Industrial Control Systems

Chapter Two:
Mathematical Modeling

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Types of Systems

- Static System: If a system does not change with time, it is called a static system.
- Dynamic System: If a system changes with time, it is called a dynamic system.

Dynamic Systems

- A system is said to be dynamic if its current output may depend on the past history as well as the present values of the input variables.
- Mathematically:

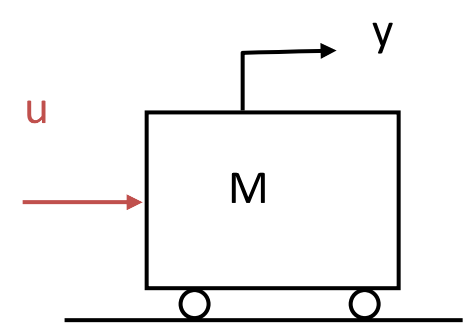
$$y(t) = \varphi[u(\tau), 0 \le \tau \le t]$$

u : Input, t : Time

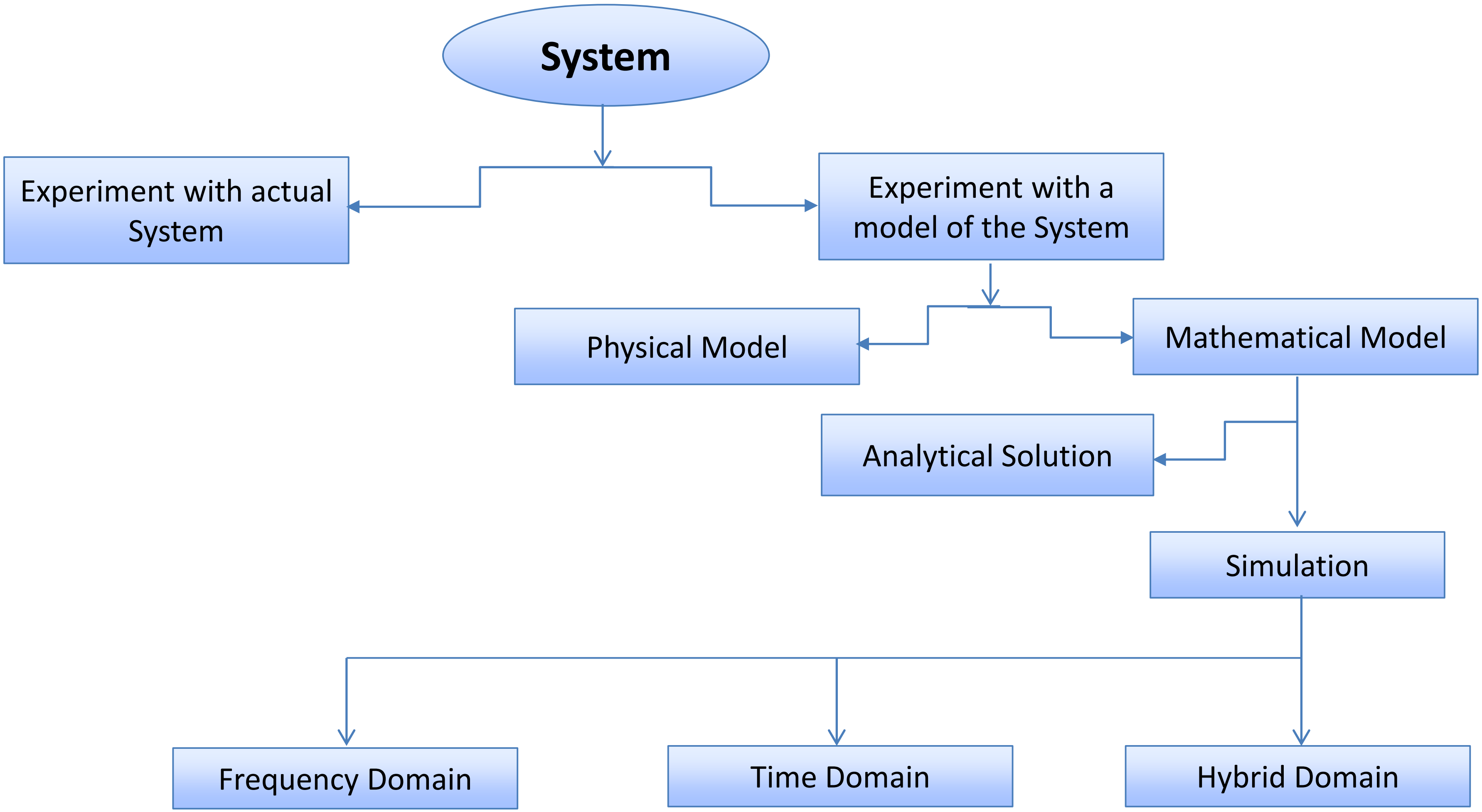
Example: A moving mass

Model: Force=Mass x Acceleration

$$M\ddot{y} = u$$



Ways to Study a System



Model

- "Dictionary.com" defines a model as *"A systematic description of an object or phenomenon that shares important characteristics with the object or phenomenon."*
- So, models present a systematic, and most often simplified description of what they represent.
- Such a description is a helpful instrument to study the characteristics of what the model represents.

Model

- A model is a simplified representation or abstraction of reality.
- Reality is generally too complex to copy exactly.
- Much of the complexity is actually irrelevant in problem solving.

Mathematical Model

- A set of mathematical equations (e.g., differential eqs.) that describes the input-output behaviour of a system.
- Mathematical models of physical systems are key elements in the design and analysis of control systems.
- The dynamic behaviour is generally described by ordinary differential equations.
- The differential equations describing the dynamic performance of a physical system are obtained by utilizing the physical laws of the process

Mathematical Model

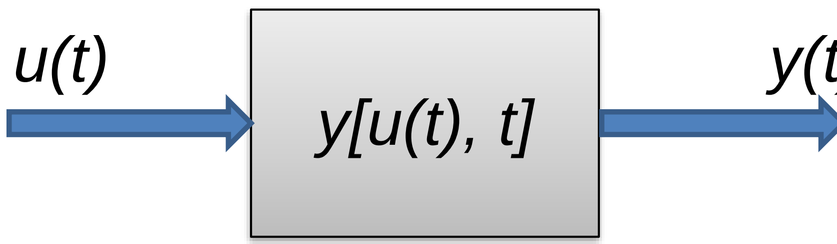
- What is a model used for?
- Simulation
- Prediction/Forecasting
- Prognostics/Diagnostics
- Design/Performance Evaluation
- Control System Design

Black Box Model

- When only input and output are known.
- Internal dynamics are either too complex or unknown.
- Easy to Model

Grey Box Model

- When input and output and some information about the internal dynamics of the system is known.
- Easier than white box Modelling.



