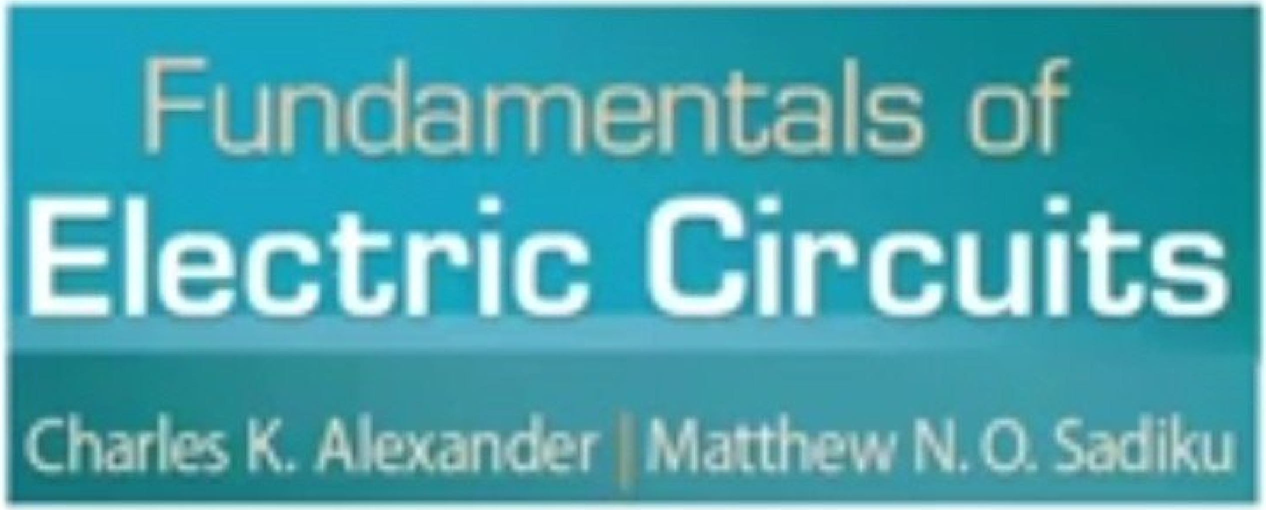
Chapter 6: Capacitors and Inductors

Lecture#1

Reference:



Fundamentals of
Electric Circuits
Charles K. Alexander | Matthew N. O. Sadiku

☐ Two new *linear* (passive) circuit elements will be introduced:

✓ The capacitor.

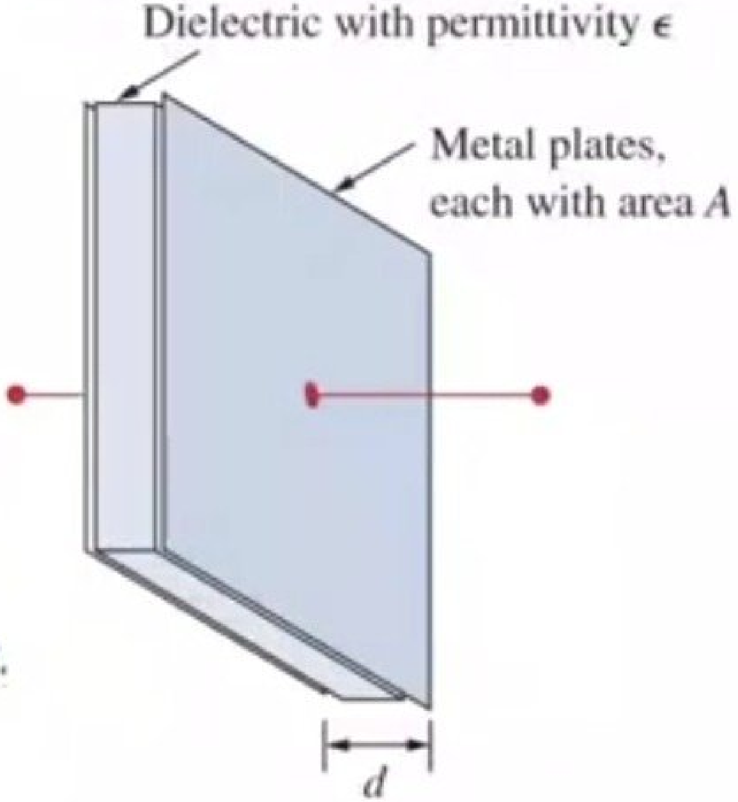
✓ The inductor.

☐ They do not dissipate energy, but instead, they store energy.

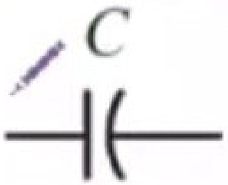
☐ Called storage elements.

☐ Practical circuit application are composed of resistor, capacitors and inductors.

Capacitors



- It is a passive element.
- It stores energy in its electric field.
- It consists of two conducting plates separated by an insulator (or dielectric).
- The plates are typically aluminum foil.
- The dielectric is often air, ceramic, paper, plastic, or mica.
- Each capacitor has a capacitance C measured in Farads (F).
- The symbol:



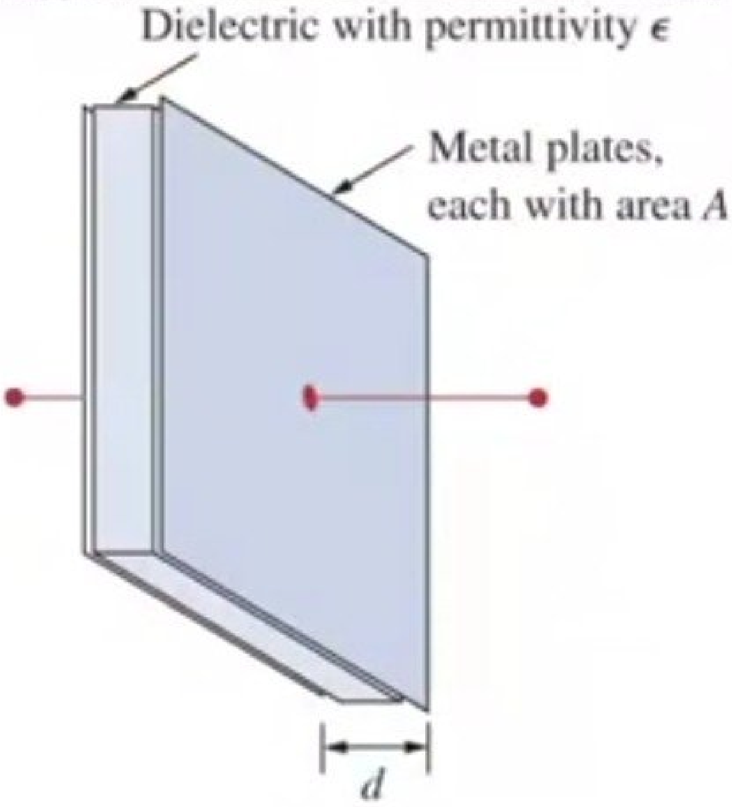
Capacitors

- ❑ **1 F = 1 Coulomb/Volt.**
- ❑ **Most capacitors are rated in pF and μF.**
- ❑ **Capacitance is determined by the *geometery* of the capacitor:**
  - ✓ **Proportional to the area of the plates (A).**
  - ✓ **Inversely proportional to the space between them (d).**

$$C = \frac{\varepsilon A}{d}$$

where ε is the permittivity of the dielectric.

- **<u>Note</u>:** **more details will be covered in EM courses.**

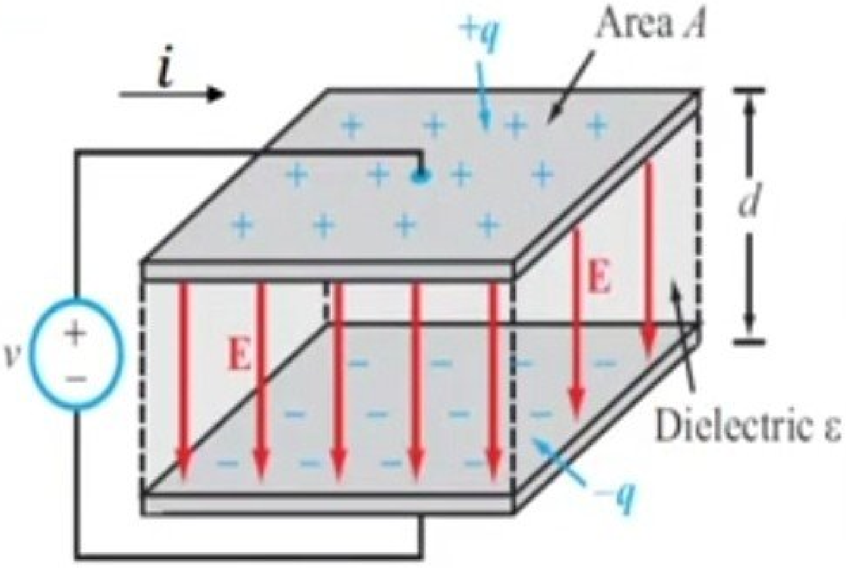


Capacitors

☐ **A capacitor stores charges (energy) on its plates! How??**

- When a voltage source v is connected to the capacitor, the source deposits a positive charge q on one plate and a negative charge $-q$ on the other (these charges will be equal in magnitude).
- The amount of stored charge is <u>proportional</u> to the voltage:

$$\boxed{q = Cv}$$



- In general $q = q(t)$ and $v = v(t)$, and $i = i(t)$.
- **<u>Note</u>**: if $v = V$ (DC), then $q = Q$ and $i = 0$,i.e., the capacitor = O.C (see later!)

<u>Note</u>: The value of C does not depend on v or q.

