + 1)}$
23. $1 + \frac{\tau_3 - \tau_1}{\tau_1 - \tau_2} e^{-t/\tau_1} + \frac{\tau_3 - \tau_2}{\tau_2 - \tau_1} e^{-t/\tau_2}$ $(\tau_1 \neq \tau_2)$	$\frac{\tau_3 s + 1}{s(\tau_1 s + 1)(\tau_2 s + 1)}$
24. $\frac{df}{dt}$	$sF(s) - f(0)$
25. $\frac{d^n f}{dt^n}$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$
26. $f(t - t_0)S(t - t_0)$	$e^{-t_0 s} F(s)$

[a]Note that $f(t)$ and $F(s)$ are defined for $t \ge 0$ only.

Example 3.1

Solve the ODE,

$$5\frac{dy}{dt}+4y=2 \qquad y(0)=1 \qquad (3\text{-}26)$$

First, take $\mathcal{L}$ of both sides of (3-26),

$$5\big(sY(s)-1\big)+4Y(s)=\frac{2}{s}$$

Rearrange,

$$Y(s)=\frac{5s+2}{s(5s+4)} \qquad (3\text{-}34)$$

Take $\mathcal{L}^{-1}$,

$$y(t)=\mathcal{L}^{-1}\left[\frac{5s+2}{s(5s+4)}\right]$$

From Table 3.1 (line 11),

$$\boxed{y(t)=0.5+0.5e^{-0.8t}} \qquad (3\text{-}37)$$

- Example 3.2: $y' - 4y = 1, \; y(0) = 1$

$$L[y' - 4y](s) = L[y'](s) - 4L[y](s) = (sY(s) - y(0)) - 4Y(s)$$

$$= (s-4)Y(s) - 1$$

$$L[1](s) = \frac{1}{s}$$

$$(s-4)Y(s) - 1 = \frac{1}{s}, \quad Y(s) = \frac{1}{s-4} + \frac{1}{s(s-4)}$$

$$y = L^{-1}[Y] = L^{-1}\left[\frac{1}{s-4} + \frac{1}{s(s-4)}\right] = \mathrm{L}^{-1}\left[\frac{1}{s-4}\right] + L^{-1}\left[\frac{1}{s(s-4)}\right]$$

From Table 3.1, entries (5) and (9)

$$(5) \quad L[e^{at}] = \frac{1}{s-a}, \quad (8) \quad L\left[\frac{e^{at} - e^{bt}}{a-b}\right] = \frac{1}{(s-a)(s-b)}$$

$$y(t) = e^{4t} + \frac{1}{4}(e^{4t} - 1) = \frac{5}{4}e^{4t} - \frac{1}{4}$$

Example 3.3:

$$L^{-1}\left[\frac{4}{s^2+4s+20}\right]=L^{-1}\left[\frac{4}{(s+2)^2+16}\right]$$

$$\because\ L[\sin at]=\frac{a}{s^2+a^2},\quad L[\sin 4t]=\frac{4}{s^2+16}$$

$$L[e^{-2t}\sin 4t]=\frac{4}{(s+2)^2+16}$$

$$\therefore\ L^{-1}\left[\frac{4}{(s+2)^2+16}\right]=e^{-2t}\sin 4t$$

Factoring the denominator polynomial

1. $$\frac{2}{3s^2+4s+1}$$

$$3s^2+4s+1=(3s+1)(s+1)=3(s+\frac{1}{3})(s+1)$$

From 9 in the table, Transforms to $e^{-t/3} - e^{-t}$

Real roots = no oscillation

2. $$\frac{2+s}{s^2+s+1}$$

$$s^2+s+1=(s+\frac{1}{2}+\frac{\sqrt{3}}{2}j)(s+\frac{1}{2}-\frac{\sqrt{3}}{2}j)=(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2$$

Transforms to $e^{-0.5t}\sin\frac{\sqrt{3}}{2}t,\ e^{-0.5t}\cos\frac{\sqrt{3}}{2}t$

Complex roots = oscillation

$$L^{-1}[\sqrt{3}(\frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2})+(\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2})]$$

From Table 3.1, line 17 and 18

$$y(t)=e^{-\frac{t}{2}}(\sin(\frac{\sqrt{3}}{2}t)+\sqrt{3}\cos(\frac{\sqrt{3}}{2}t))$$