White Box Model

- When input and output and internal dynamics of the system is known.
- One should know have complete knowledge of the system to derive a white box model.

u(t) →

$$\frac{dy(t)}{dt} = 3\frac{du(t)}{dt} - \frac{d^2y(t)}{dt^2}$$

→ *y(t)*

Six Step Approach to Dynamic System Modeling

- Define the system and its components.
- Formulate the mathematical model and list the necessary assumptions.
- Write the differential equations describing the model.
- Solve the equations for the desired output variables.
- Examine the solutions and the assumptions.
- If necessary, reanalyse or redesign the system.

Mathematical modeling of Physical Systems (Electrical Systems)

Basic Elements of Electrical Systems

Symbol


- The time domain expression relating voltage and current for the resistor is given by Ohm’s law i-e

$$v_R(t) = i_R(t)R$$

- The Laplace transform of the above equation is

$$V_R(s) = I_R(s)R$$

Basic Elements of Electrical Systems

- The time domain expression relating voltage and current for the Capacitor is given as:

$$v_c(t) = \frac{1}{C}\int i_c(t)dt$$

- The Laplace transform of the above equation (assuming there is no charge stored in the capacitor) is

$$V_c(s) = \frac{1}{Cs} I_c(s)$$

Basic Elements of Electrical Systems

- The time domain expression relating voltage and current for the inductor is given as:

$$v_L(t) = L\frac{di_L(t)}{dt}$$

- The Laplace transform of the above equation (assuming there is no energy stored in inductor) is

$$V_L(s) = LsI_L(s)$$

V-I and I-V relations

Component	Symbol	V-I Relation	I-V Relation
Resistor		$v_R(t) = i_R(t)R$	$i_R(t) = \frac{v_R(t)}{R}$
Capacitor		$v_c(t) = \frac{1}{C}\int i_c(t)dt$	$i_c(t) = C\frac{dv_c(t)}{dt}$
Inductor		$v_L(t) = L\frac{di_L(t)}{dt}$	$i_L(t) = \frac{1}{L}\int v_L(t)dt$

Electrical Systems

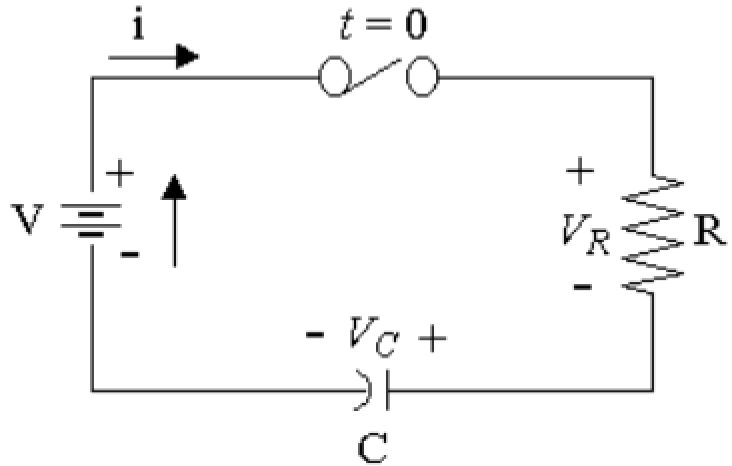
Kirchhoff's Current Law

$$\sum_k i_k(t) = 0$$

Kirchhoff's Voltage Law

$$\sum_k v_k(t) = 0$$

RC Circuit



An RC series circuit

$$V_R = Ri$$

and

$$V_C = \frac{1}{C}\int i\,dt$$

Kirchhoff's voltage law says the total voltages must be zero. So applying this law to a series RC circuit results in the equation:

$$Ri + \frac{1}{C}\int i\,dt = V$$

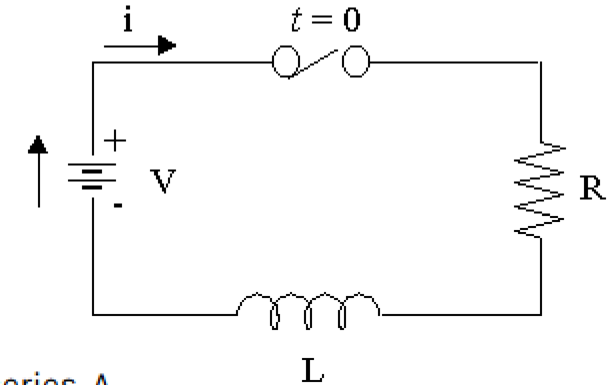
One way to solve this equation is to turn it into a **differential equation**, by differentiating throughout with respect to t:

$$R\frac{di}{dt} + \frac{i}{C} = 0$$

Solving the equation gives us:

$$i = \frac{V}{R}e^{-t/RC}$$

RL Circuit



RL circuit diagram

The RL circuit shown above has a resistor and an inductor connected in series. A constant voltage V is applied when the switch is closed.