Capacitors

- ☐ **For capacitor Types, shapes, and applications see the textbook!**

(a) (b) (c)

Courtesy of Tech America

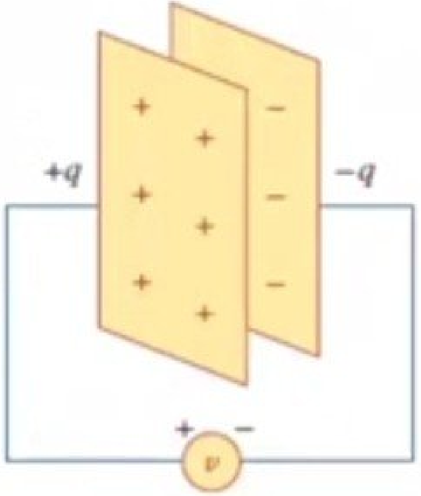
Capacitor Current-Voltage Relationship

- By taking the first derivative of

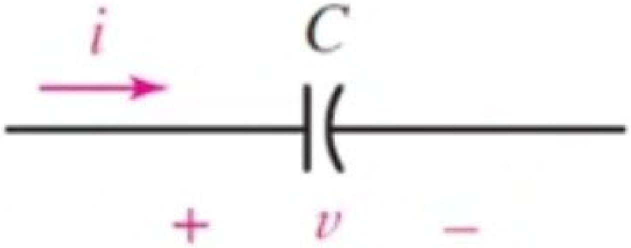
$$q(t) = Cv(t)$$

with respect to time t, we can have:

$$\boxed{i = C\frac{dv}{dt}}$$

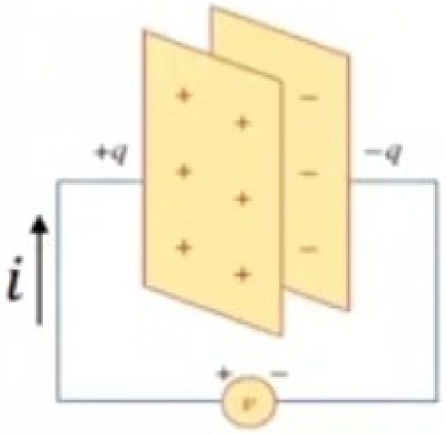


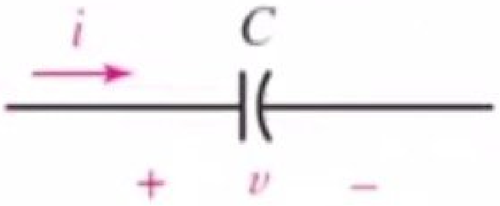
- **Note**: this assumes the passive sign convention.



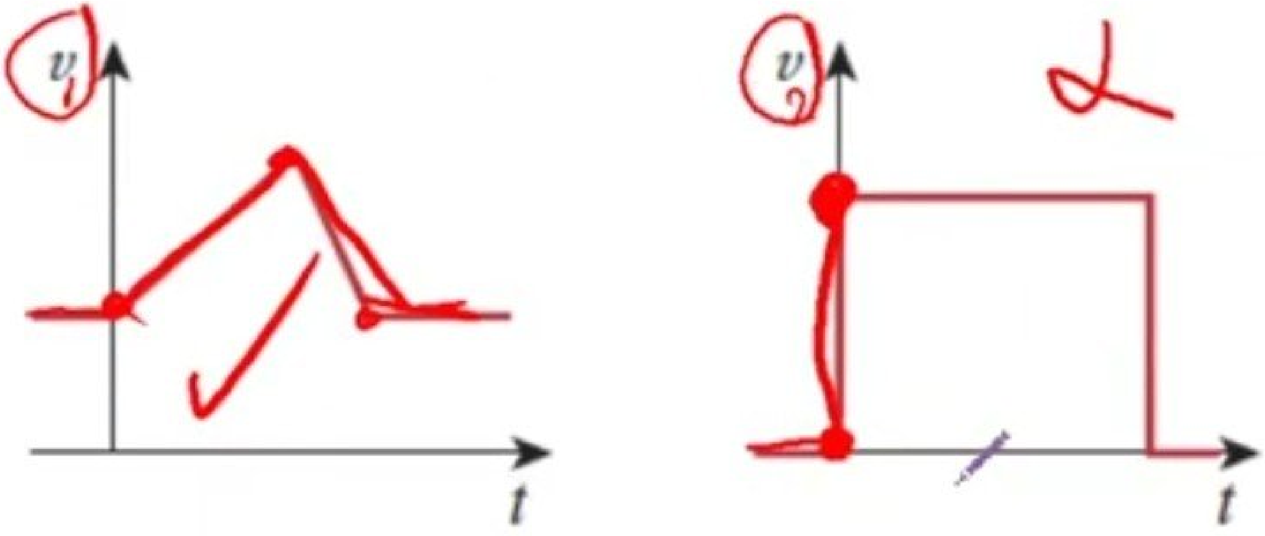
Note: if $vi > 0$ → the capacitor is charging.
if $vi < 0$ → the capacitor is discharging

- If $v = V$ (DC), then $i = C\dfrac{dv}{dt} = 0$.
- Also $q = Q$ (constant stored charge).
- This means that with DC voltage applied to the terminals no current will flow, i.e., the capacitor is open circuit (o.c).
- However, the voltage on the capacitor's plates can't change instantaneously.
- The capacitor's voltage is <u>continuous</u> function of t
- An abrupt (sudden) change in voltage would require an infinite current (see the equation)!





Capacitor Current-Voltage Relationship



(a)

(b)

Figure 6.7

Voltage across a capacitor: (a) allowed, (b) not allowable; an abrupt change is not possible.

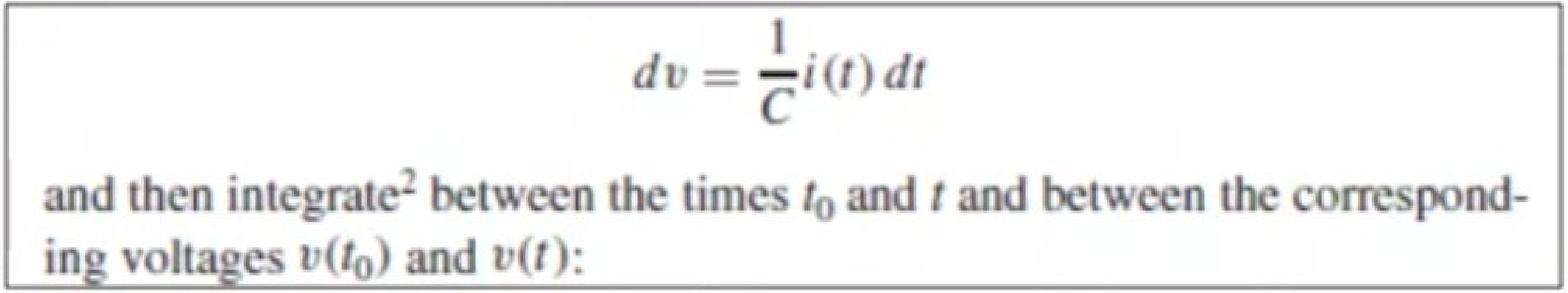
Capacitor Current-Voltage Relationship

- The cap’s voltage can be obtained from the current by:

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau + v(t_0)$$

where $v(t_o)$ is the cap’s voltage at the initial time t_o. It can be obtained from the initial stored charge by $v(t_o) = \frac{q(t_o)}{C}$.

- **Proof**:



$$dv = \frac{1}{C} i(t)\, dt$$

and then integrate[2] between the times t_0 and t and between the corresponding voltages $v(t_0)$ and $v(t)$:

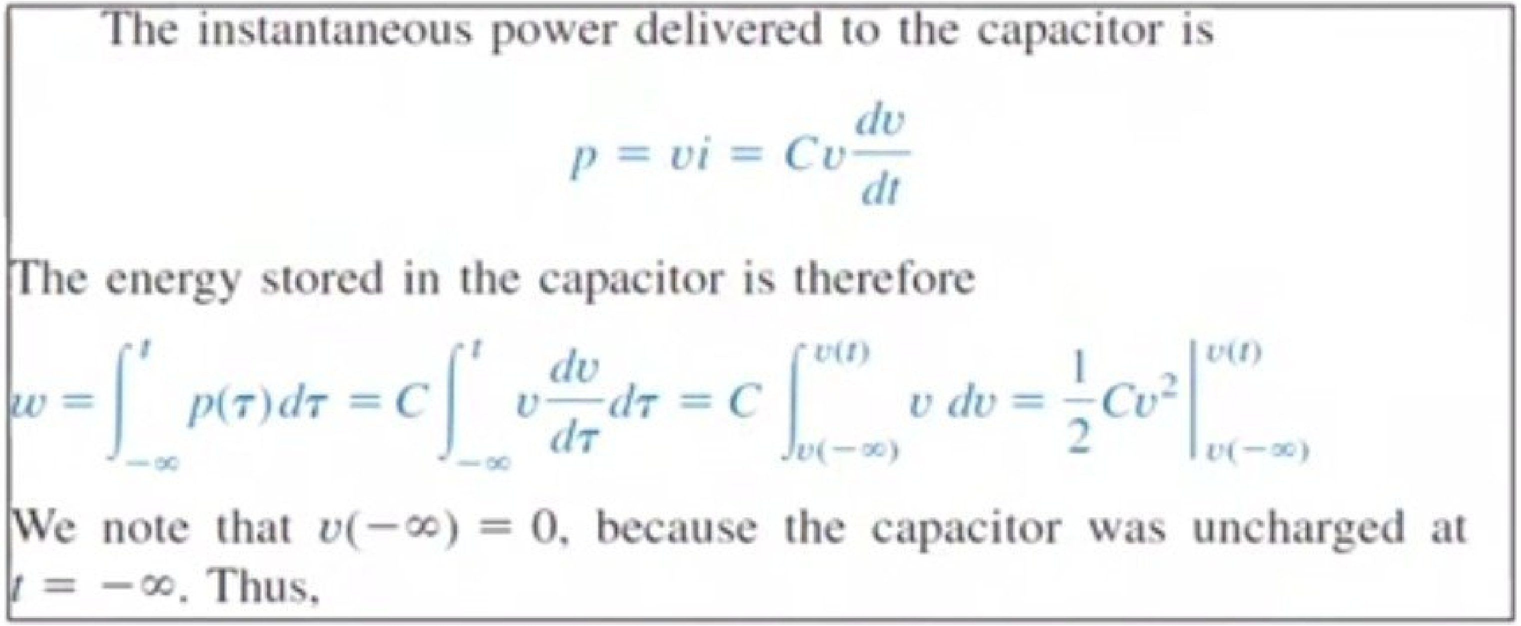
- **Note**: this shows the capacitor has a memory, which is often

Energy Stored in the Capacitor

- The energy stored in the capacitor is computed by:

$$\boxed{w = \frac{1}{2}Cv^2} \quad \xrightarrow{\text{Or}} \quad w = \frac{q^2}{2C}$$

- **Proof**:



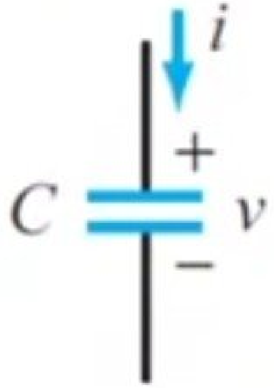
The instantaneous power delivered to the capacitor is

$$p = vi = Cv\frac{dv}{dt}$$

The energy stored in the capacitor is therefore

$$w = \int_{-\infty}^{t} p(\tau)\,d\tau = C\int_{-\infty}^{t} v\frac{dv}{d\tau}\,d\tau = C\int_{v(-\infty)}^{v(t)} v\,dv = \frac{1}{2}Cv^2\Big|_{v(-\infty)}^{v(t)}$$

We note that $v(-\infty) = 0$, because the capacitor was uncharged at $t = -\infty$. Thus,



Capacitors

Example 6.1

(a) Calculate the charge stored on a 3-pF capacitor with 20 V across it.
(b) Find the energy stored in the capacitor.

