اغلب الامتحان فهم مش خطوات حل

Ch2: diff equ.

Ch3: - Laplace transform
- inverse (partial expansion)
- tables. (اخر جدول بيجي دائماً بالامتحان)

Industrial Control

Chapter Five:
First Order & Second Order Systems

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System's Response

- We can find the time response of dynamic systems for arbitrary initial conditions and inputs

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}[G(s)\overset{\text{input}}{U(s)}]$$

- Classifying the response of some standard systems to standard inputs can provide insight
  - Ex Systems: first order, second order

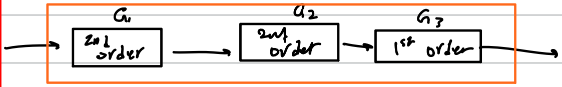
    Standard inputs بنستخدمها لفحص ال response for the output of the syst.
  - Ex Inputs: impulse, step, ramp, sinusoid

كل جزء انه linear time invariant system.

4th order transfer function

$G = G_1 * G_2 = 4^{th}$ order system

5th order transfer function.



$G = G_1 * G_2 * G_3$

System's Order

The order of the system is given by the maximum power of **s** in the denominator polynomial, Q(s).

Here, $Q(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots\dots + a_{n-1} s + a_n$.

Now, **n** is the order of the system

When n = 0, the system is zero order system.

When n = 1, the system is first order system.

When n = 2, the system is second order system and so on.

The numerator and denominator polynomial of equation (2.10) can be expressed in the factorized form as shown in equation (2.11).

order of the trans function (syst) = # of the poles

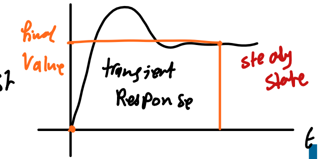
$$T(s) = \frac{P(s)}{Q(s)} = \frac{(s+z_1)(s+z_2)\dots\dots(s+z_m)}{(s+p_1)(s+p_2)\dots\dots(s+p_n)} \qquad \dots..(2.11)$$

where, $z_1, z_2, \dots\dots z_m$ are zeros of the system.

$p_1, p_2, \dots\dots p_n$ are poles of the system.

Now, the value of **n** gives the number of poles in the transfer function. Hence the order is also given by the number of poles of the transfer function.

from the start point (zero) until stedy state of the syst



Transient Response and Steady-State Response.

The time response of a control system consists of two parts:

- the transient response
- and the steady-state response.

By transient response, we mean that which goes from the initial state to the final state.

By steady-state response, we mean the manner in which the system output behaves as t approaches infinity. Thus the system response c(t) may be written as

$$c(t) = c_{tr}(t) + c_{ss}(t)$$

Dynamic Behavior

In analyzing process dynamic and process control systems, it is important to know how the process responds to changes in the process inputs.

A number of standard types of input changes are widely used for two reasons:

1. They are representative of the types of changes that occur in plants.

2. They are easy to analyze mathematically.

Laplace Transform of Standard Inputs

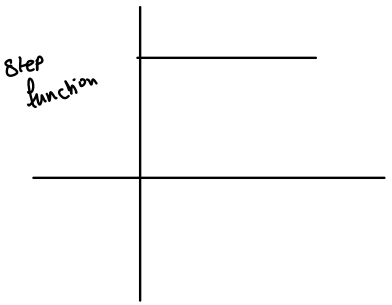
Response of the syst ≡ output of the syst.

Step Function:

The unit step function is,

$\frac{1}{s}$

step function



$$
\begin{aligned}
u(t) &= 1 \quad \text{for } t \geq 0 \\
&= 0 \quad \text{for } t < 0
\end{aligned}
$$

$$L\{u(t)\} = \int_0^\infty u(t) \cdot e^{-st}\, dt = \int_0^\infty 1 \cdot e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

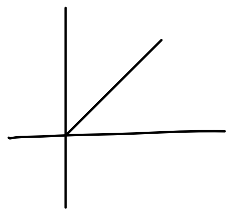
$$= \left[\frac{e^{-\infty}}{-s} - \frac{e^{0}}{-s}\right] \quad \text{as } e^{-\infty} = 0$$

$$L\{u(t)\} = \frac{1}{s}$$

Laplace Transform of Standard Inputs

Ramp Function:

linear function



The unit ramp function is defined as,

$\frac{1}{s^2}$

$$r(t) = t \qquad \text{for } t \geq 0$$
$$= 0 \qquad \text{for } t < 0$$

$$L\{r(t)\} = \int_0^\infty r(t)\, e^{-st}\, dt = \int_0^\infty t\, e^{-st}\, dt$$

Integrating by parts,

$$= \left[\frac{t \,.\, e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} \,.\, 1\, dt = [0 - 0] + \frac{1}{s}\int_0^\infty e^{-st}\, dt$$

$$= \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}\left[\frac{e^{-\infty}}{-s} - \frac{e^{0}}{-s}\right] \qquad \text{as } e^{-\infty} = 0$$

$$L\{r(t)\} = \frac{1}{s^2}$$

$$L\{t\, u(t)\} = \frac{1}{s^2} \qquad \text{as } r(t) = t\, u(t)$$

Laplace Transform of Standard Inputs

Ramp Function:

The unit ramp function is defined as,

$$r(t) = t \qquad \text{for } t \geq 0$$
$$= 0 \qquad \text{for } t < 0$$

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Integrating by parts,

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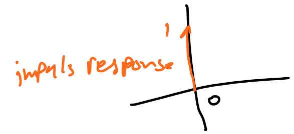
$$= \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}\left[\frac{e^{-\infty}}{-s} - \frac{e^{0}}{-s}\right] \qquad \text{as } e^{-\infty} = 0$$

$$L\{r(t)\} = \frac{1}{s^2}$$

$$L\{t\, u(t)\} = \frac{1}{s^2} \qquad \text{as } r(t) = t\, u(t)$$

Laplace Transform of Standard Inputs



Impulse Function:

The unit impulse function is δ(t) and defined as,

$$\delta(t) = 1 \quad \text{for } t = 0$$
$$= 0 \quad \text{for } t \neq 0$$

We know the relation between unit step and unit impulse.

$$\delta(t) = \frac{d\,u(t)}{dt}$$

Taking Laplace transform of both sides,

$$L\{\delta(t)\} = L\left\{\frac{d\,u(t)}{dt}\right\}$$

$$L\left\{\frac{d\,f(t)}{dt}\right\} = s\,F(s) - f\left(0^-\right)$$

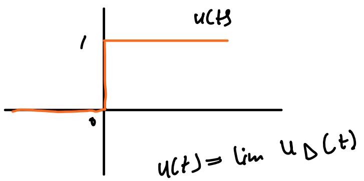
$$L\{\delta(t)\} = s\,.\,L\{u(t)\} - u(t)\big|_{t=0^-}$$

$$u(t)\big|_{t=0^-} = 0$$

$$L\{u(t)\} = \frac{1}{s}$$

$$L\{\delta(t)\} = s\,.\,\frac{1}{s} - 0$$

$$L\{\delta(t)\} = 1$$



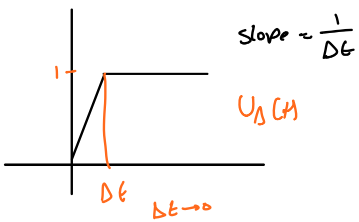
$u(t) = \lim_{D \to 0} u_D(t)$

المشتقة ← التغير مع الزمن

→ in reality there's nothing that goes from 0 → 1 (slope = zero) in otime

↓

reality



$\delta(t) = \lim_{\Delta t \to 0}^{-1} = 1$

general forms:

① $G(s) = \frac{1}{Ts+1}$

τ : time const

② $G(s) = \frac{a}{s+a} = \frac{1}{a}\left[\frac{1}{\frac{1}{a}s+1}\right]$

$\frac{1}{a}$ = time const

Can be impresed by a transfer function with order 1.

eg:

$G(s) = \frac{10}{s+10}$

$\frac{10}{10}\left[\frac{1}{\frac{1}{10}s+1}\right]$

τ

if transfer function is in this form

$\boxed{G(s) = \frac{a}{s+a} \quad \tau = 1/a}$

FIRST-ORDER SYSTEMS

- Consider the first-order system shown in Figure 5–1(a). Physically, this system may represent an *RC* circuit, thermal system, or the like