The (variable) voltage across the **resistor** is given by:

$$V_R = iR$$

The (variable) voltage across the **inductor** is given by:

$$V_L = L\frac{di}{dt}$$

Kirchhoff's voltage law says that the directed sum of the voltages around a circuit must be zero. This results in the following differential equation:

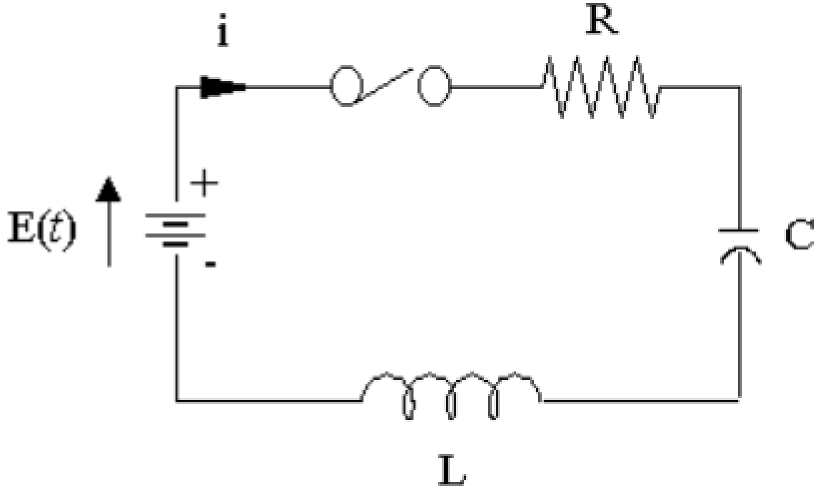
$$Ri + L\frac{di}{dt} = V$$

The solution of the differential equation $Ri + L\frac{di}{dt} = V$ is:

$$i = \frac{V}{R}\left(1 - e^{-(R/L)t}\right)$$

Once the switch is closed, the current in the circuit is not constant. Instead, it will build up from zero to some steady state.

RLC Circuit



Consider a series RLC circuit (one that has a resistor, an inductor and a capacitor) with a **constant** driving electro-motive force (emf) E. The current equation for the circuit is

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\ dt = E$$

This is equivalent: $L\frac{di}{dt} + Ri + \frac{1}{C}q = E$

Differentiating, we have

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = 0$$

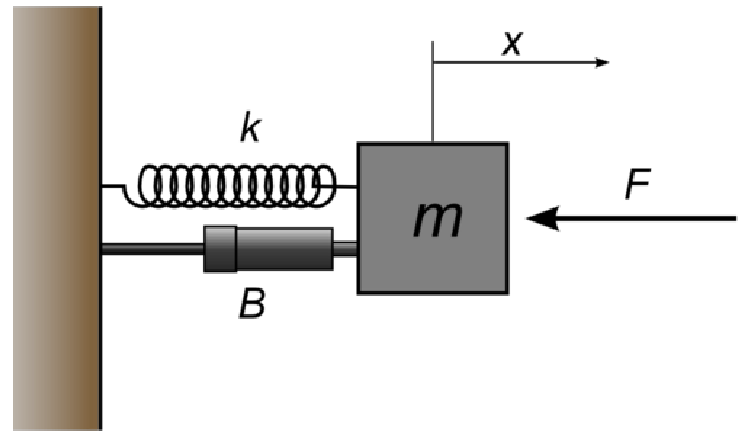
Mathematical modeling of Physical Systems (Mechanical Systems)

- **Part-I:** Translational Mechanical System
- **Part-II:** Rotational Mechanical System

Basic Types of Mechanical Systems

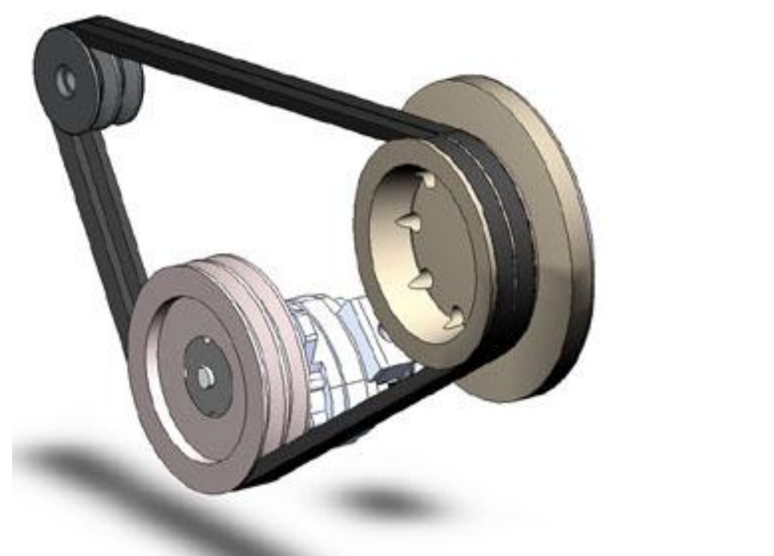
Translational

Linear Motion



Rotational

Rotational Motion



Part-I

TRANSLATIONAL MECHANICAL SYSTEMS

Basic Elements of Translational Mechanical Systems

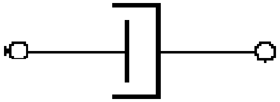
Translational Spring

i)

Translational Mass

ii)

iii) **Translational Damper**

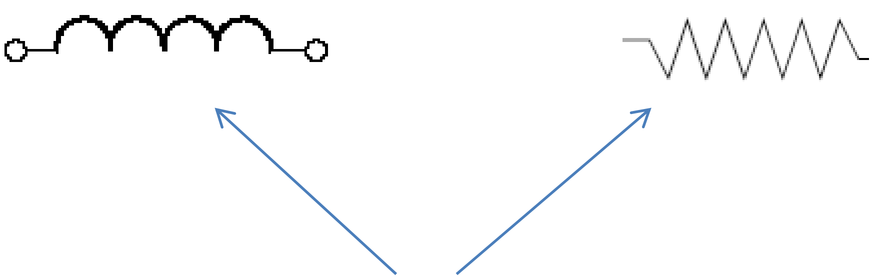


Translational Spring

A translational spring is a mechanical element that can be deformed by an external force such that the deformation is directly proportional to the force applied to it.

Translational Spring

i)

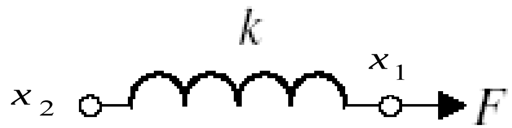


Circuit Symbols

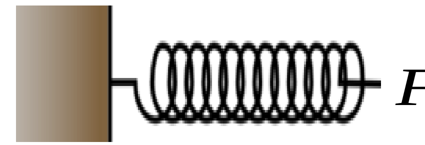
Translational Spring

Translational Spring

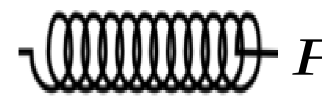
If F is the applied force



Then x_1 is the deformation if $x_2 = 0$



Or $(x_1 - x_2)$ is the deformation.