Solution:

(a) Since $q = Cv$,

$$q = 3 \times 10^{-12} \times 20 = 60 \text{ pC}$$

(b) The energy stored is

$$w = \frac{1}{2}Cv^2 = \frac{1}{2} \times 3 \times 10^{-12} \times 400 = 600 \text{ pJ}$$

What is the

Capacitors

Example 6.2

The voltage across a 5-μF capacitor is

$$v(t) = 10 \cos 6000t \text{ V}$$

Calculate the current through it.

Solution:

By definition, the current is

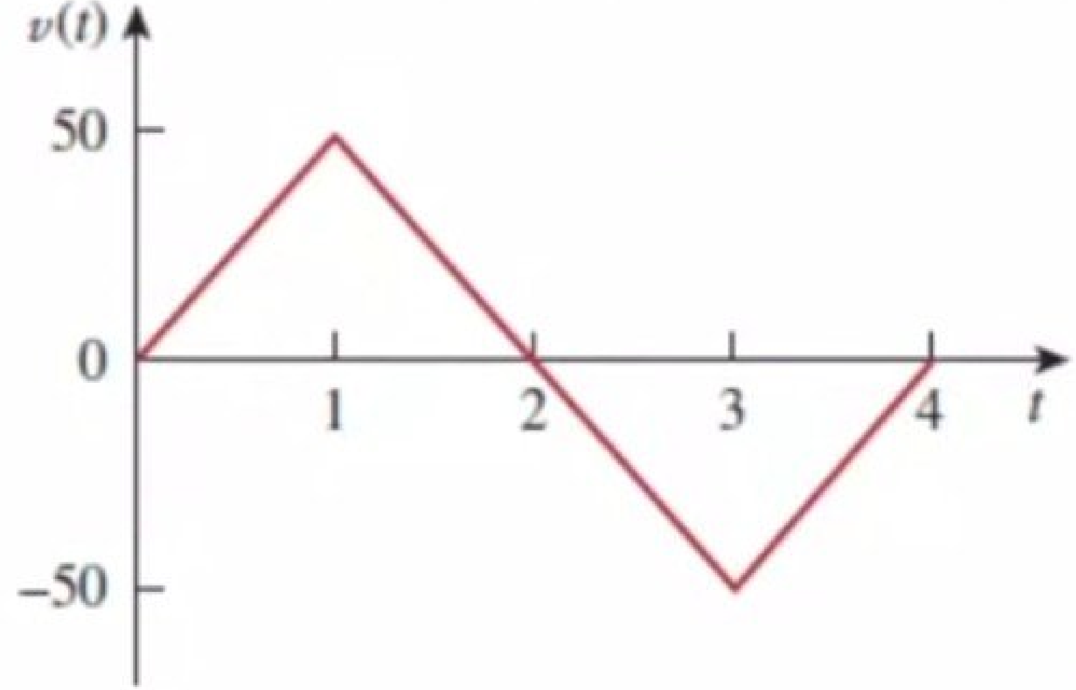
$$i(t) = C\frac{dv}{dt} = 5 \times 10^{-6}\frac{d}{dt}(10 \cos 6000t)$$

$$= -5 \times 10^{-6} \times 6000 \times 10 \sin 6000t = -0.3 \sin 6000t \text{ A}$$

Capacitors

Example 6.4

Determine the current through a 200-μF capacitor whose voltage is



Capacitors

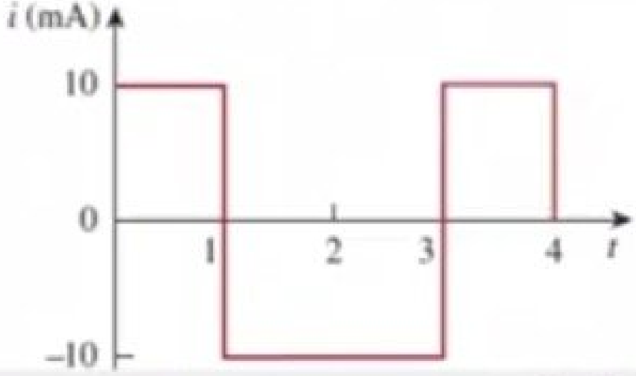
Example 6.4

Solution:

$$v(t) = \begin{cases} 50t \text{ V} & 0 < t < 1 \\ 100 - 50t \text{ V} & 1 < t < 3 \\ -200 + 50t \text{ V} & 3 < t < 4 \\ 0 & \text{otherwise} \end{cases}$$

$C = 200\ \mu\text{F}$

$$i = C\frac{dv}{dt} = 200 \times 10^{-6} \times \begin{cases} 50 & 0 < t < 1 \\ -50 & 1 < t < 3 \\ 50 & 3 < t < 4 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 10 \text{ mA} & 0 < t < 1 \\ -10 \text{ mA} & 1 < t < 3 \\ 10 \text{ mA} & 3 < t < 4 \\ 0 & \text{otherwise} \end{cases}$$



Example 6.3

Determine the voltage across a 2-μF capacitor if the current through it is

$$i(t) = 6e^{-3000t}\ \text{mA}$$

Assume that the initial capacitor voltage is zero.

Solution:

Since $v = \frac{1}{C}\int_0^t i\,dt + v(0)$ and $v(0) = 0$,

$$v = \frac{1}{2 \times 10^{-6}}\int_0^t 6e^{-3000t}\,dt \cdot 10^{-3}$$

$$= \frac{3 \times 10^{3}}{-3000}e^{-3000t}\Big|_0^t = (1 - e^{-3000t})\ \text{V}$$

Capacitors

Practice Problem 6.4

An initially uncharged 1-mF capacitor has the current shown

Find $v(t)$, $v(2ms)$, $v(5ms)$

Solution:

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau + v(t_0)$$

$$i(\tau) = \begin{cases} 5\tau, & 0 \cancel{<} \tau < 2\text{ ms} \\ 100\text{ mA}, & \tau > 2\text{ ms} \end{cases}$$

Case: $0 < t < 2$ ms:

$$\text{v(t)} = 1000\int_0^{t\text{ (ms)}} 50\tau\, d\tau + \cancel{v}(0) = 25000t^2\text{ V};\ (t\text{ in ms})$$

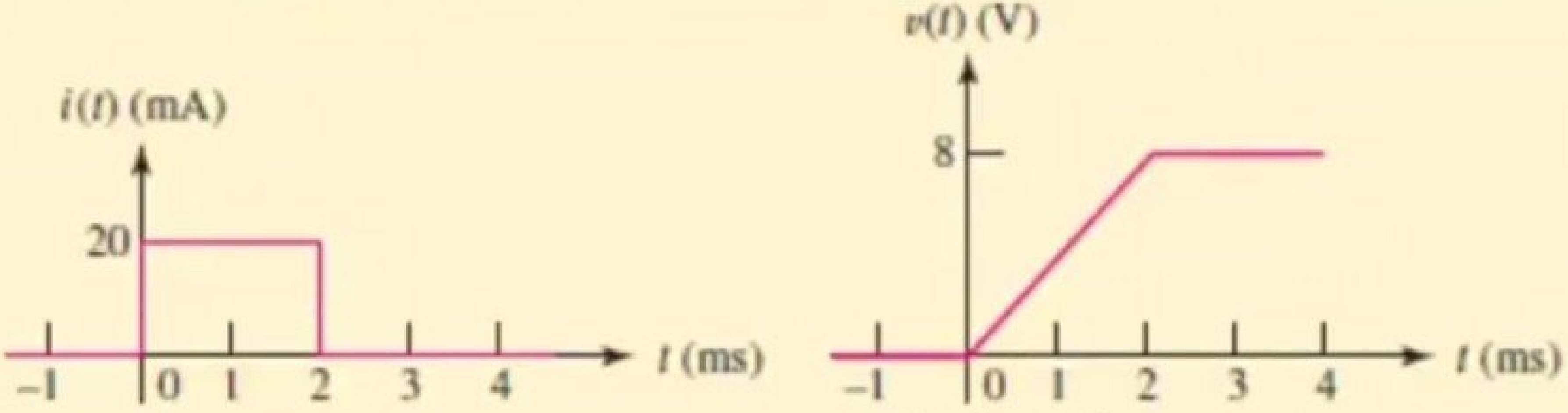
Case: 2 ms < t:

$$\text{v(t)} = 1\cancel{00}0\int_{2\text{ ms}}^{t\text{ (ms)}} 100\,\cancel{\text{m}}\text{A}\, d\tau + v(2\text{ ms}) = 100(\text{t} - 0.002) + 0.1\text{ V},\ (t\text{ in ms}$$

Capacitors

HW:

Find the capacitor voltage that is associated with the current shown graphically in Fig. 7.5*a*. The value of the capacitance is 5 μF.