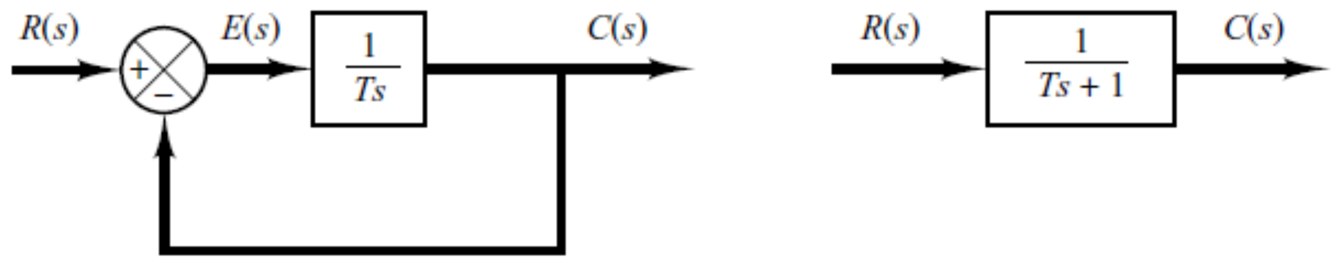
transfer function for all 1st order syst is the same

$$\frac{C(s)}{R(s)} = \frac{1}{Ts+1}$$

general form transfer function

يعني

τ : time const of the syst



- **Unit-Step Response of First-Order Systems.** Since the Laplace transform of the unit-step function is 1/s, substituting R(s)=1/s into Equation, we obtain

$$C(s) = \frac{1}{Ts+1} * \frac{1}{s}$$

Expanding $C(s)$ into partial fractions gives

$$C(s) = \frac{1}{s} - \frac{T}{Ts+1} = \frac{1}{s} - \frac{1}{s+(1/T)}$$

by partial fraction expansion

طالعة T عادي مرتين

$\mathcal{L}^{-1} C(s)$

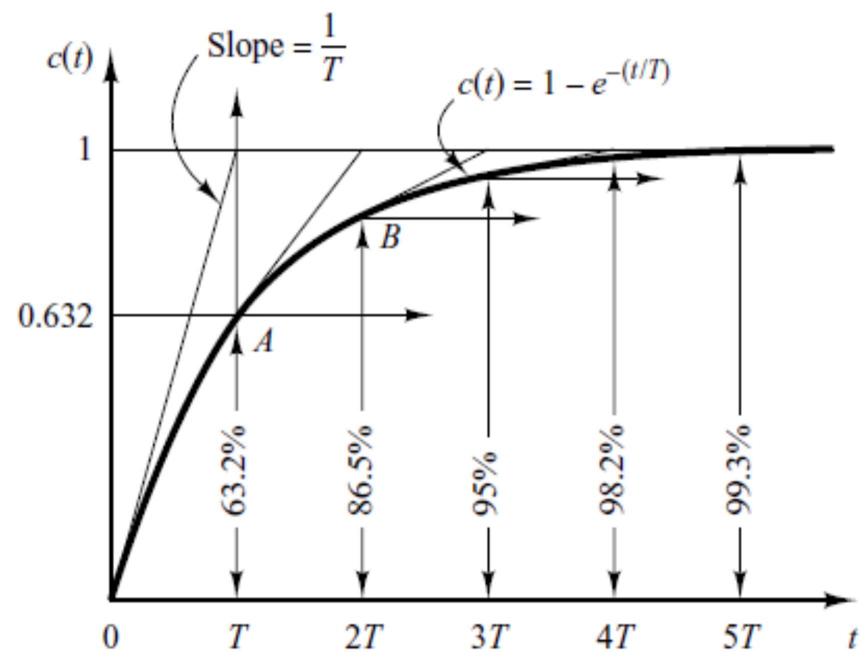
$$c(t) = 1 - e^{-t/T}, \quad \text{for } t \geq 0$$

stable syst

$K_1 = 1$

$K_2 = -T$

- $$c(t) = 1 - e^{-t/T}, \qquad \text{for } t \geq 0$$
- Equation states that initially the output c(t) is zero and finally it becomes unity.
- One important characteristic of such an exponential response curve c(t) is that at t=T the value of c(t) is 0.632, or the response c(t) has reached 63.2% of its total change
- This may be easily seen by substituting t=T in c(t).That is,
- $c(T) = 1 - e^{-1} = 0.632$
- The exponential response curve c(t) is shown.
- In one time constant, the exponential response curve has gone from 0 to 63.2%of the final value.
- In two time constants, reaches 86.5%.
- At t=3T, 4*T*, and 5T, the response
- reaches 95%, 98.2%, and 99.3%,
- Thus, for t > 4T, the response
- remains within 2%.
- As seen from Equation
- , the steady state is reached
- mathematically only after an
- infinite time.

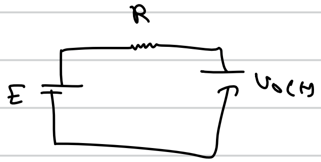


RC circuit هواد كليه ابسط مثال (1^{st}. order)

Mathmatical Model ال موجود في شابتر

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{1+RCs}$$

$\tau = RC$



$V_o(t) = V_i\,(1-e^{-t/RC})$

$V_i = E$

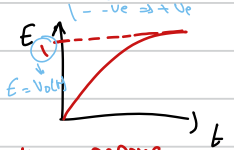
$V_o(t) = E\,(1-e^{-t/RC})$

→ step resp

$E = 1$

$V_o(t) = 1 - e^{-t/RC}$

(− −ve ⇒ +ve)



time responce for any 1^{st}. order-syst

when $t = 0$, $t = \tau$, $t = 4\tau$

$V_o(0) = 1 - e^{0}$

$= 0$

$V_o(\tau) = 1 - e^{-\tau/\tau}$

$= 1 - e^{-1}$

$= 1 - 0.36$

$= 0.632$

→ τ : the time it takes the first order system to reach 0.632 of its final Value.

⇒ Syst reach $\frac{2}{3}$ of the final value

→ final value $= 10$ at τ → $E = 6.32$

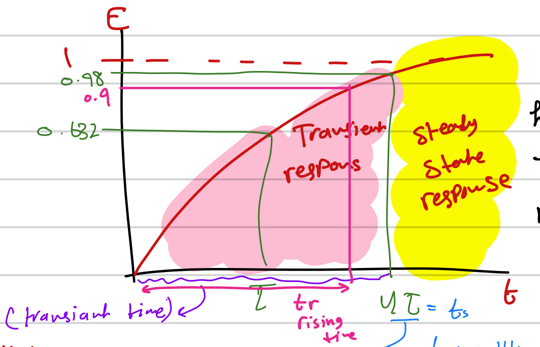
$V_o(4\tau) = 1 - e^{-4}$

$= 1 - 0.018$

$= 0.98$

as a result :

t_s : Settling time, the time that it takes to reach 98% of its final value, or remains with 2% changes of its final Value



for $t > 4\tau$ The response remain with 2%

sometimes it's 4τ or 5τ

the terminal $e^{-t/\tau}$ will be almost zero (very small) so it's neglected that's why it's considered as stedy state

t_r : rising time, The time difference between $t_{0.9}$ and $t_{0.1}$

كل ما كان أقل كل ما كان أحسن

- **Unit-Ramp Response of First-Order Systems.**
- Since the Laplace transform of the unit-ramp function is $1/s^2$, we obtain the output of the system of

$$C(s) = \frac{1}{Ts+1}\frac{1}{s^2}$$

Expanding $C(s)$ into partial fractions gives

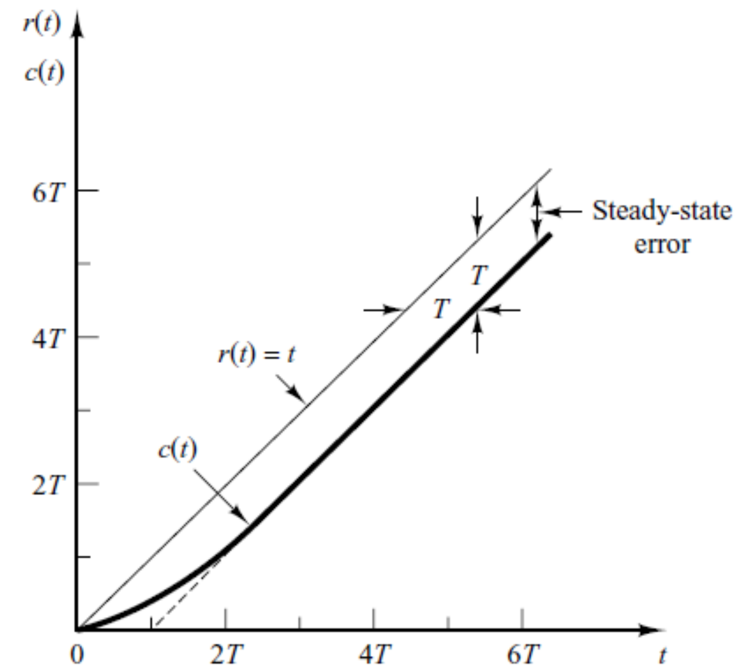
$$C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts+1}$$

Taking the inverse Laplace transform:

$$c(t) = t - T + Te^{-t/T}, \qquad \text{for } t \geq 0$$

The error signal $e(t)$ is then

$$\begin{aligned} e(t) &= r(t) - c(t) \\ &= T\left(1 - e^{-t/T}\right) \end{aligned}$$