The equation of motion is given as

$$F = k(x_1 - x_2)$$

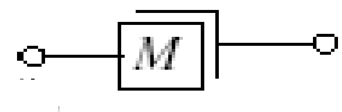
Where k is stiffness of spring expressed in N/m

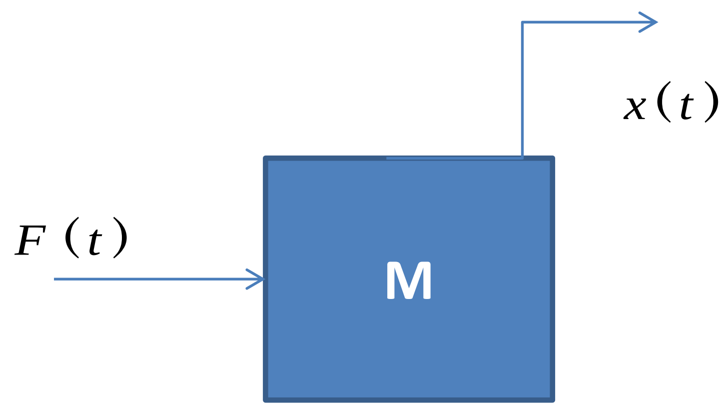
Translational Mass

- Translational Mass is an inertia element.
- A mechanical system without mass does not exist.
- If a force *F* is applied to a mass and it is displaced to *x* meters then the relation b/w force and displacements is given by Newton's law.

$$F = M\ddot{x}$$

ii) **Translational Mass**





Translational Damper

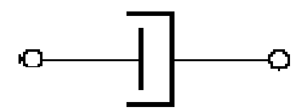
- When the viscosity or drag is not negligible in a system, we often model them with the damping force.

- All the materials exhibit the property of damping to some extent.

- If damping in the system is not enough then extra elements (e.g. Dashpot) are added to increase damping.

iii)

Translational Damper



Common Uses of Dashpots

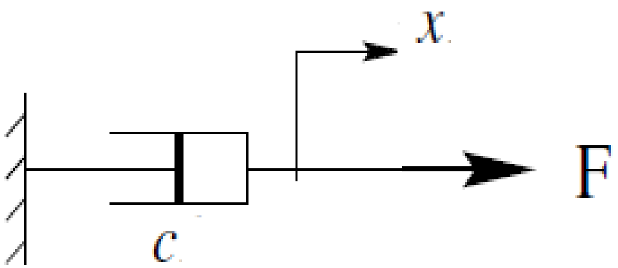
Door Stoppers

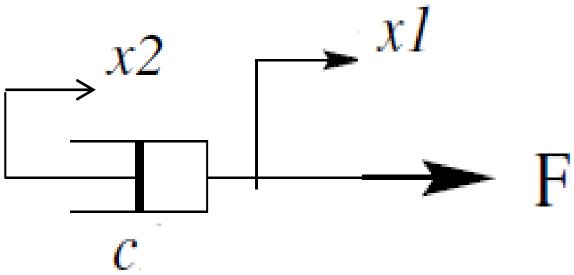
Vehicle Suspension

Bridge Suspension

Flyover Suspension

Translational Damper





$$F = C\dot{x}$$

$$F = C(\dot{x}_1 - \dot{x}_2)$$

- Where *C* is damping coefficient (*N/ms*$^{-1}$).

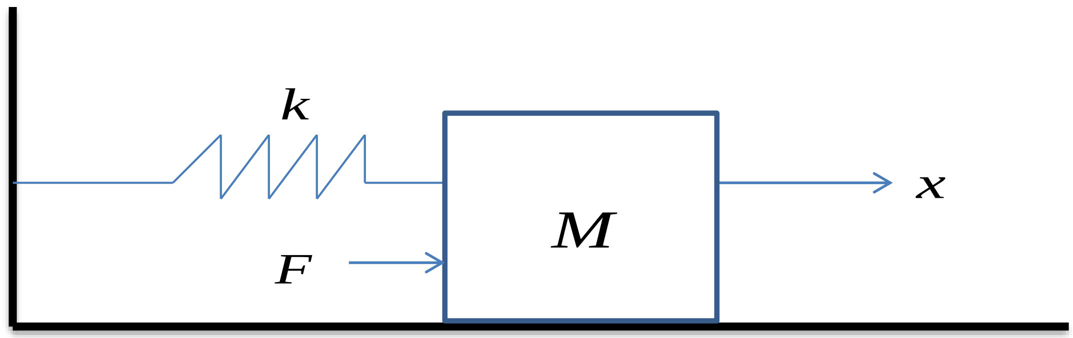
Mechanical Translational Systems

Newton's 2nd Law for Translational Systems

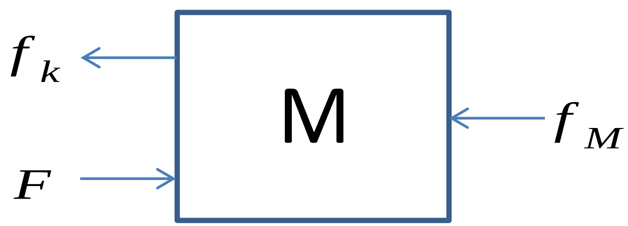
$$\sum_k F_k(t) = M\,\ddot{x}(t)$$

Mass-Spring System

Consider the following system (friction is negligible)

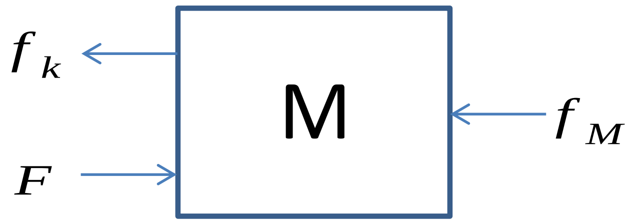


Free Body Diagram



Where f_k and f_M are forces applied by the spring and inertial force respectively.