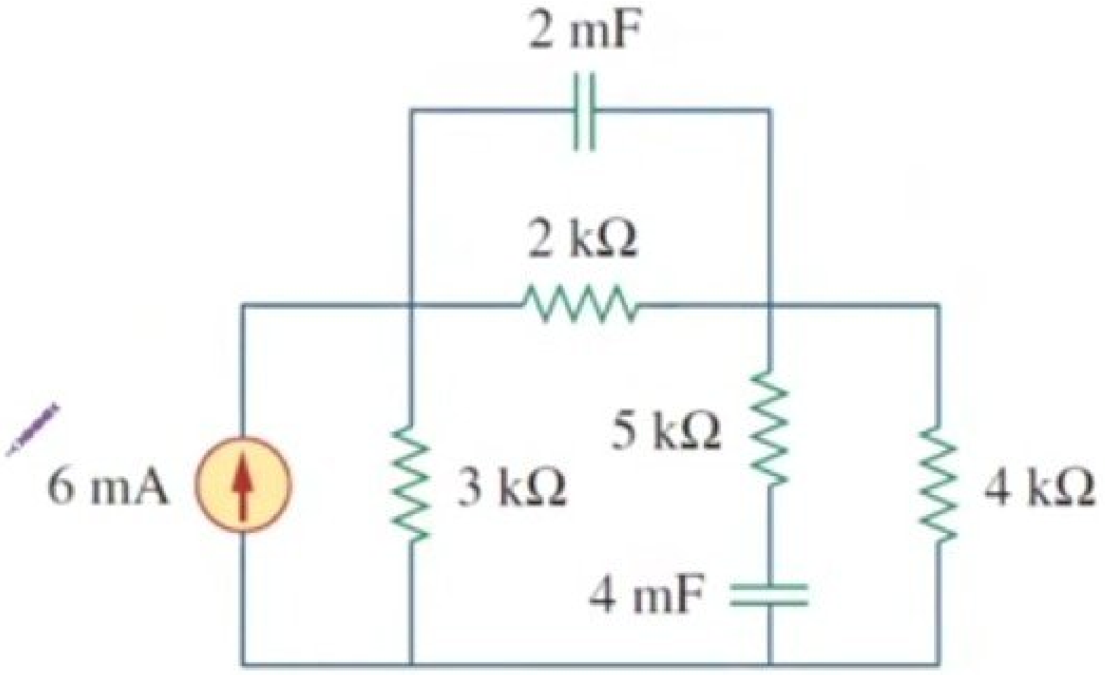
$v(t) = 0 \qquad t \leq 0$

$v(t) = 4000t \qquad 0 \leq t \leq 2\ \text{ms}$

$v(t) = 8 \qquad t \geq 2\ \text{ms}$

Capacitors

Example 6.5 Obtain the energy stored in each capacitor



Example 6.5 **Solution:**

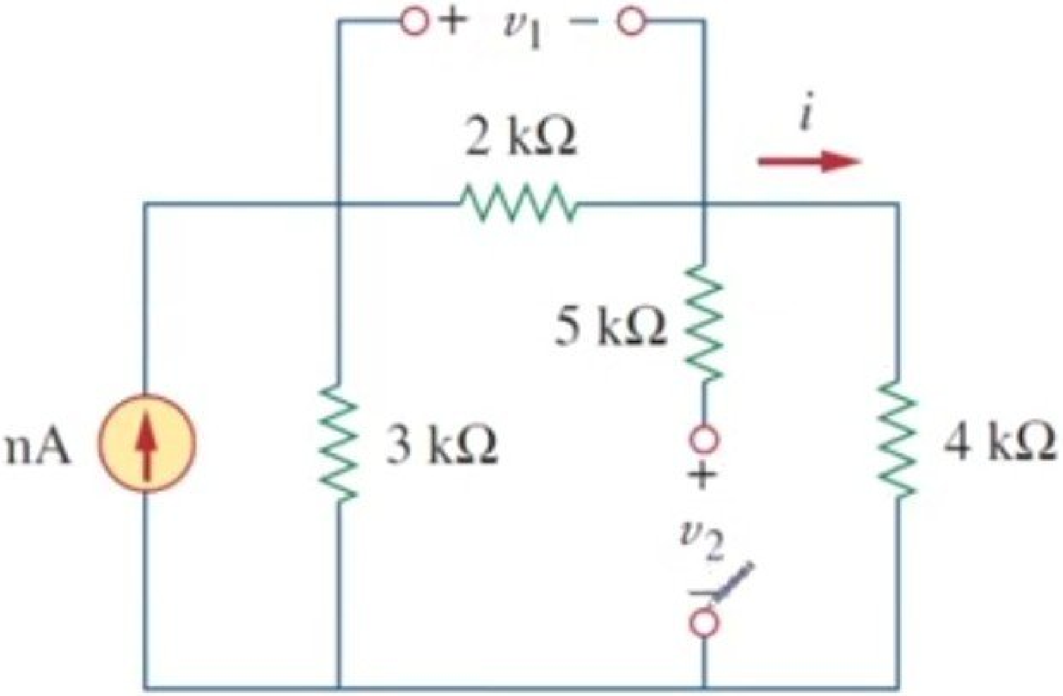
Under dc conditions, we replace each capacitor with an open circuit,

$$i = \frac{3}{3+2+4}(6 \text{ mA}) = 2 \text{ mA}$$

$$v_1 = 2000i = 4 \text{ V} \qquad v_2 = 4000i = 8 \text{ V}$$

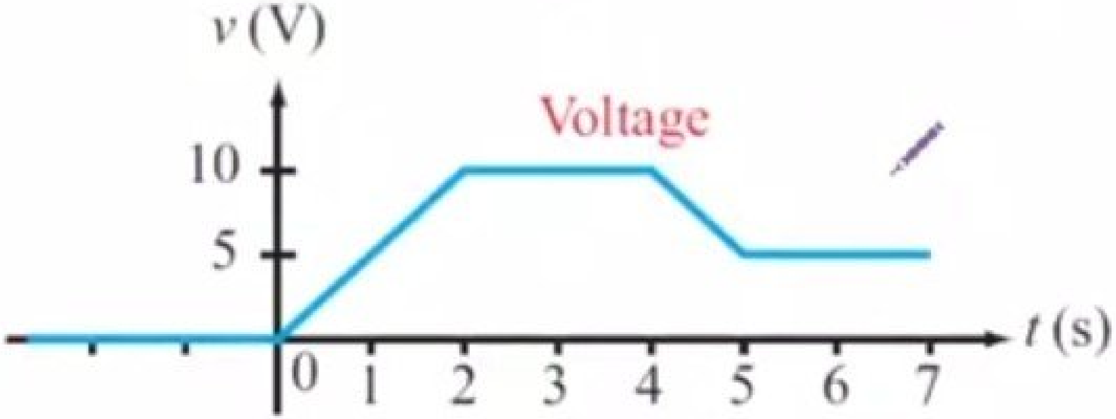
$$w_1 = \frac{1}{2}C_1 v_1^2 = \frac{1}{2}(2 \times 10^{-3})(4)^2 = 16 \text{ mJ}$$

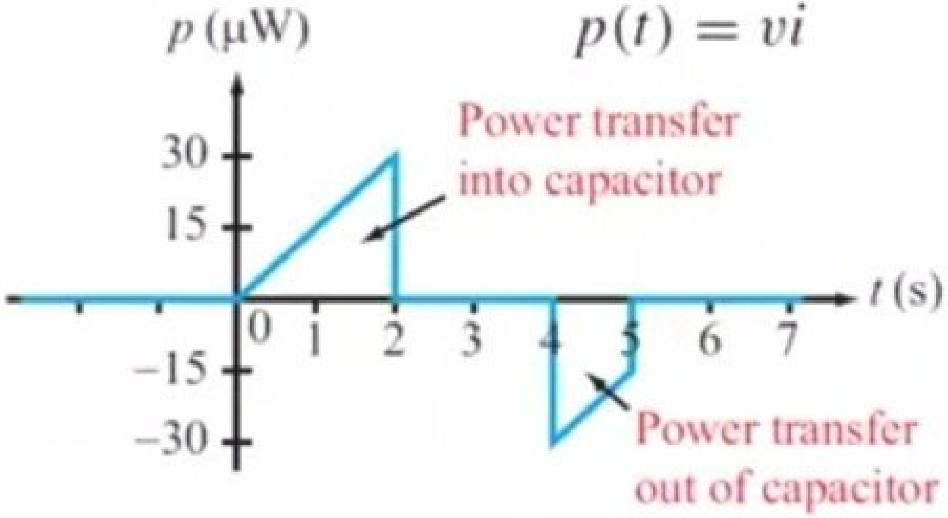
$$w_2 = \frac{1}{2}C_2 v_2^2 = \frac{1}{2}(4 \times 10^{-3})(8)^2 = 128 \text{ mJ}$$

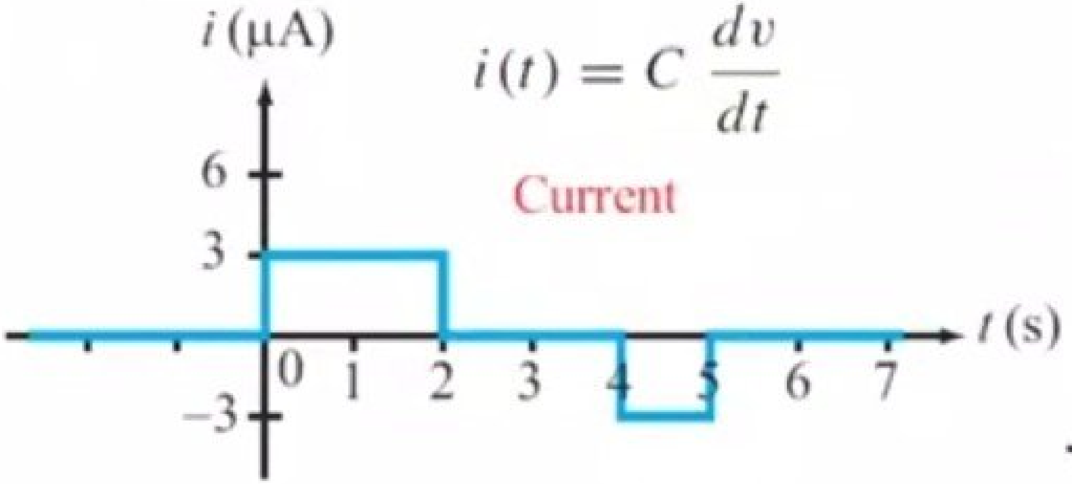


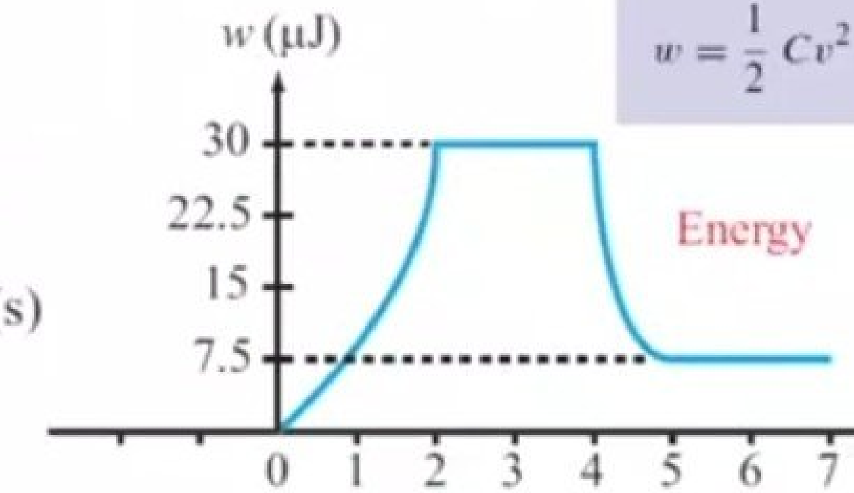
Capacitor Response: Given $v(t)$, determine $i(t)$, $p(t)$, and $w(t)$