Above Equation states that initially the output c(t) is zero and finally it becomes unity

Unit-Impulse Response of First-Order Systems. For the unit-impulse input, $R(s) = 1$ and the output of the system of Figure 5–1(a) can be obtained as

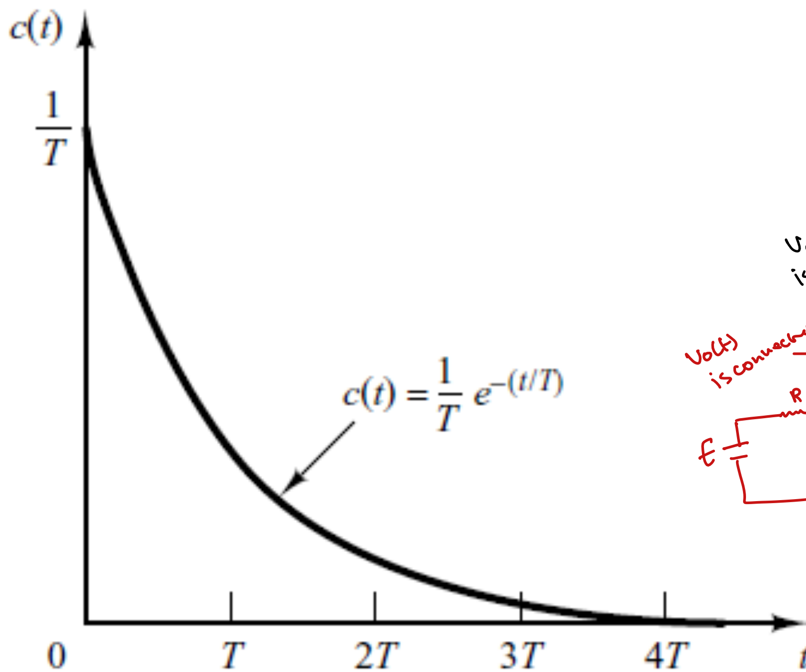
$$C(s) = \frac{1}{Ts + 1} \tag{5–7}$$

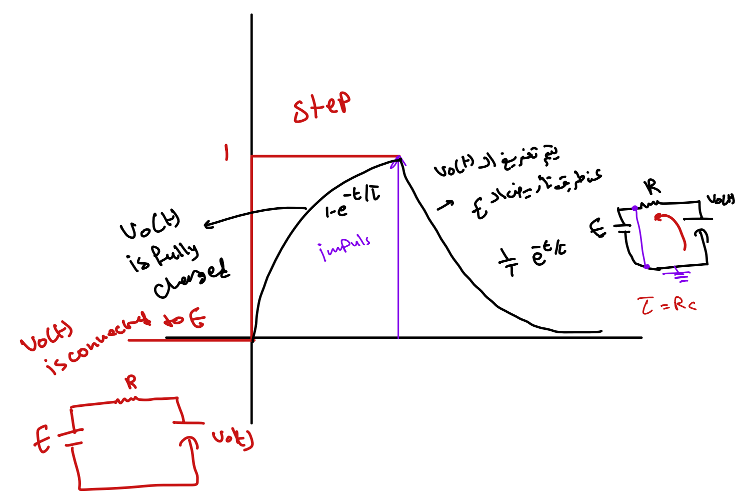
The inverse Laplace transform of Equation (5–7) gives

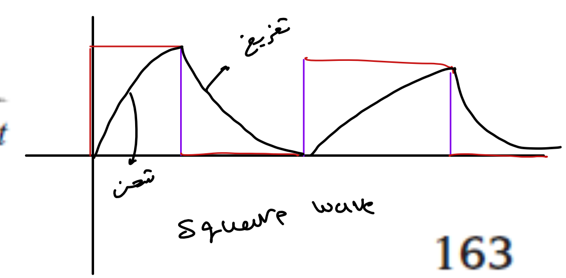
in DC:

$$c(t) = \frac{1}{T} e^{-t/T}, \qquad \text{for } t \geq 0 \qquad \Rightarrow c(t) = \frac{K}{T} e^{-t/T} \tag{5–8}$$

The response curve given by Equation (5–8) is shown in Figure 5–4.







- for the unit-ramp input the output c(t) is

$$c(t) = t - T + Te^{-t/T}, \qquad \text{for } t \geq 0$$

- For the unit-step input, which is the derivative of unit-ramp input, the output c(t) is

$$c(t) = 1 - e^{-t/T}, \qquad \text{for } t \geq 0$$

- Finally, for the unit-impulse input, which is the derivative of unit-step input, the output c(t) is

$$c(t) = \frac{1}{T}e^{-t/T}, \qquad \text{for } t \geq 0$$

Comparing the system responses to these three inputs clearly indicates that the response to the derivative of an input signal can be obtained by differentiating the response of the system to the original signal. It can also be seen that the response to the integral of the original signal can be obtained by integrating the response of the system to the original signal and by determining the integration constant from the zero-output initial condition. This is a property of linear time-invariant systems. Linear time-varying systems and nonlinear systems do not possess this property.

- <u>First-Order Systems:</u>

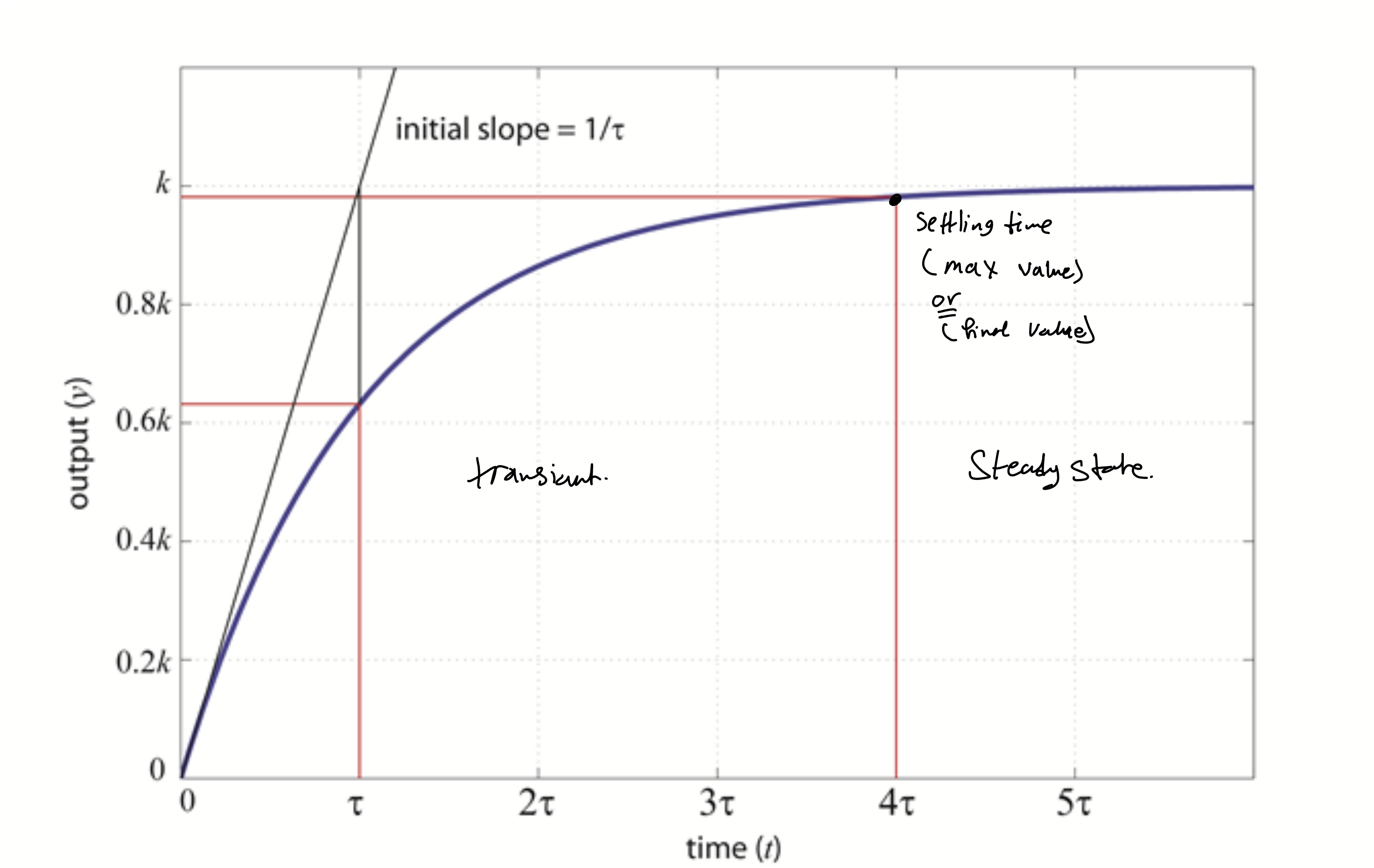
general form for 1st order.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{k}{\tau s + 1}$$

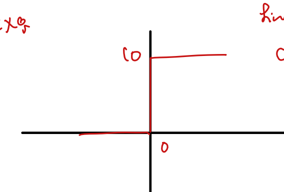
$k \rightarrow$ DC Value

input signal غالبًا بتساوي مقدار

- <u>Step Response:</u>



ex:



final value for step resp = 10

$c(t) = 10(1 - e^{-t/T})$

$\lim_{t \to \infty} c(t) = 10(1-0)$

$= 10 \Rightarrow$ Steady State value

or

DC value.