EXAMPLE 3.5

Find the inverse transform of

$$Y(s) = \frac{1 + e^{-2s}}{(4s+1)(3s+1)} \tag{3-72}$$

SOLUTION

Equation 3-72 can be split into two terms:

$$Y(s) = Y_1(s) + Y_2(s) \tag{3-73}$$

$$= \frac{1}{(4s+1)(3s+1)} + \frac{e^{-2s}}{(4s+1)(3s+1)} \tag{3-74}$$

The inverse transform of $Y_1(s)$ can be obtained directly from Table 3.1:

$$y_1(t) = e^{-t/4} - e^{-t/3} \tag{3-75}$$

Because $Y_2(s) = e^{-2s}Y_1(s)$, its inverse transform can be written immediately by replacing t by $(t-2)$ in (3-75), and then multiplying by the shifted step function:

$$y_2(t) = [e^{-(t-2)/4} - e^{-(t-2)/3}]S(t-2) \tag{3-76}$$

Thus, the complete inverse transform is

$$y(t) = e^{-t/4} - e^{-t/3} + [e^{-(t-2)/4} - e^{-(t-2)/3}]S(t-2)$$

$$= y_1(t) + y_2(t) \tag{3-77}$$

3.16 (a) The differential equation

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = \cos t$$

has initial conditions, $y(0) = 1$, $y'(0) = 2$. Find $Y(s)$ and, without finding $y(t)$, determine what functions of time will appear in the solution.

(b) If $Y(s) = \dfrac{s+1}{s(s^2+4s+8)}$, find $y(t)$.

a) Take the Laplace transform:

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + 6\left[sY(s) - y(0)\right] + 9Y(s) = \frac{s}{s^2+1}$$

$$(s^2 + 6s + 9)Y(\mathrm{s}) - \mathrm{s}(1) - 2 - (6)(1) = \frac{s}{s^2+1}$$

$$(s^2 + 6s + 9)Y(s) = \frac{s}{s^2+1} + s + 8$$

$$(s^2 + 6s + 9)Y(\mathrm{s}) = \frac{s + s^3 + s + 8s^2 + 8}{s^2+1}$$

$$Y(s) = \frac{s^3 + 8s^2 + 2s + 8}{(s+3)^2 (s^2 + 1)}$$

To find $y(t)$ we have to expand $Y(s)$ into its partial fractions

$$Y(s) = \frac{A}{(s+3)^2} + \frac{B}{s+3} + \frac{Cs}{s^2+1} + \frac{D}{s^2+1}$$

$$y(t) = Ate^{-3t} + Be^{-3t} + C\cos t + D\sin t$$

b) $$Y(s) = \frac{s+1}{s(s^2+4s+8)}$$

Since $\frac{4^2}{4} < 8$, there are complex factors.

$\therefore$ complete the square in denominator

$$s^2 + 4s + 8 = s^2 + 4s + 4 + 8 - 4$$

$$= s^2 + 4s + 4 + 4 = (s+2)^2 + (2)^2 \qquad \{ b = 2 \ , \ \omega = 2 \}$$

$\therefore$ Partial fraction expansion gives

$$Y(s) = \frac{A}{s} + \frac{B(s+2)}{s^2+4s+8} + \frac{C}{s^2+4s+8} = \frac{s+1}{s(s^2+4s+8)}$$

Multiply by s and let $s \rightarrow 0$

$$A = 1/8$$

Multiply by $s(s^2+4s+8)$

$$A(s^2+4s+8) + B(s+2)s + Cs = s + 1$$

$$As^2 + 4As + 8A + Bs^2 + 2Bs + Cs = s + 1$$

s^2: $\quad A + B = 0 \quad \rightarrow \quad B = -A = -\dfrac{1}{8}$

s^1: $\quad 4A + 2B + C = 1 \quad \rightarrow \quad C = 1 + 2\left(\dfrac{1}{8}\right) - 4\left(\dfrac{1}{8}\right) = \dfrac{3}{4}$

s^0: $\quad 8A = 1 \quad \rightarrow A = \dfrac{1}{8}$ (This checks with above result)

$$Y(s) = \frac{1/8}{s} + \frac{(-1/8)(s+2)}{(s+2)^2 + 2^2} + \frac{3/4}{(s+2)^2 + 2^2}$$

$$y(t) = \left(\frac{1}{8}\right) - \left(\frac{1}{8}\right) e^{-2t} \cos 2t + \left(\frac{3}{8}\right) e^{-2t} \sin 2t$$