$C = 0.6\text{-}\mu F$




$p(t) = vi$

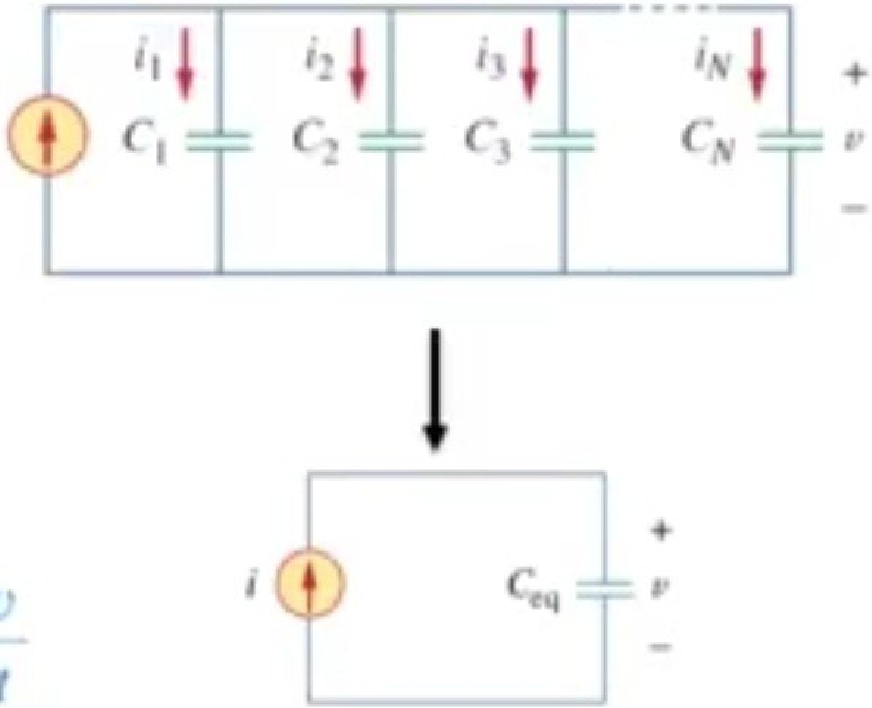

$$i(t) = C\,\frac{dv}{dt}$$


$$w = \frac{1}{2} C v^2 \qquad \text{(J)}.$$

Capacitors

❑ Capacitors in parallel:

$$\boxed{C_{\text{eq}} = C_1 + C_2 + C_3 + \cdots + C_N}$$



Proof:

$$i = i_1 + i_2 + i_3 + \cdots + i_N$$

$$\rightarrow i = C_1\frac{dv}{dt} + C_2\frac{dv}{dt} + C_3\frac{dv}{dt} + \cdots + C_N\frac{dv}{dt}$$

$$= \left(\sum_{k=1}^{N} C_k\right)\frac{dv}{dt} = C_{\text{eq}}\frac{dv}{dt}$$

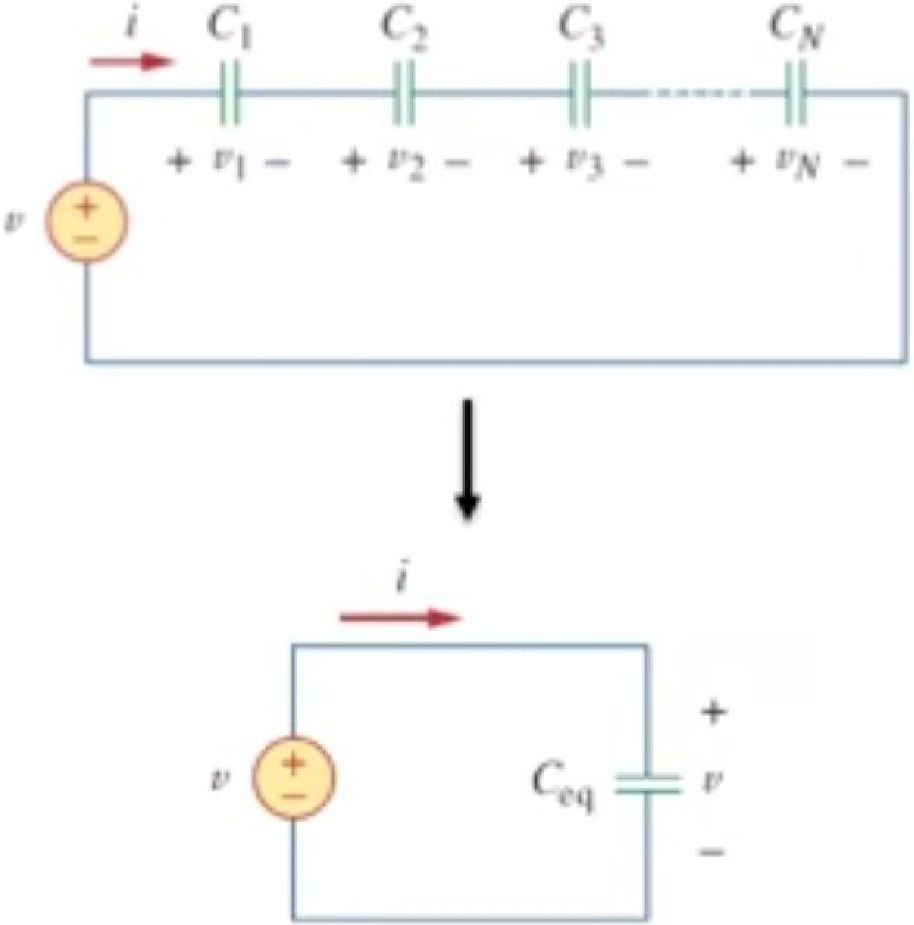
Capacitors

☐ Capacitors in series:

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots + \frac{1}{C_N}$$

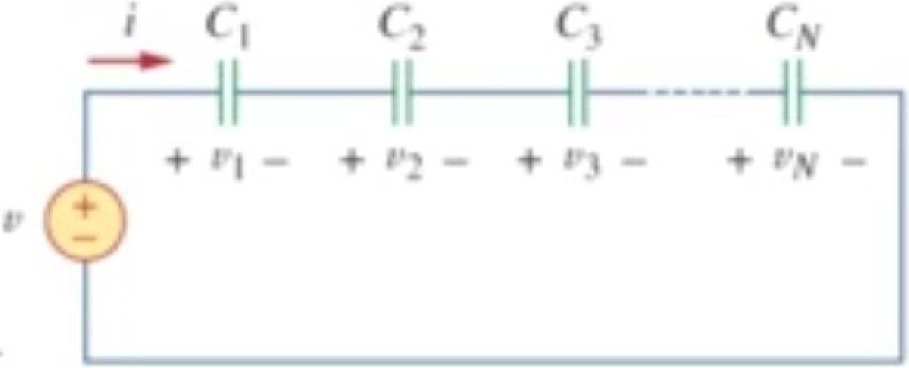
For two caps in parallel

$$C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$$



Capacitors

❑ Capacitors in series:



Proof:

$$\boxed{v = v_1 + v_2 + v_3 + \cdots + v_N}$$

$$\begin{aligned}
v &= \frac{1}{C_1}\int_{t_0}^{t} i(\tau)d\tau + v_1(t_0) + \frac{1}{C_2}\int_{t_0}^{t} i(\tau)d\tau + v_2(t_0) \\
&\qquad + \cdots + \frac{1}{C_N}\int_{t_0}^{t} i(\tau)d\tau + v_N(t_0) \\
&= \left(\frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}\right)\int_{t_0}^{t} i(\tau)d\tau + v_1(t_0) + v_2(t_0) \\
&\qquad + \cdots + v_N(t_0) \\
&= \frac{1}{C_{\text{eq}}}\int_{t_0}^{t} i(\tau)d\tau + v(t_0)
\end{aligned}$$

$$\boxed{v(t_0) = v_1(t_0) + v_2(t_0) + \cdots + v_N(t_0)}$$

Series and Parallel Caps

- ☐ Another way to think about the combinations of capacitors is this:
- ☐ Combining capacitors in parallel is equivalent to increasing the surface area of the capacitors:
- ☐ This would lead to an increased overall capacitance (as is observed)
- ☐ A series combination can be seen as increasing the total plate separation
- ☐ This would result in a decrease in capacitance (as is observed)

Capacitors

Example 6.6 Find the equivalent capacitance seen between terminals *a* and *b*

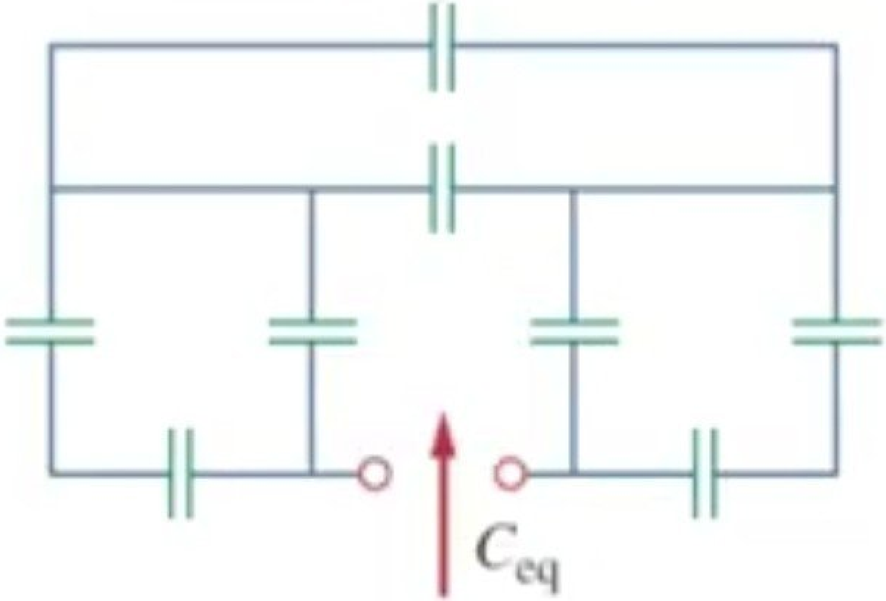
$$\frac{20 \times 5}{20 + 5} = 4\ \mu\text{F}$$

$$4 + 6 + 20 = 30\ \mu\text{F}$$

$$C_{eq} = \frac{30 \times 60}{30 + 60} = 20\ \mu\text{F}$$

Capacitors

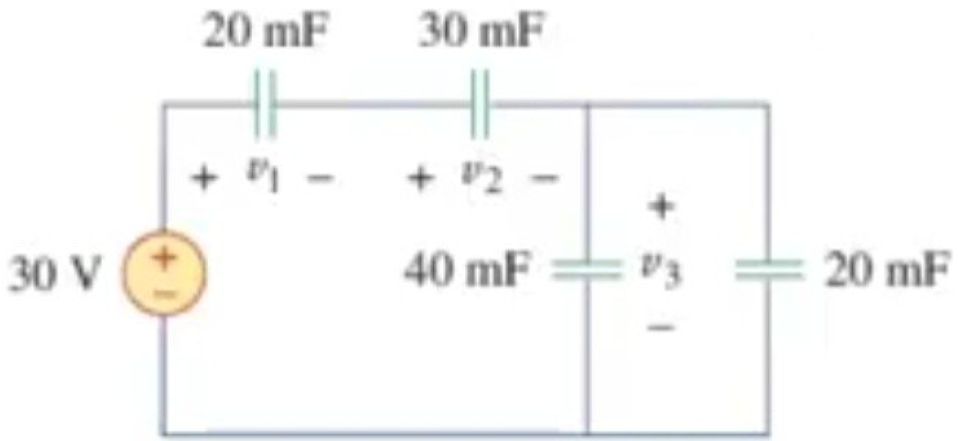
6.18 Find C_{eq} in the circuit of Fig. 6.52 if all capacitors are 4 μF.