$\Rightarrow G(t)$ in s domain

$G(s) = \frac{10}{Ts+1}$

$\lim_{s \to 0} \frac{10}{Ts+1} = 10$

Transient Response Specifications: Rise Time

Let's first take a look at *1st-order step response*

$$H(s) = \frac{a}{s+a}, \qquad a > 0 \qquad \text{(stable pole)}$$

DC gain = 1 (by FVT)

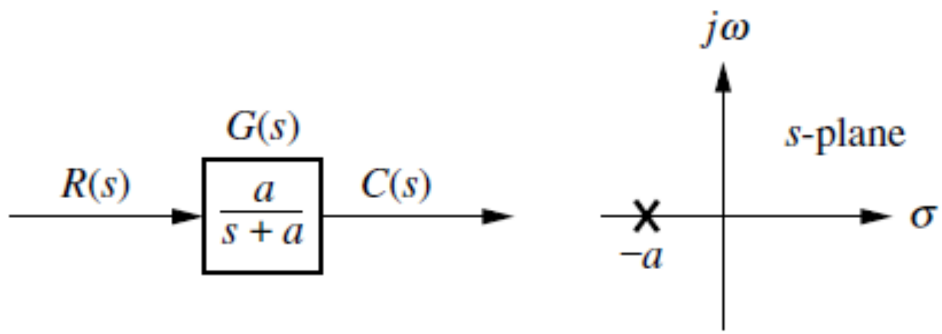
Step response: $$Y(s) = \frac{H(s)}{s} = \frac{a}{s(s+a)} = \frac{1}{s} - \frac{1}{s+a}$$

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = 1(t) - e^{-at}$$

$e^{-t/\tau}$

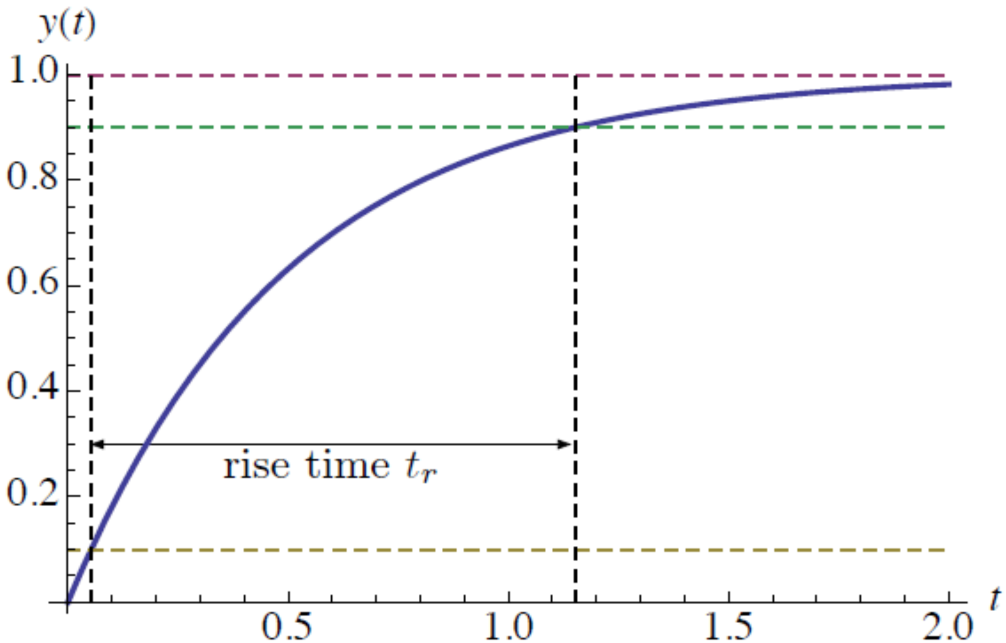
$\tau = 1/a$

$\Rightarrow e^{-a\tau}$



Rise Time

Step response: $y(t) = 1(t) - e^{-at}$



Rise time t_r: the time it takes to get from 10% of steady-state value to 90%

In this example, it is easy to compute t_r analytically:

$$1 - e^{-at_{0.1}} = 0.1 \qquad e^{-at_{0.1}} = 0.9 \qquad t_{0.1} = -\frac{\ln 0.9}{a}$$

$$1 - e^{-at_{0.9}} = 0.9 \qquad e^{-at_{0.9}} = 0.1 \qquad t_{0.9} = -\frac{\ln 0.1}{a}$$

$$t_r = t_{0.9} - t_{0.1} = \frac{\ln 0.9 - \ln 0.1}{a} = \frac{\ln 9}{a} \approx \frac{2.2}{a} = 2.2\,\tau$$

$\tau = \frac{1}{a}$

PROBLEM: A system has a transfer function, $G(s) = \frac{50}{s+50}$. Find the time constant, T_c, settling time, T_s, and rise time, T_r.

ANSWER: $T_c = 0.02$ s, $T_s = 0.08$ s, and $T_r = 0.044$ s.

The complete solution is located at www.wiley.com/college/nise.

Summary of 1st. order:

① $G(s) = \frac{1}{Ts+1}$

T: time const

② $G(s) = \frac{a}{s+a} = \left(\frac{1}{a}\right)\left[\frac{1}{\frac{1}{a}s+1}\right]$

$\frac{1}{a}$: time Const.

③ $G(s) = \frac{C(s)}{R(s)}$

$= \frac{1}{s}$

$C(s) = \frac{1}{s} \cdot \left[\frac{1}{Ts+1}\right]$

$c(t) = 1 - e^{-t/T}$

time Const

↳ settling time $(4T)$

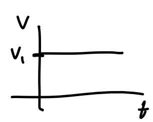
↳ Rising time $\frac{2.2}{a}$

Note:

DC: Dirict current → that's why it's a Steady state Value

(Constant with time) او بنسمي ال SS ← DC

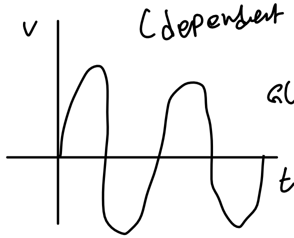
(independent of time and frequency)



$G(s) = V_1$

AC: Alternating current

(dependent on time and frequency)



$G(s) = \sin \omega t$

ex:

$G(s) = \frac{15}{s+3}$

to solve it (always compare with the standard form)

$G(s) = \frac{K}{T s+1}$

Devid by ③

$= \frac{\cancel{3}[5]}{\cancel{3}\left[\frac{1}{3}s+1\right]}$

$= \frac{5}{\frac{1}{3}s+1}$

$T = 1/3$

$K = 5$

- Step responce:

$c(t) = K(1 - e^{-t/T})$

- Impulse Response:

$c(t) = \frac{K}{T} e^{-t/T}$

⊛ final value theory

$$\boxed{\lim_{t\to\infty} g(t) = \lim_{s\to\infty} G(s)}$$

ex:

$$H(s) = \frac{(3s+6)}{s^3+3s^2+7s+5}$$

sol:

- order of H(s)
- zeros $\quad 3s=-6$
- poles $\quad s=-2$
- Z-P Map

$P(s) = s^3+3s^2+7s+5$

Trial and error

0, 1, -1

$s=-1$, $P(s)=0$

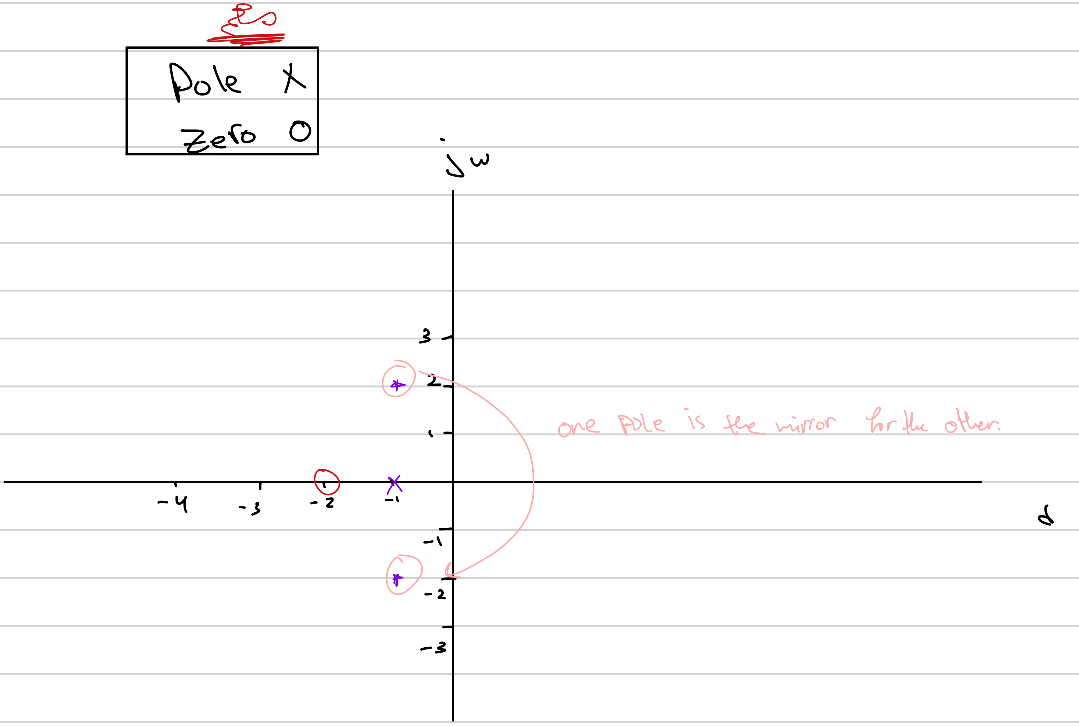
$(s+1)(s^2+2s+5)$

$$s+1 \overline{\big)\, s^3+3s^2+7s+5} \quad = s^2+2s+5$$

$$\frac{-B \pm \sqrt{B^2-4AC}}{2A}$$

$= -1 \pm 2j$

$$H(s) = \frac{(s-(-2))}{(s-(-1))(s-(-1-2j))(s-(-1+2j))}$$



Geo gebra
web site
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Second-Order Systems

Standard form

$$H(s) = \frac{\omega_n^2}{s^2 + 2\underset{\text{Zaita}}{\zeta}\omega_n s + \omega_n^2}$$

ω_n: natural frequency

$\omega = 2\pi f$ rad/sec

frequency (Hz)

ζ: damping ratio

By the quadratic formula, the poles are:

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$
$$= -\omega_n\left(\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

The nature of the poles changes depending on ζ: poles dep. on the value of ζ

- $\zeta > 1$ both poles are real and negative
- $\zeta = 1$ one negative pole (poles are real)
- $\zeta < 1$ two complex poles with negative real parts

all poles are real no imaginary part

$\zeta = 0 \to$ real part $= 0$ (un damped syst)

$0 < \zeta < 1 \to$ under damping

$\zeta = 1 \to$ critical damping

$$s = \underbrace{-\sigma}_{\text{real part}} \pm \underbrace{j\omega_d}_{\text{imaginary part}}$$

damping frequency

where $\sigma = \zeta\omega_n$, $\omega_d = \omega_n\sqrt{1 - \zeta^2}$

example : reduction for the following second order syst.

بنرجع ال SO لل general form.

① $\dfrac{32}{s^2+8s+16}$

$\Rightarrow 2\left[\dfrac{16}{s^2+8s+16}\right]$

$w_n = \sqrt{16} = 4$ rad/sec

$2\zeta w_n = 8$

$2\zeta(4) = 8$

$2\zeta = 2 \rightarrow \boxed{\zeta = 1}$

② $\dfrac{2+s}{s^2+8s+4}$

$\dfrac{2}{s^2+8s+4} + \dfrac{s}{s^2+8s+4}$

$\dfrac{1}{2}\left[\dfrac{4}{s^2+8s+4}\right] + \dfrac{s}{4}\left(\dfrac{4}{s^2+8s+4}\right)$

Prototype 2nd-Order System

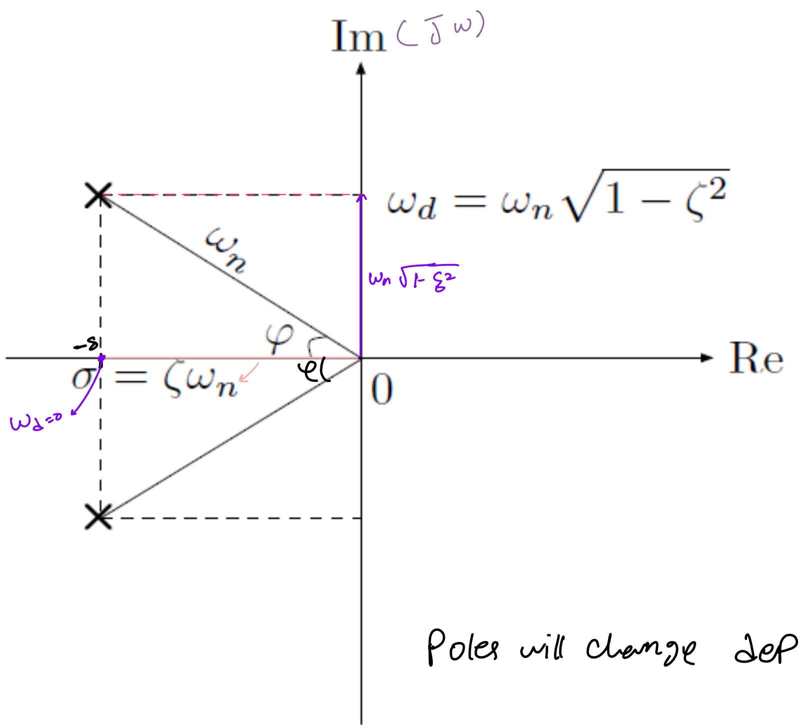
Syst is stable because both poles on the -ve side

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \qquad \zeta < 1$$

The poles are

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\sigma \pm j\omega_d$$

freq domain for the second order



Note that

$$\sigma^2 + \omega_d^2 = \zeta^2\omega_n^2 + \omega_n^2 - \zeta^2\omega_n^2$$
$$= \omega_n^2$$

$$\cos\varphi = \frac{\zeta\omega_n \rightarrow \text{المجاور}}{\omega_n \rightarrow \text{الوتر}} = \zeta$$

Poles will change dep on ξ ωn

2nd-Order Response

Let's compute the system's impulse and step response:

بعمله بالشكل هاد عشان ناخذ $\mathcal{L}^{-1}$ اله

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \boxed{\frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}}$$

- Impulse response:

Strainder form for $\mathcal{L}^{-1}$ for the Impulse response

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{(\omega_n^2/\omega_d)\omega_d}{(s+\sigma)^2 + \omega_d^2}\right\}$$

$$= \frac{\omega_n^2}{\omega_d} e^{-\sigma t} \sin(\omega_d t) \qquad \text{(table, \# 20)}$$

- Step response:

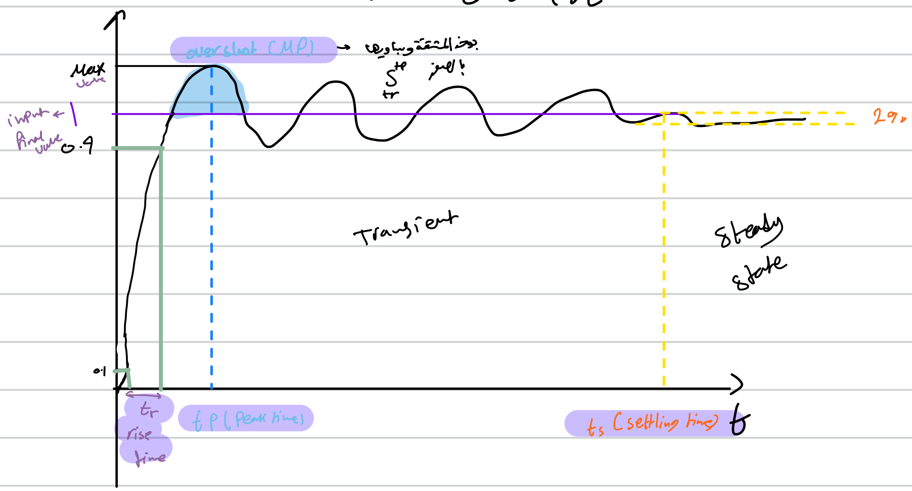
$$\mathcal{L}^{-1}\left\{\frac{H(s)}{s}\right\} = \mathcal{L}^{-1}\left\{\frac{\sigma^2 + \omega_d^2}{s[(s+\sigma)^2 + \omega_d^2]}\right\}$$

solved using the partial fraction expansion.

$$\frac{K_1}{s} + \frac{K_2}{(s+\sigma)^2 + \omega_d^2}$$

$$= 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right) \qquad \text{(table, \#21)}$$