Mass-Spring System

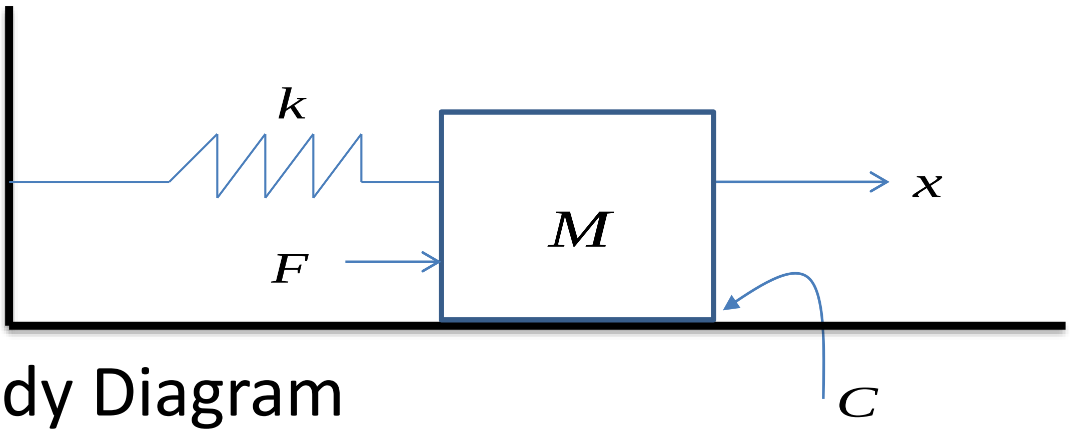


Then the differential equation of the system is:

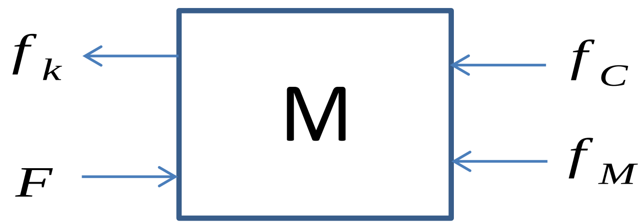
$$F = M\ddot{x} + kx$$

Mass-Spring-Friction System

Consider the following system



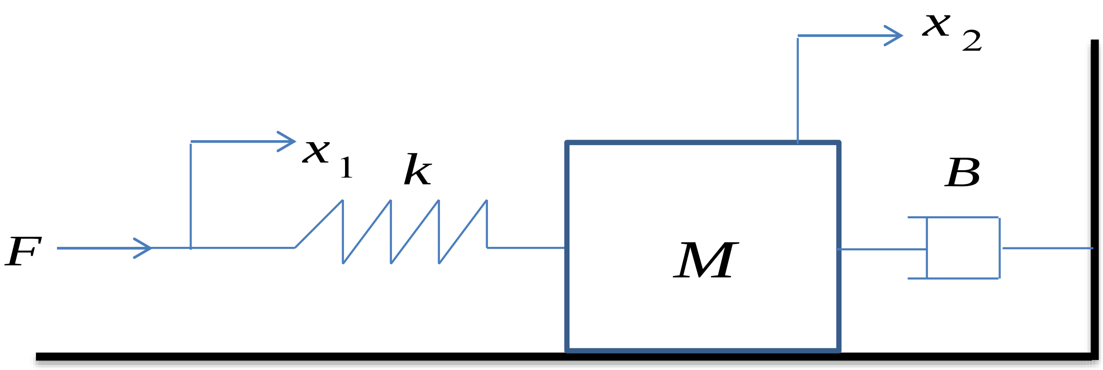
- Free Body Diagram



$$F = f_k + f_M + f_C$$

Mass-Spring- Damper System

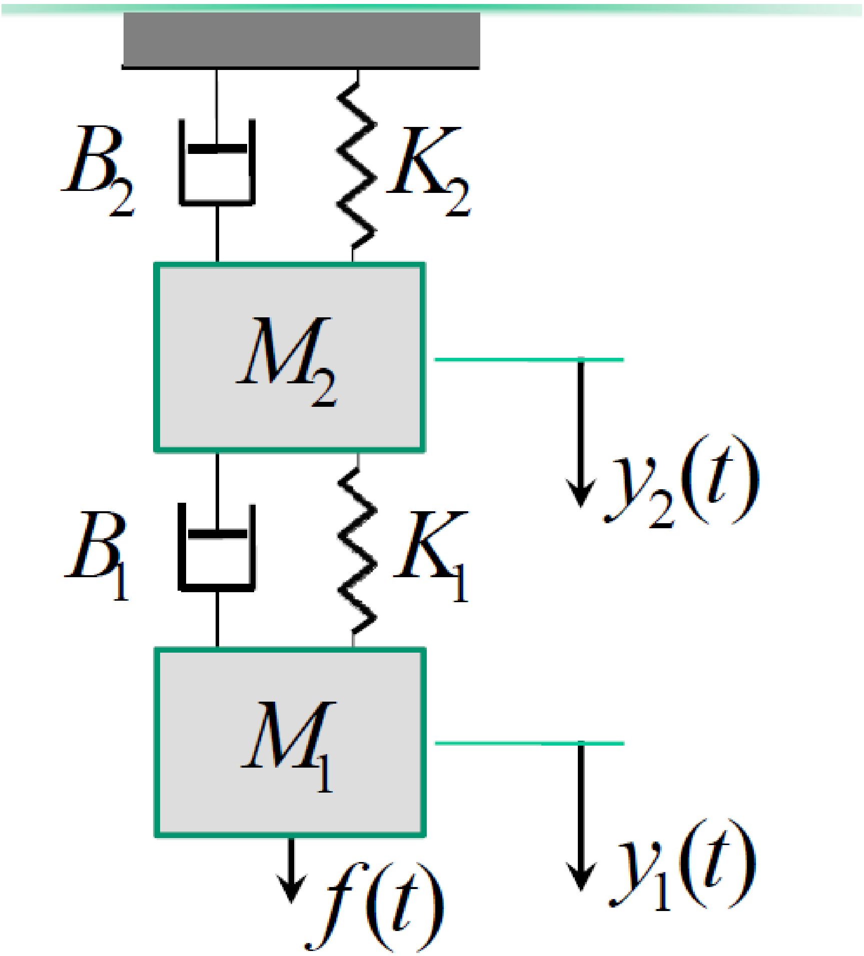
Consider the following system



$$F = k(x_1 - x_2)$$

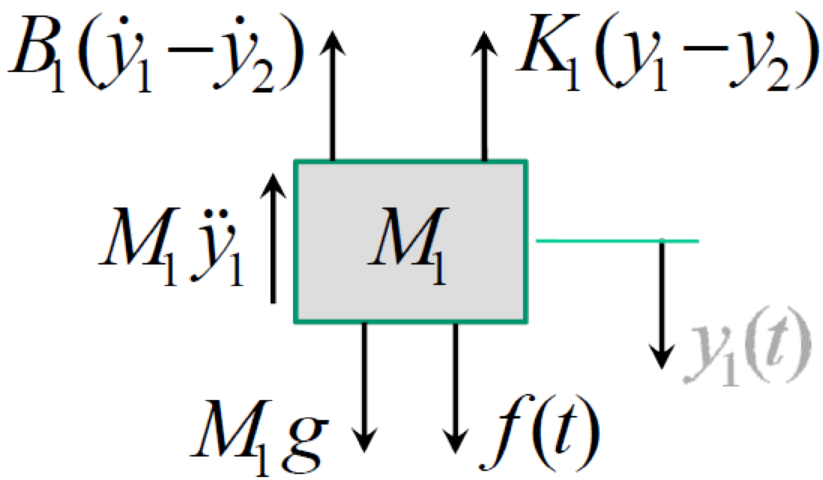
$$0 = k(x_2 - x_1) + M\ddot{x}_2 + B\dot{x}_2$$

Mechanical System



Mechanical System

Free-body diagram for Mass 1:

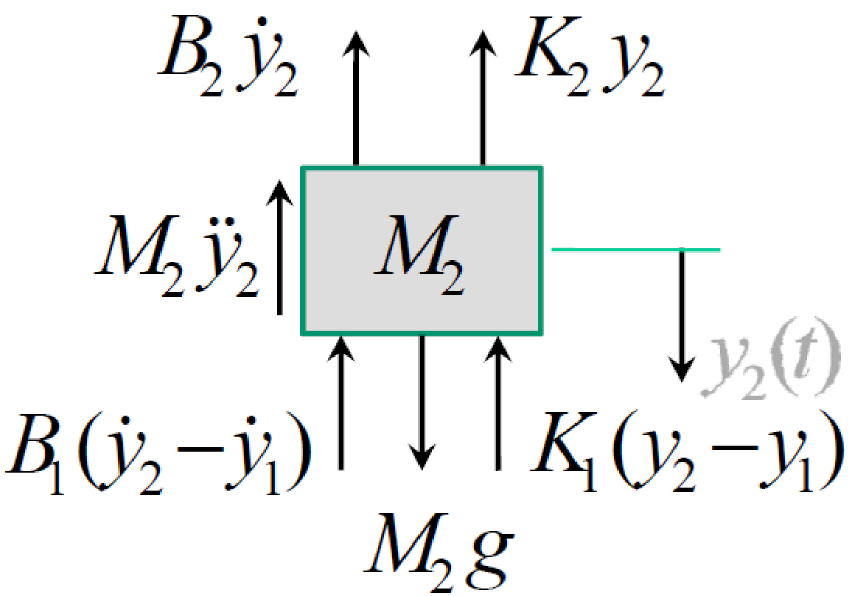


$$-M_1\ddot{y}_1 - B_1(\dot{y}_1-\dot{y}_2) - K_1(y_1-y_2) + M_1 g + f(t) = 0$$

$$M_1\ddot{y}_1 + B_1\dot{y}_1 + K_1 y_1 - B_1\dot{y}_2 - K_1 y_2 = M_1 g + f(t)$$

Mechanical System

Free-body diagram for Mass 2:



$$-M_2\ddot{y}_2-B_2\dot{y}_2-K_2y_2-B_1(\dot{y}_2-\dot{y}_1)$$
$$-K_1(y_2-y_1)+M_2g=0$$

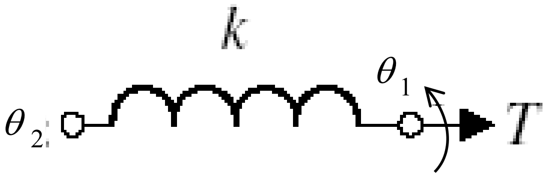
$$-B_1\dot{y}_1-K_1y_1+M_2\ddot{y}_2+(B_1+B_2)\dot{y}_2+(K_1+K_2)y_2=M_2g$$

Part-II

ROTATIONAL MECHANICAL SYSTEMS

Basic Elements of Rotational Mechanical Systems

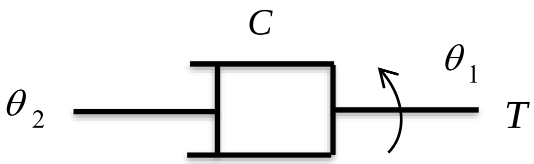
Rotational Spring



$$T = k(\theta_1 - \theta_2)$$

Basic Elements of Rotational Mechanical Systems

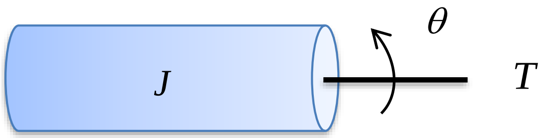
Rotational Damper



$$T = C(\dot{\theta}_1 - \dot{\theta}_2)$$

Basic Elements of Rotational Mechanical Systems

Moment of Inertia



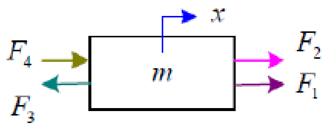
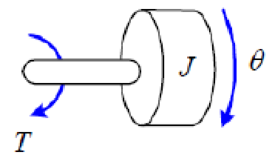
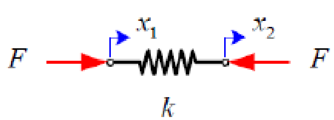
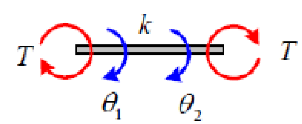
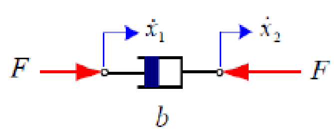
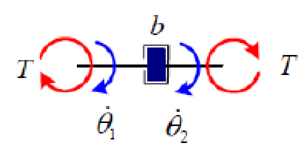
$$T = J\ddot{\theta}$$