$C_{eq} = 2.1818\ \mu\text{F}$

Capacitors

Example 6.7 For the circuit in Fig. 6.18, find the voltage across each capacitor.

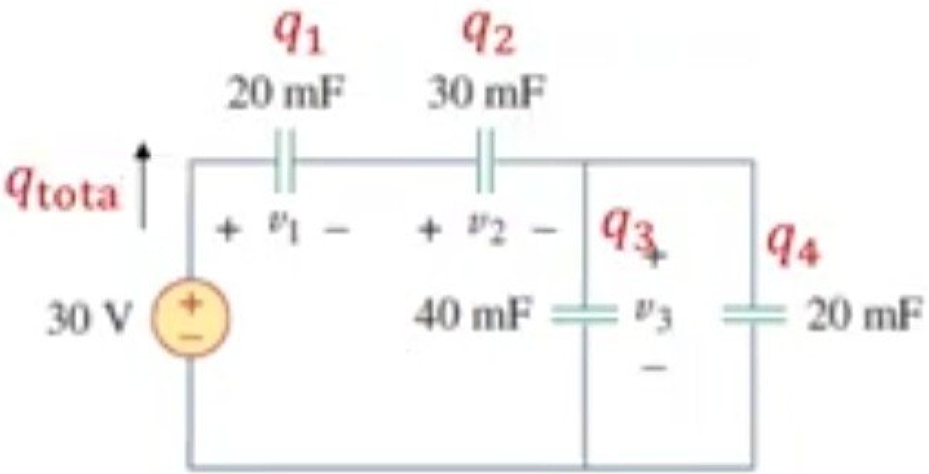


Capacitors

Example 6.7 For the circuit in Fig. 6.18, find the voltage across each capacitor.

$$C_{eq} = \frac{1}{\frac{1}{60}+\frac{1}{30}+\frac{1}{20}} \text{ mF} = 10 \text{ mF}$$

$$q_{\text{total}} = C_{eq}(30) = 0.3 \text{ C}$$



charge acts like current, so:
$q_1 = q_2 = q_{\text{total}}$

and then: $v_1 = \frac{q_1}{C_1} = 15 \text{ V}$
and $v_2 = \frac{q_2}{C_2} = 10 \text{ V}$

$$v_3 = 30 - v_1 - v_2 = 5 \text{ V}$$

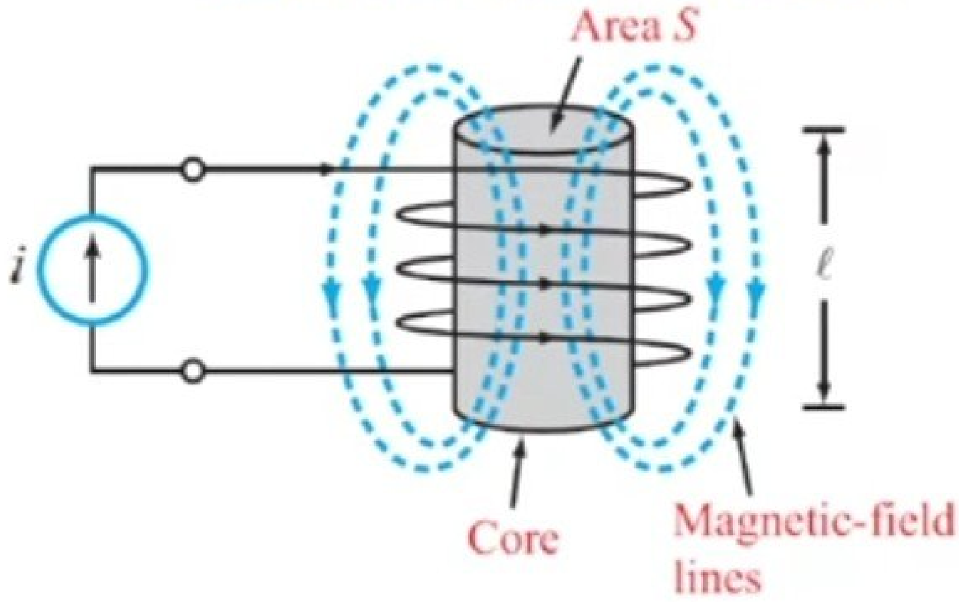
OR:

$$q_3 = \frac{C_3}{C_3 + C_4} q_{\text{total}} = 0.2 \text{ C}$$
$$q_4 = \frac{C_4}{C_3 + C_4} q_{\text{total}} = 0.1 \text{ C}$$

Inductors

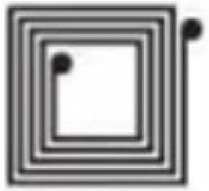
- ☐ An inductor is a passive element that stores energy in its magnetic field
- ☐ They have applications in power supplies, transformers, radios, TVs, radars, and electric motors.
- ☐ Any conductor has inductance L, but the effect is typically enhanced by coiling the wire up.

Solenoid Wound Inductor

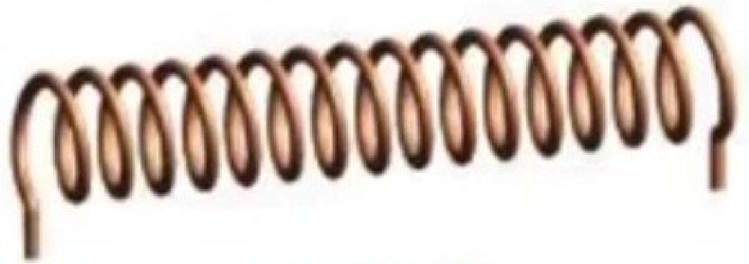


❑ Several Types

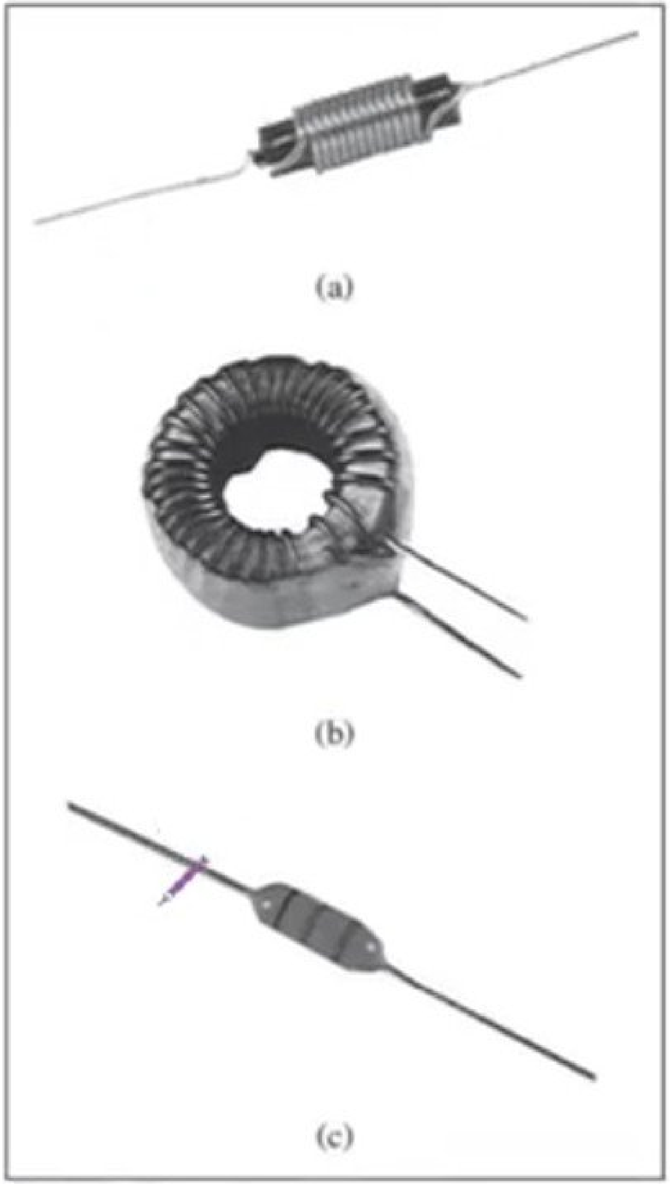
High current inductor



Planar inductor



Solenoid



(a)

(b)

(c)

Figure 6.22

Various types of inductors: (a) solenoidal wound inductor, (b) toroidal inductor, (c) chip inductor.

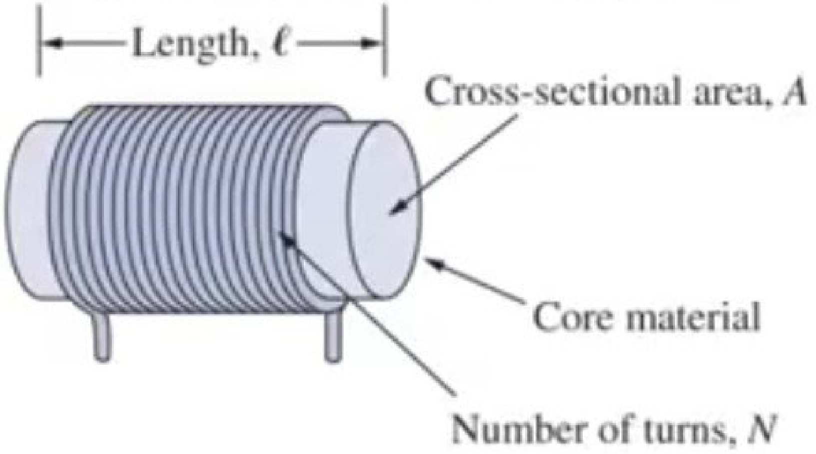
Courtesy of Tech America.