Syst will stay stable as long as σ is always +ve when δ is +ve



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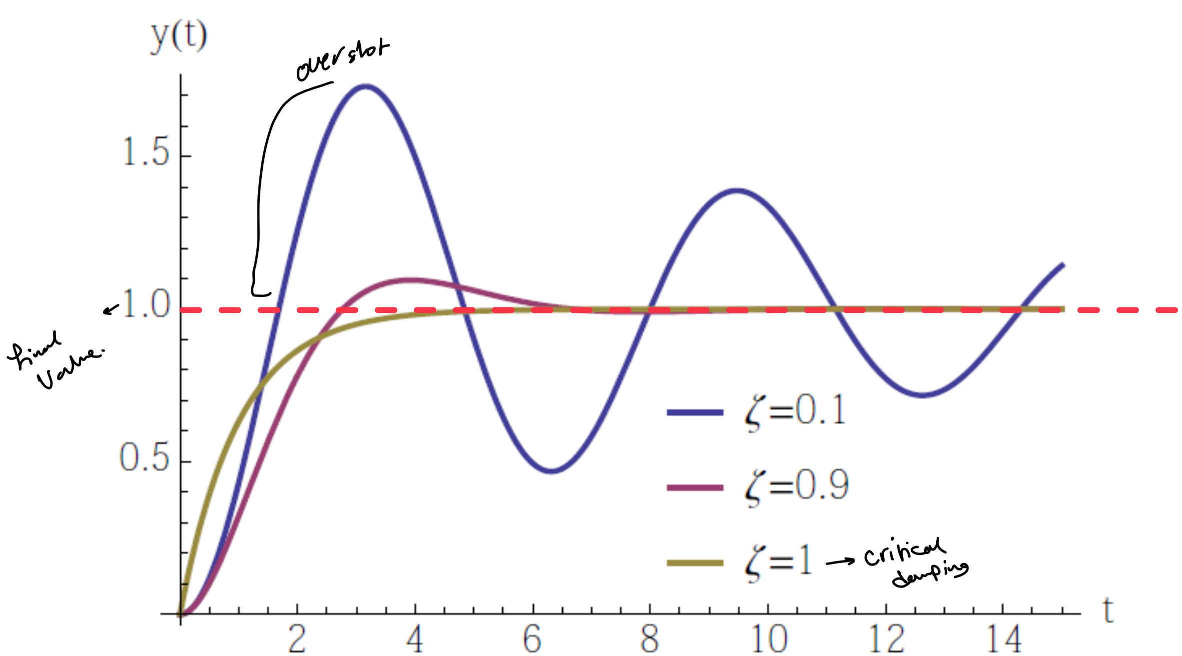
$$MP = \frac{\text{max value} - \overset{\text{usually } 1}{\text{input}}}{\text{input}}$$

2nd-Order Step Response

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

$$u(t) = 1(t) \qquad \longrightarrow \qquad y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

where $\sigma = \zeta\omega_n$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$ (damped frequency)



The parameter ζ is called the *damping ratio*

- $\zeta > 1$: system is overdamped
- $\zeta < 1$: system is underdamped
- $\zeta = 0$: no damping $\Rightarrow$ poles are imaginary
  ($\omega_d = \omega_n$) oselations
  (syst is almost unstable)
- $\zeta = 1$: critical damping. (real poles)

2nd-Order Step Response

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

$$u(t) = 1(t) \qquad \longrightarrow \qquad y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

where $\sigma = \zeta\omega_n$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$ (damped frequency)

We will see that the parameters ζ and ω_n determine certain important features of the transient part of the above step response.

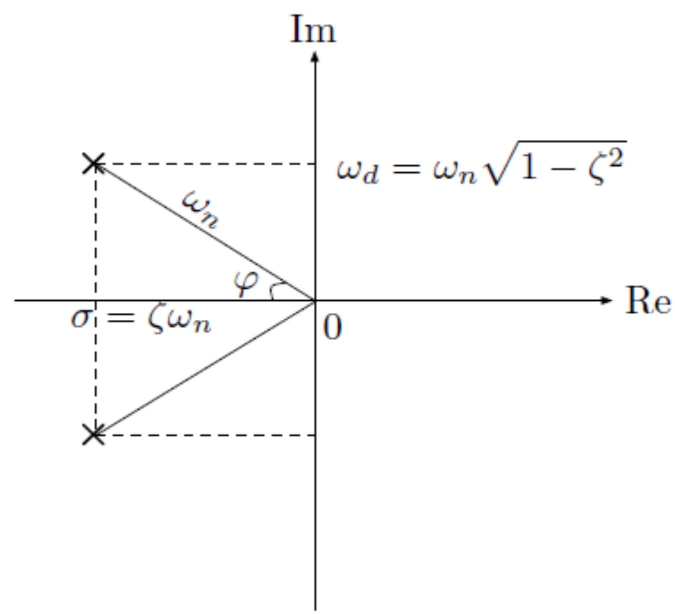
We will also learn how to pick ζ and ω_n in order to *shape* these features according to given *specifications*.

Transient Response Specs

Now let's consider the more interesting case: *2nd-order response*

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

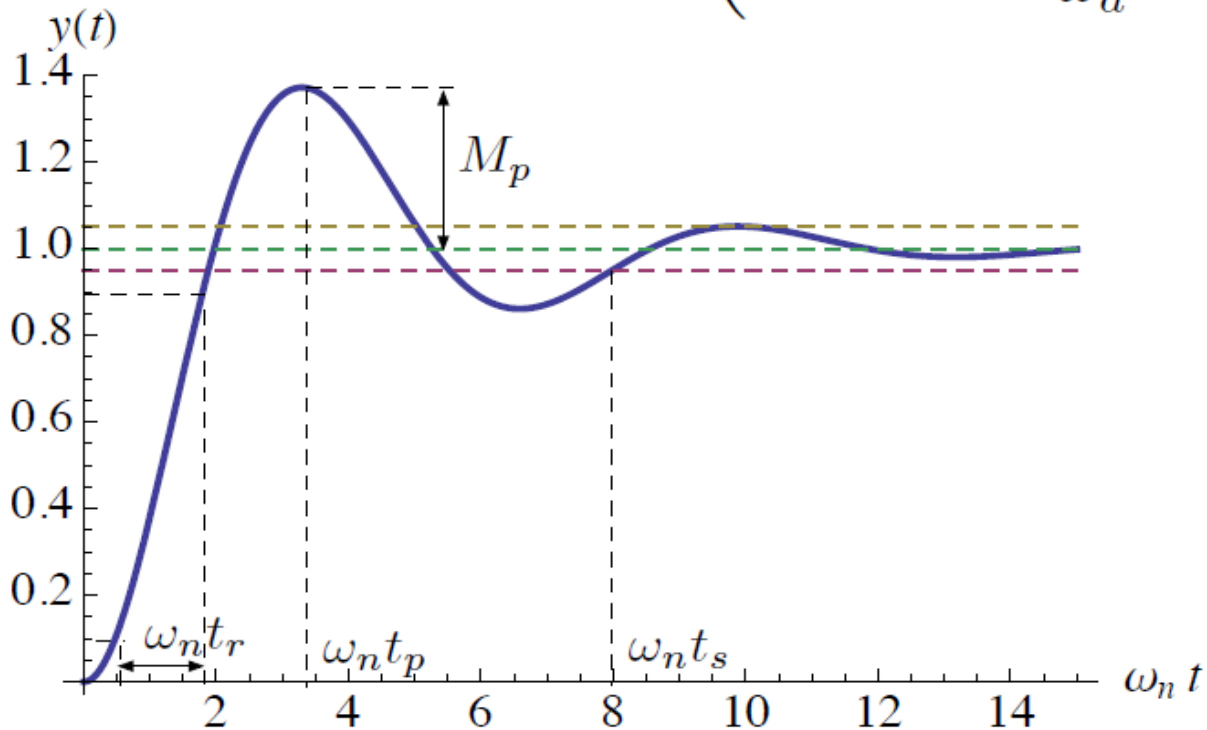
where $\sigma = \zeta\omega_n \; \omega_d = \omega_n\sqrt{1-\zeta^2} \qquad (\zeta < 1)$