Mechanical Rotational Systems

Newton's 2nd Law for Rotational Systems

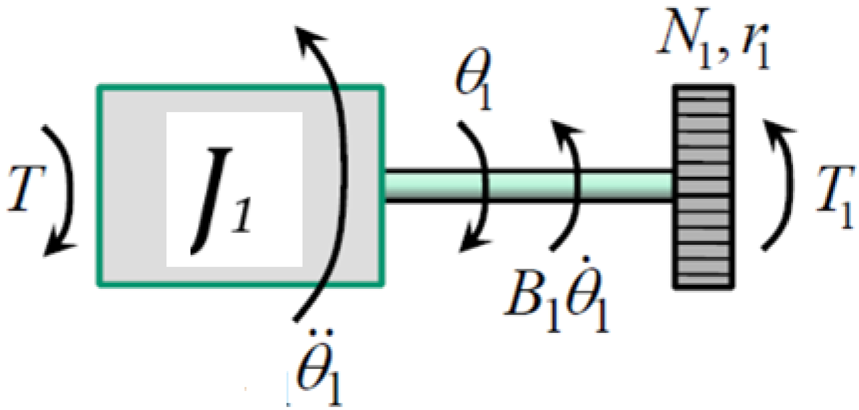
$$\sum_k T_k(t) = J\,\ddot{\theta}(t)$$

Basic Elements of Rotational Mechanical Systems

Element	Translation	Rotation
Inertia	 $\sum F = m\,a$	 $\sum T = J\,\alpha$
Spring	 $F = k(x_1 - x_2) = kx$	 $T = k(\theta_1 - \theta_2) = k\theta$
Damper	 $F = b(\dot{x}_1 - \dot{x}_2) = b\dot{x}$	 $T = b(\dot{\theta}_1 - \dot{\theta}_2) = b\dot{\theta}$

Example 1

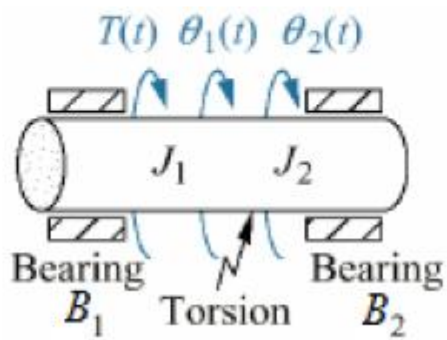
Free-body diagrams for inertia J_1 :

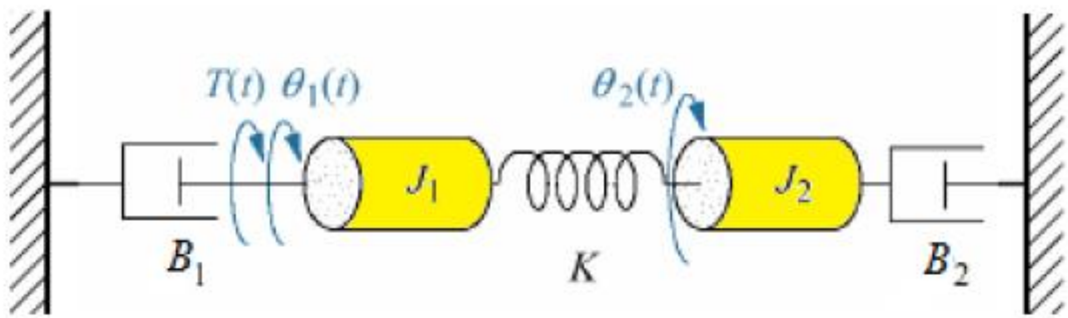


Newton's 2nd law: $T(t) - J_1\ddot{\theta}_1(t) - B_1\dot{\theta}_1(t) - T_1(t) = 0$

$$\Rightarrow \quad T(t) - J_1\ddot{\theta}_1(t) - B_1\dot{\theta}_1(t) = T_1(t)$$