Inductors

- ☐ ***An inductor consists of a coil of conducting wire.***
- ☐ **Calculating the inductance depends on the geometry:**
- ▪ **For example, for a solenoid the inductance is:**

$$\boxed{L = \frac{N^2 \mu A}{l}}$$

- ✓ **Where N is the number of turns of the wire around the core of cross sectional area A and length l.**
- ✓ **The material used for the core has a magnetic property called the permeability, μ.**



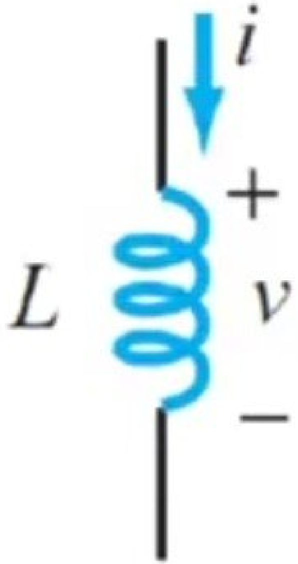
- ☐ Inductance, measured in Henries, H.
- ☐ 1 Henry = 1 volt-second per ampere.

Inductors current-voltage relationship

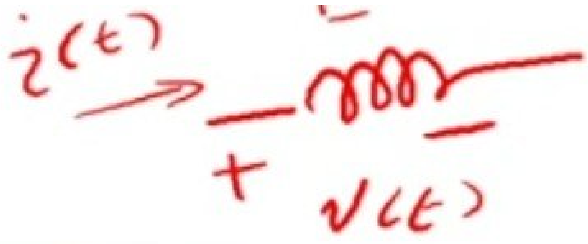
- ❑ ***The inductor opposes the change of current flowing through it.***
- ❑ If a current is passed through an inductor, the voltage across it is directly proportional to the time rate of change in current:

$$\boxed{v = L\frac{di}{dt}}$$

with passive sign convention.



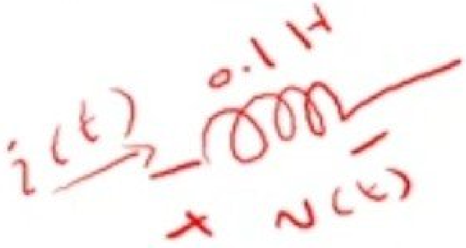
Inductors



Example 6.8 The current through a 0.1-H inductor is $i(t) = 10te^{-5t}$ A. Find the voltage across the inductor and the energy stored in it.

$v(t) = L\frac{di(t)}{dt}$

$\boxed{v = L\,di/dt}$



$i(t) = I$

$\hookrightarrow v = 0$

$= s.c$

$L = 0.1$ H,

$$v = 0.1\frac{d}{dt}(10te^{-5t}) = e^{-5t} + t(-5)e^{-5t} = \underline{e^{-5t}(1 - 5t)}\text{ V}$$

$w = \frac{1}{2} L i^2$

The energy stored is

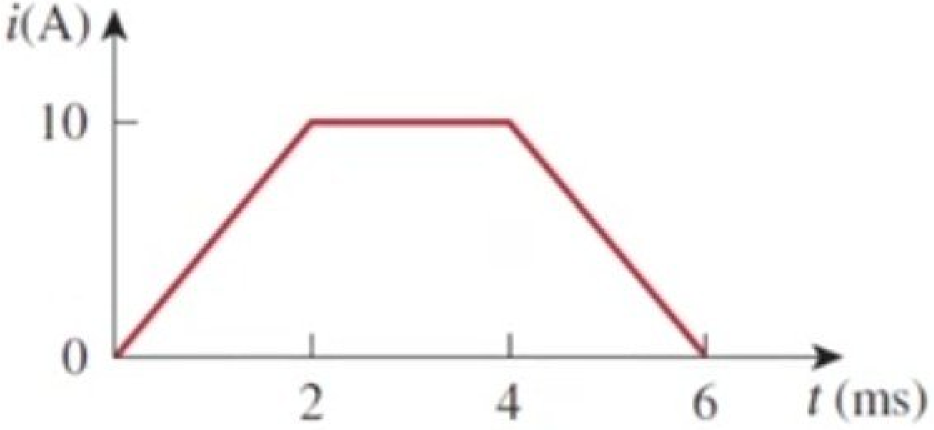
$$w = \frac{1}{2}Li^2 = \frac{1}{2}(0.1)100t^2e^{-10t} = 5t^2e^{-10t}\text{ J}$$

$p(t) = v(t)\,i(t)$

or $p(t) = \frac{dw(t)}{dt}$

Inductors

6.40 The current through a 5-mH inductor is shown in Fig. 6.66. Determine the voltage across the inductor at $t = 1$, 3, and 5 ms.



Inductors

Example 6.9 Find the current through a 5-H inductor if the voltage across it is

$$v(t) = \begin{cases} 30t^2, & t > 0 \\ 0, & t < 0 \end{cases}$$

Also, find the energy stored at $t = 5$ s. Assume $i(v) > 0$.

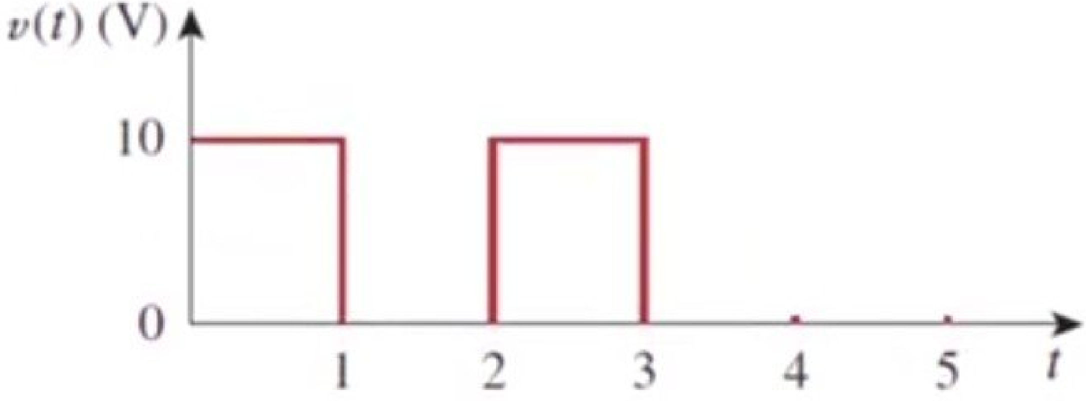
Since $i = \frac{1}{L}\int_{t_0}^{t} v(t)\,dt + i(t_0)$ and $L = 5$ H,

$$i = \frac{1}{5}\int_{0}^{t} 30t^2\,dt + 0 = 6 \times \frac{t^3}{3} = 2t^3 \text{ A}$$

$$w\big|_0^5 = \frac{1}{2}Li^2(5) - \frac{1}{2}Li(0) = \frac{1}{2}(5)(2 \times 5^3)^2 - 0 = 156.25 \text{ kJ}$$

Inductors

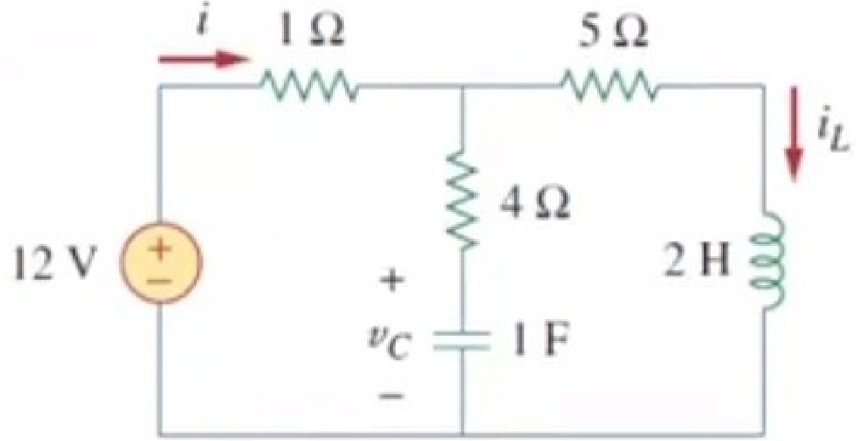
6.42 If the voltage waveform in Fig. 6.67 is applied across the terminals of a 5-H inductor, calculate the current through the inductor. Assume $i(0) = -1$ A.



Inductors

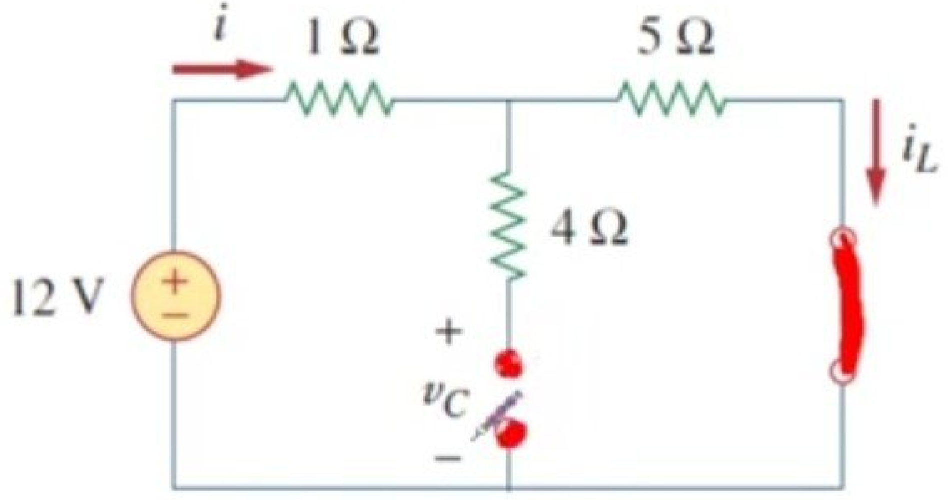
Example 6.10

Under dc conditions, find: (a) i, v_C, and i_L, (b) the energy stored in the capacitor and inductor.