Step response: $\quad y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$

Transient-Response Specs

Step response: $$y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

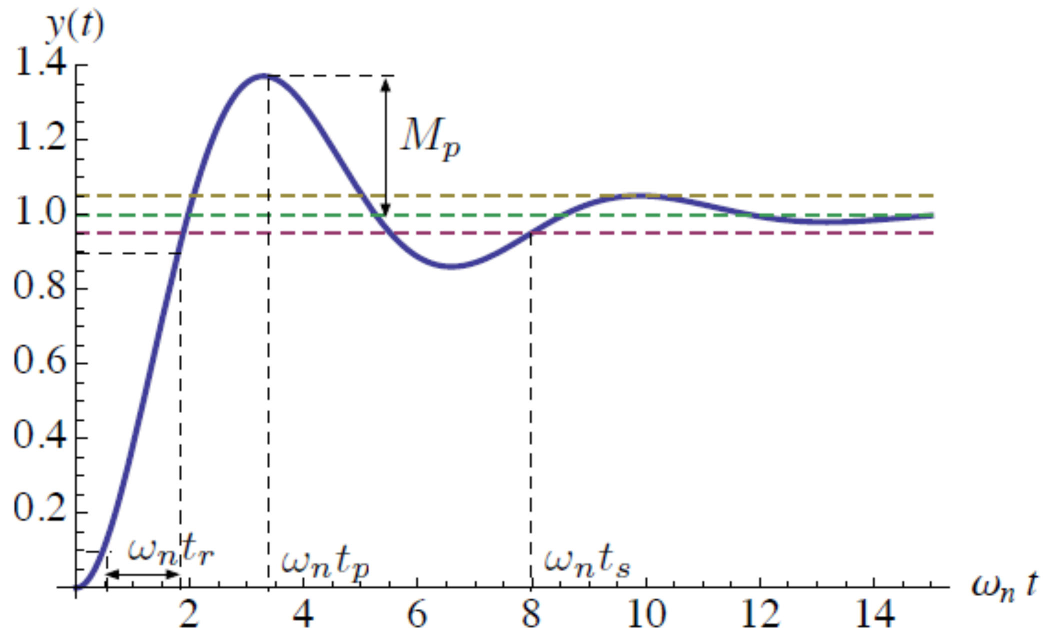


- rise time t_r — time to get from $0.1y(\infty)$ to $0.9y(\infty)$
- overshoot M_p and peak time t_p
- settling time t_s — first time for transients to decay to within a specified small percentage of $y(\infty)$ and stay in that range (we will usually worry about 5% settling time)

سواء 2% أو 5% صح

اعلم انه اذ t_p قليلة وكل ما كانت

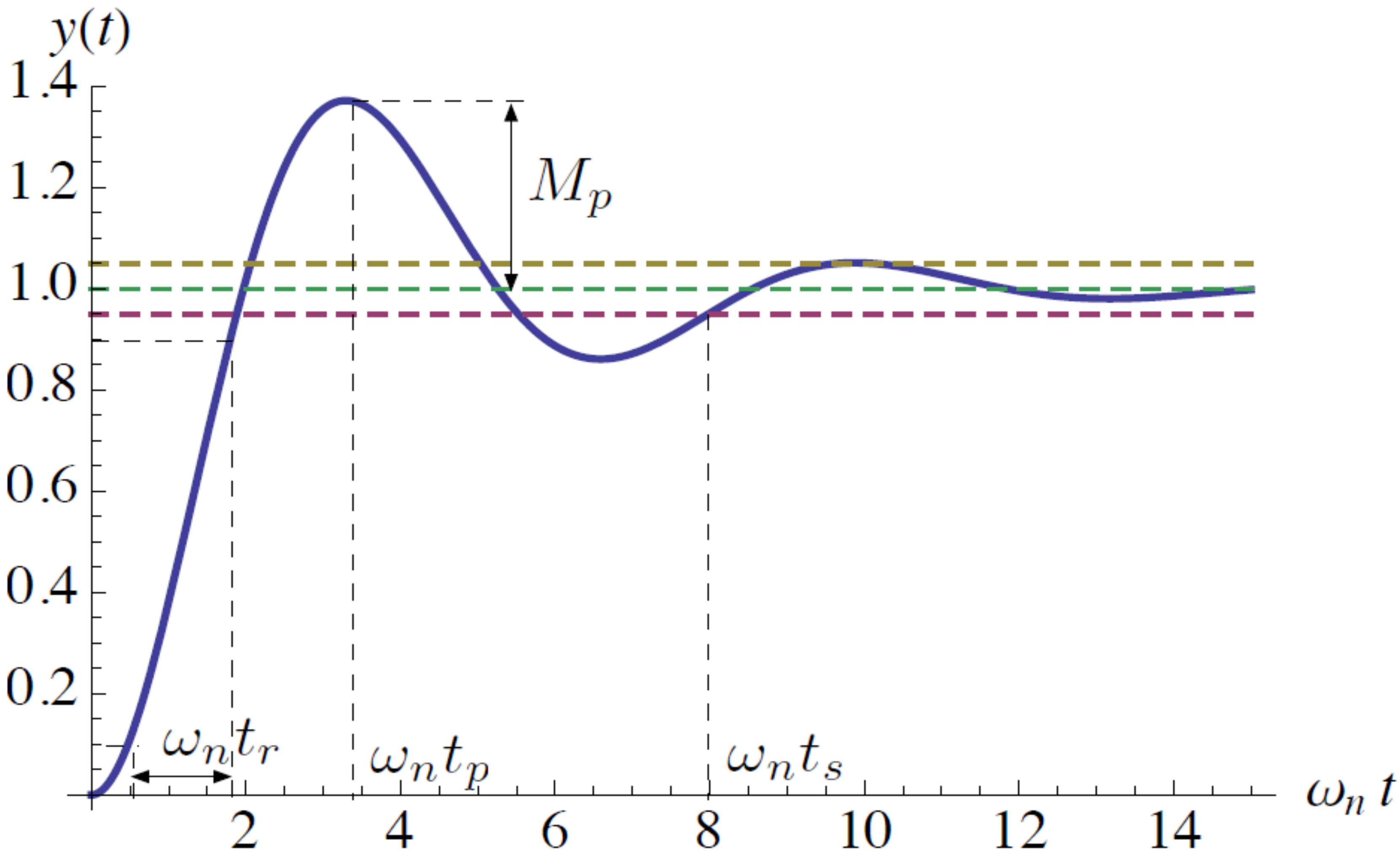
Transient-Response (or Time-Domain) Specs



Do we want these quantities to be large or small?

- t_r small
- M_p small
- t_p small
- t_s small

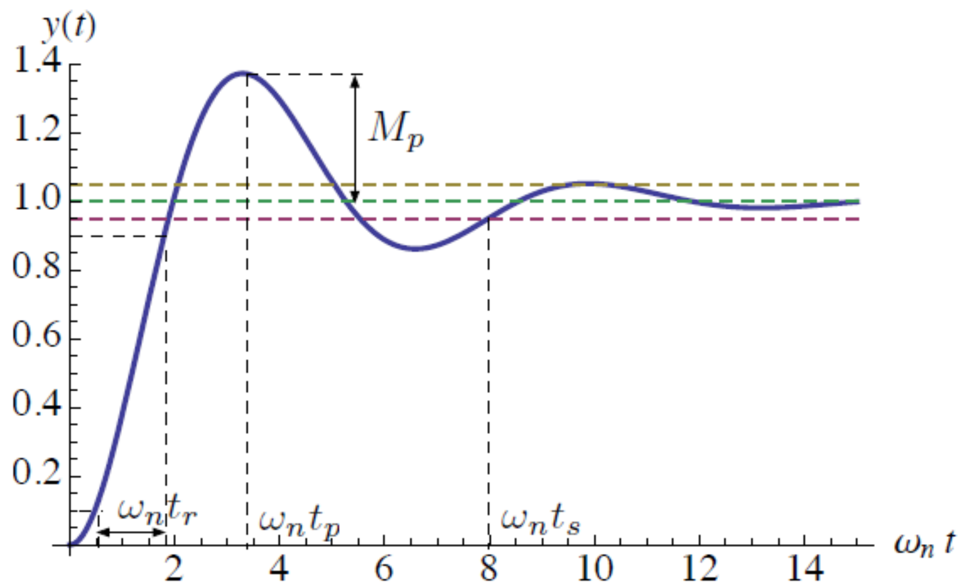
Trade-offs among specs: decrease $t_r \longrightarrow$ increase M_p, etc.



y(t)
1.4
1.2
1.0
0.8
0.6
0.4
0.2
M_p
ω_n t_r
ω_n t_p
ω_n t_s
2
4
6
8
10
12
14
ω_n t