Example 2



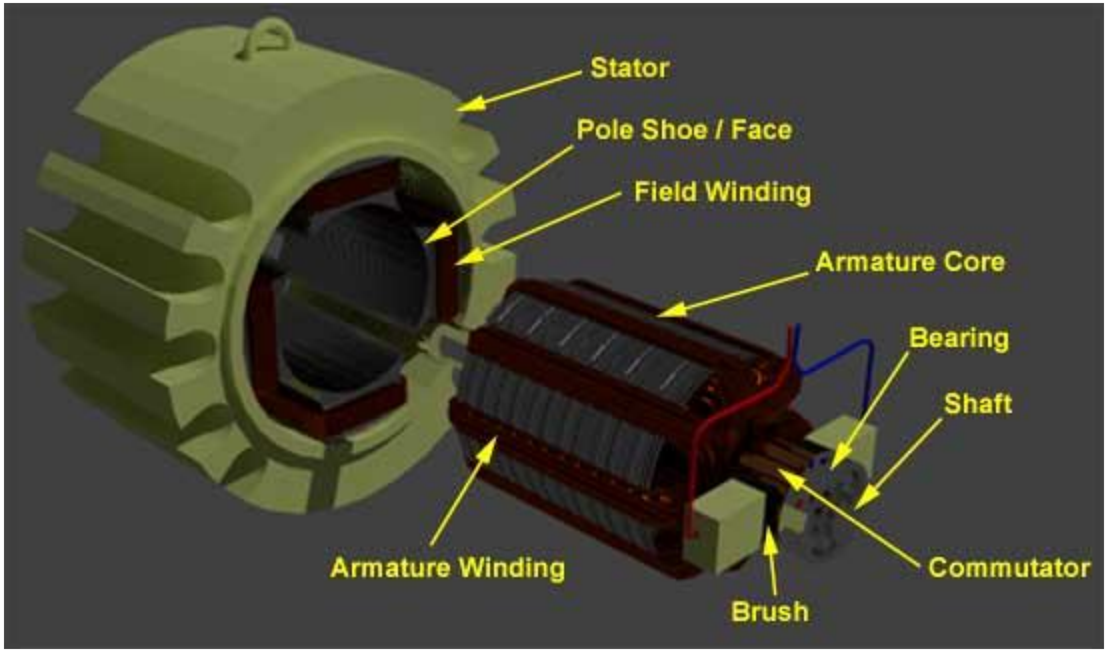


$$T(t) = J_1\ddot{\theta}_1 + B_1\dot{\theta}_1 + k(\theta_1 - \theta_2)$$

$$0 = J_2\ddot{\theta}_2 + B_2\dot{\theta}_2 + k(\theta_2 - \theta_1)$$

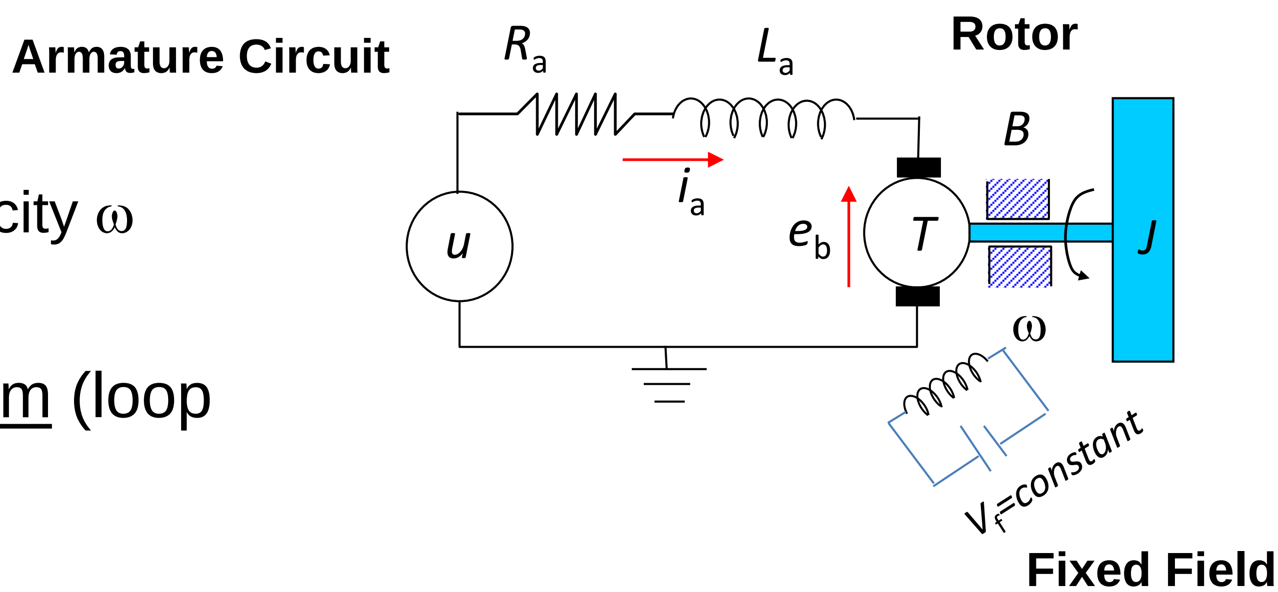
Modeling of Electromechanical Systems

D.C Drives



- Variable Voltage can be applied to the armature terminals of the DC motor.
- Another method is to vary the flux per pole of the motor.
- The first method involves adjusting the motor's armature, while the latter method involves changing the motor field. These methods are referred to as "armature control" and "field control."

Armature Controlled D.C Motor



Input: voltage *u*
Output: Angular velocity ω

Electrical Subsystem (loop method):

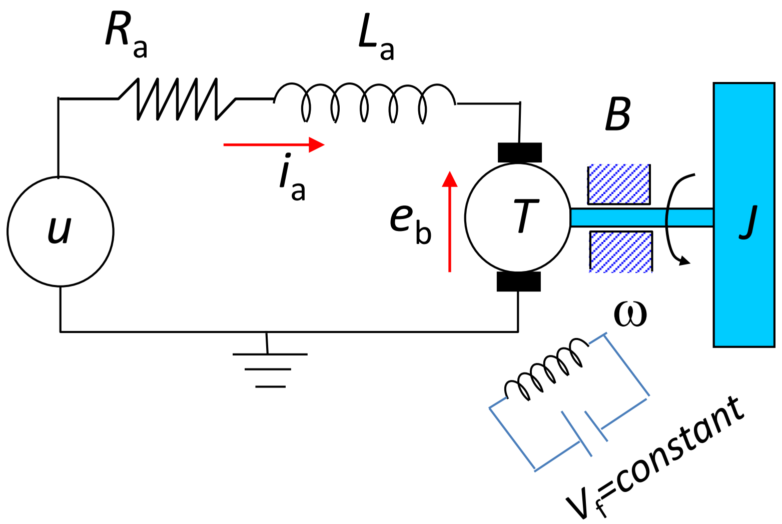
$$u = R_a i_a + L_a \frac{di_a}{dt} + e_b, \quad \text{where } e_b = \text{back-emf voltage}$$

Mechanical Subsystem

$$T_{motor} = J\dot{\omega} + B\omega$$

Back electromotive force (back emf) is a voltage generated by a rotating DC motor that opposes the supply voltage.

Armature Controlled D.C Motor



Power Transformation:

Torque-Current: $T_{motor} = K_t i_a$

Voltage-Speed: $e_b = K_b \omega$

- Combing previous equations results in the following mathematical model:

$$\begin{cases} L_a \dfrac{di_a}{dt} + R_a i_a + K_b \omega = u \\ J\dot{\omega} + B\omega - K_t i_a = 0 \end{cases}$$