Inductors

Solution:



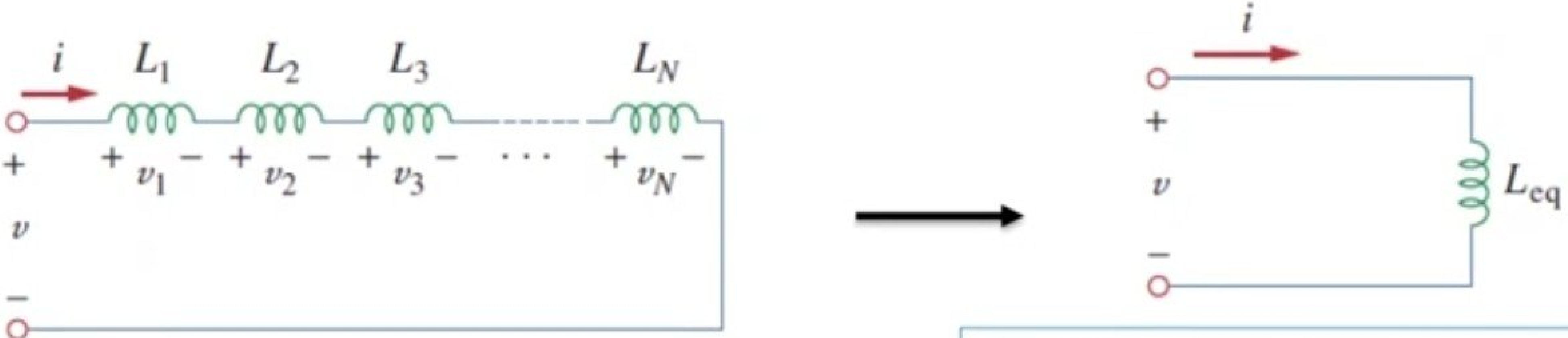
$$i = i_L = \frac{12}{1 + 5} = 2 \text{ A}$$

$$v_C = 5i = 10 \text{ V}$$

$$w_C = \frac{1}{2}Cv_C^2 = \frac{1}{2}(1)(10^2) = 50 \text{ J}$$

$$w_L = \frac{1}{2}Li_L^2 = \frac{1}{2}(2)(2^2) = 4 \text{ J}$$

Series Inductors



$$L_{eq} = L_1 + L_2 + L_3 + \cdots + L_N$$

Proof:

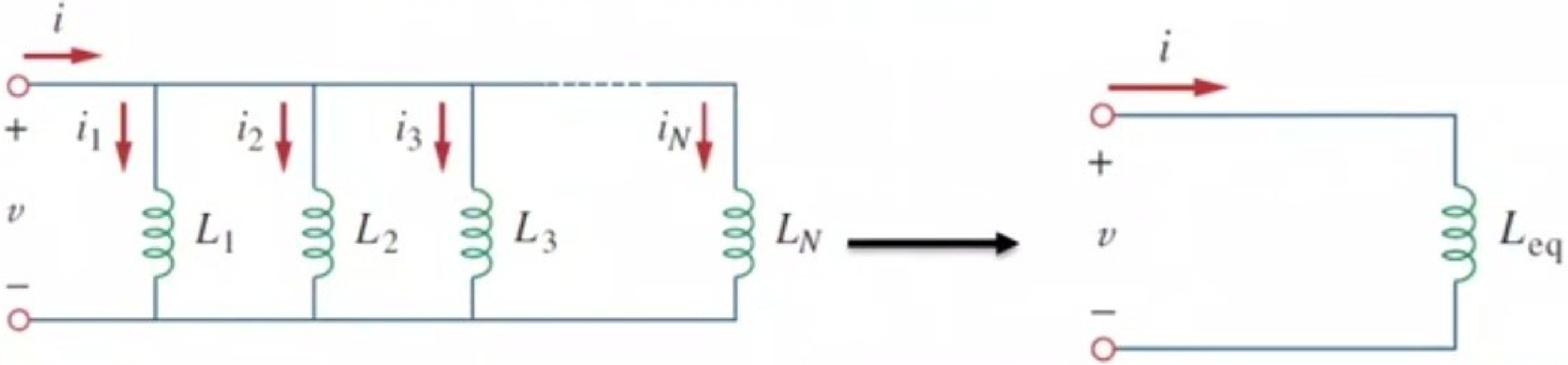
$$v = v_1 + v_2 + v_3 + \cdots + v_N$$

$$v = L_1\frac{di}{dt} + L_2\frac{di}{dt} + L_3\frac{di}{dt} + \cdots + L_N\frac{di}{dt}$$

$$= (L_1 + L_2 + L_3 + \cdots + L_N)\frac{di}{dt}$$

$$= \left(\sum_{k=1}^{N} L_k\right)\frac{di}{dt} = L_{eq}\frac{di}{dt}$$

Parallel Inductors



Proof:

$$i = i_1 + i_2 + i_3 + \cdots + i_N$$

$$i = \frac{1}{L_1}\int_{t_0}^{t} v\,dt + i_1(t_0) + \frac{1}{L_2}\int_{t_0}^{t} v\,dt + i_2(t_0) + \cdots + \frac{1}{L_N}\int_{t_0}^{t} v\,dt + i_N(t_0)$$

$$= \left(\frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N}\right)\int_{t_0}^{t} v\,dt + i_1(t_0) + i_2(t_0) + \cdots + i_N(t_0)$$

$$= \left(\sum_{k=1}^{N}\frac{1}{L_k}\right)\int_{t_0}^{t} v\,dt + \sum_{k=1}^{N} i_k(t_0) = \frac{1}{L_{eq}}\int_{t_0}^{t} v\,dt + i(t_0)$$

$$i(t_0) = i_1(t_0) + i_2(t_0) + \cdots + i_N(t_0)$$

$$\boxed{\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \cdots + \frac{1}{L_N}}$$

For two inductors in parallel

$$\boxed{L_{eq} = \frac{L_1 L_2}{L_1 + L_2}}$$

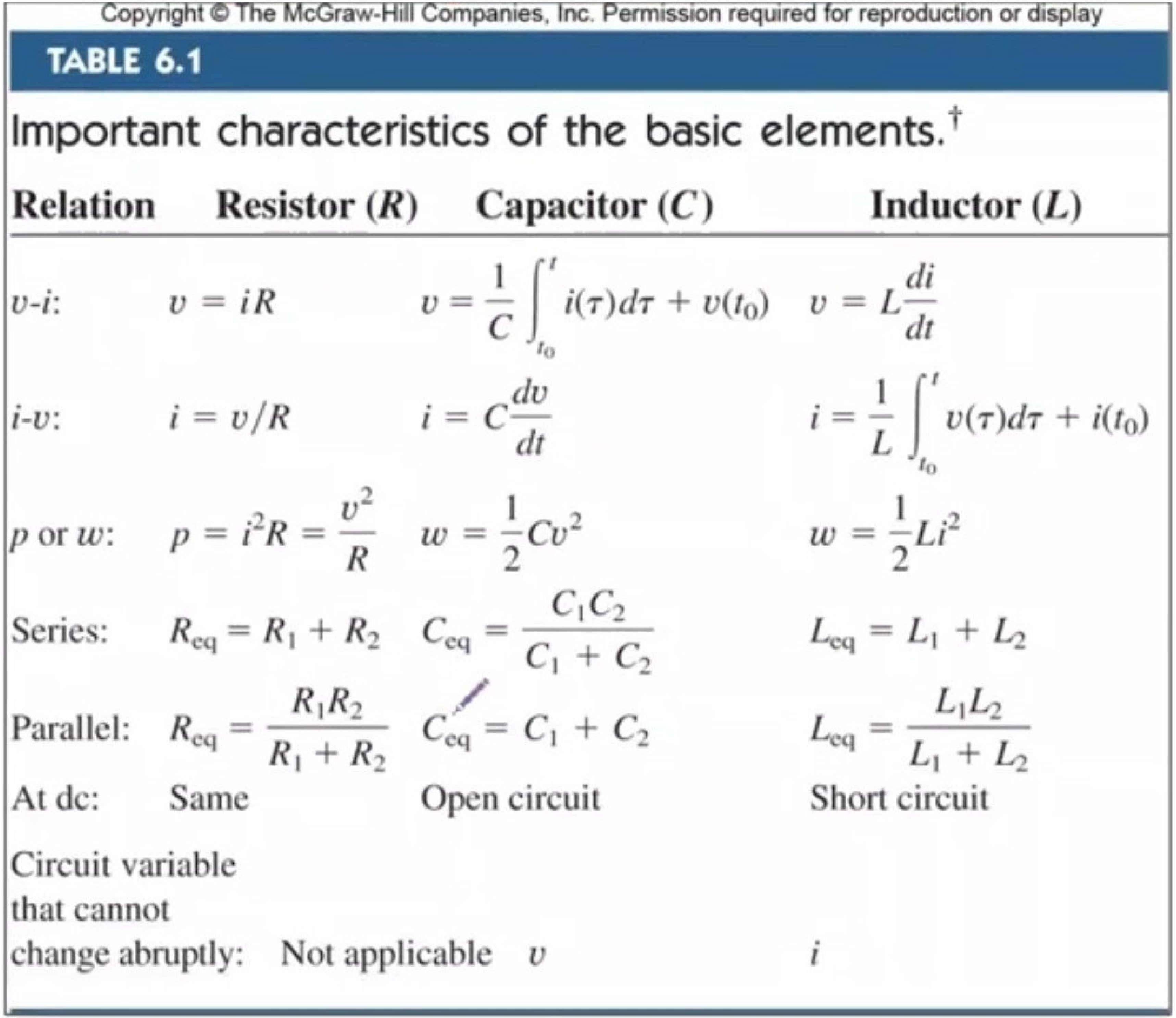


TABLE 6.1

Important characteristics of the basic elements.[†]

Relation	Resistor (R)	Capacitor (C)	Inductor (L)
v-i:	$v = iR$	$v = \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau + v(t_0)$	$v = L\frac{di}{dt}$
i-v:	$i = v/R$	$i = C\frac{dv}{dt}$	$i = \frac{1}{L}\int_{t_0}^{t} v(\tau)d\tau + i(t_0)$
p or w:	$p = i^2R = \frac{v^2}{R}$	$w = \frac{1}{2}Cv^2$	$w = \frac{1}{2}Li^2$
Series:	$R_{eq} = R_1 + R_2$	$C_{eq} = \frac{C_1C_2}{C_1 + C_2}$	$L_{eq} = L_1 + L_2$
Parallel:	$R_{eq} = \frac{R_1R_2}{R_1 + R_2}$	$C_{eq} = C_1 + C_2$	$L_{eq} = \frac{L_1L_2}{L_1 + L_2}$
At dc:	Same	Open circuit	Short circuit
Circuit variable that cannot change abruptly:	Not applicable	v	i

Inductors

Example 6.12 For the circuit in Fig. 6.33, $i(t) = 4(2 - e^{-10t})$ mA. If $i_2(0) = -1$ mA, find: (a) $i_1(0)$; (b) $v(t)$, $v_1(t)$, and $v_2(t)$; (c) $i_1(t)$ and $i_2(t)$.