الاشتقاق مش مطلوب
بس المهم القانون النهائي

Formulas for TD Specs: Rise Time



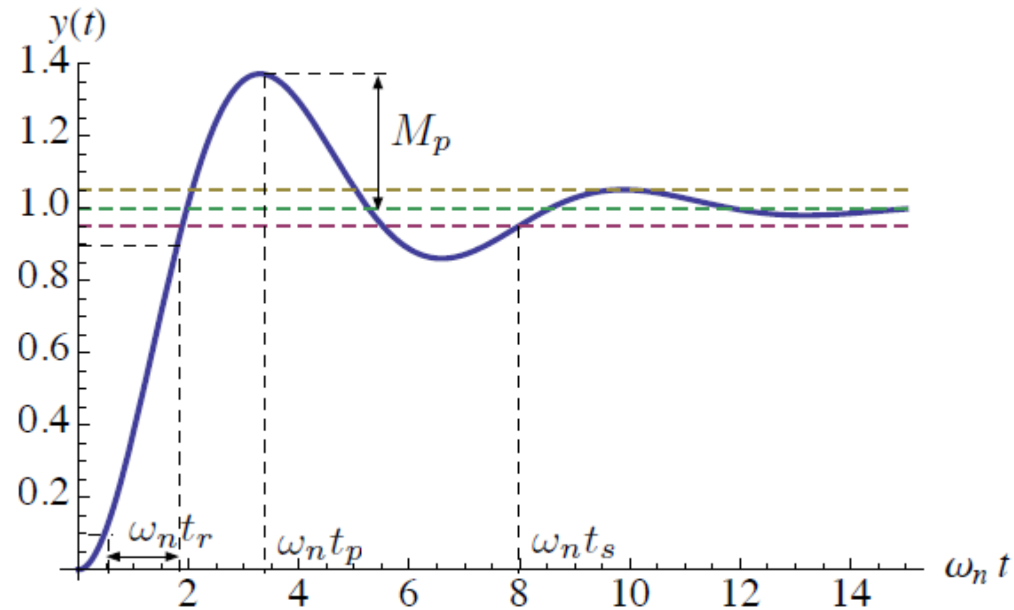
Rise time t_r — hard to calculate analytically.
Empirically, on the normalized time scale ($t \to \omega_n t$), rise times are *approximately* the same

$$w_n t_r \approx 1.8 \qquad (\text{exact for } \zeta = 0.5)$$

كل ما بعدنا
عن هاي القيمة
ال error بزيد

So, we will work with $t_r \approx \dfrac{1.8}{\omega_n}$ (good approx. when $\zeta \approx 0.5$)

Formulas for TD Specs: Overshoot & Peak Time



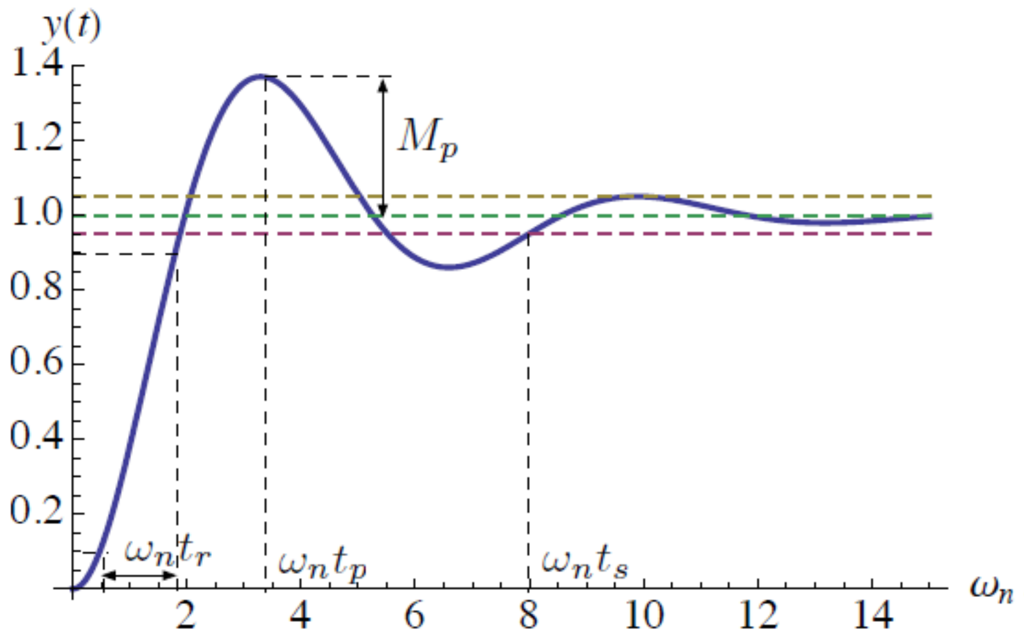
t_p is the *first time* $t > 0$ when $y'(t) = 0$

$$y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

$$y'(t) = \left(\frac{\sigma^2}{\omega_d} + \omega_d\right) e^{-\sigma t}\sin(\omega_d t) = 0 \text{ when } \omega_d t = 0, \pi, 2\pi, \ldots$$

so $t_p = \dfrac{\pi}{\omega_d}$

Formulas for TD Specs: Overshoot & Peak Time



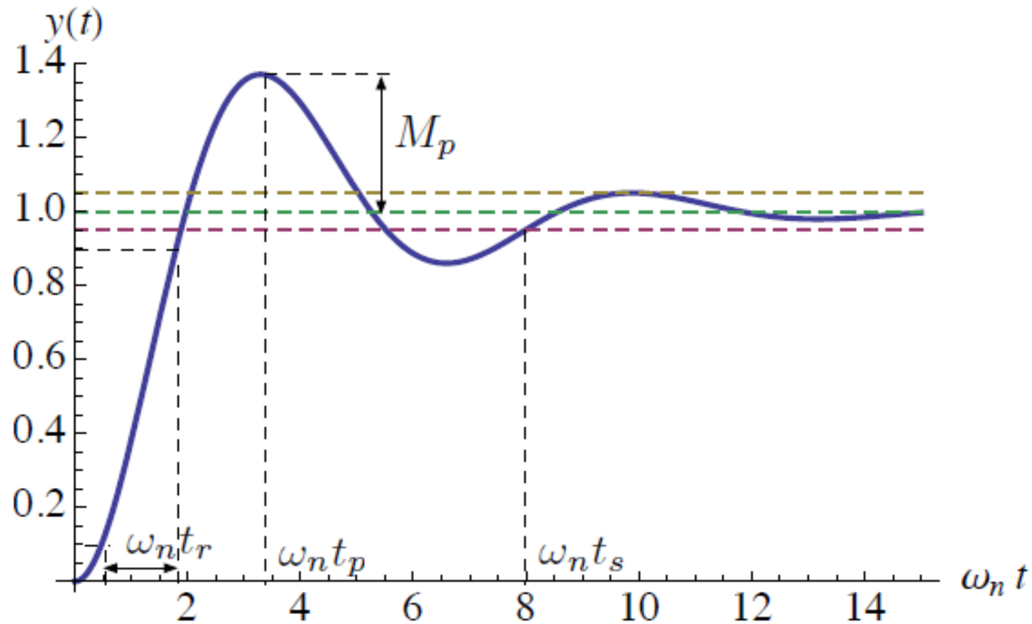
We have just computed $t_p = \dfrac{\pi}{\omega_d}$

To find M_p, plug this value into $y(t)$:

$$M_p = y(t_p) - 1 = -e^{-\frac{\sigma\pi}{\omega_d}} \left(\cos\left(\omega_d \frac{\pi}{\omega_d}\right) + \frac{\sigma}{\omega_d} \sin\left(\omega_d \frac{\pi}{\omega_d}\right)\right)$$

$$= \exp\left(-\frac{\sigma\pi}{\omega_d}\right) = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) \quad \text{— exact formula}$$

Formulas for TD Specs: Settling Time



$t_s = \min\left\{ t > 0 : \dfrac{|y(t') - y(\infty)|}{y(\infty)} \le 0.05 \text{ for all } t' \ge t \right\}$ (here, $y(\infty) = 1$)

$$|y(t) - 1| = e^{-\sigma t} \left| \cos(\omega_d t) + \frac{\sigma}{\omega_d} \sin(\omega_d t) \right|$$

here, $e^{-\sigma t}$ is what matters (sin and cos are bounded between ± 1), so $e^{-\sigma t_s} \le 0.05$ this gives $t_s = -\dfrac{\ln 0.05}{\sigma} \approx \dfrac{3}{\sigma}$

Formulas for TD Specs

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\sigma^2 + \omega_d^2}{(s+\sigma)^2 + \omega_d^2}$$

$$t_r \approx \frac{1.8}{\omega_n}$$

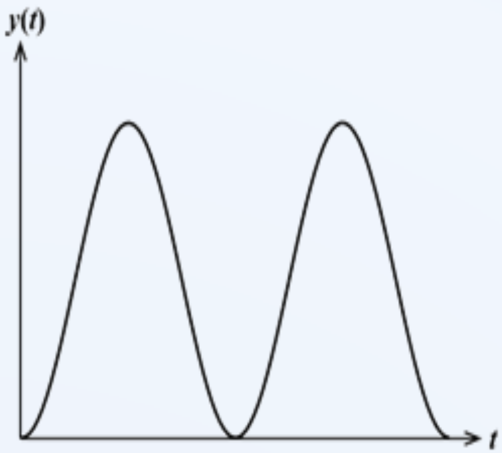
$$t_p = \frac{\pi}{\omega_d}$$

$$M_p = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

$$t_s \approx \frac{3}{\sigma}$$

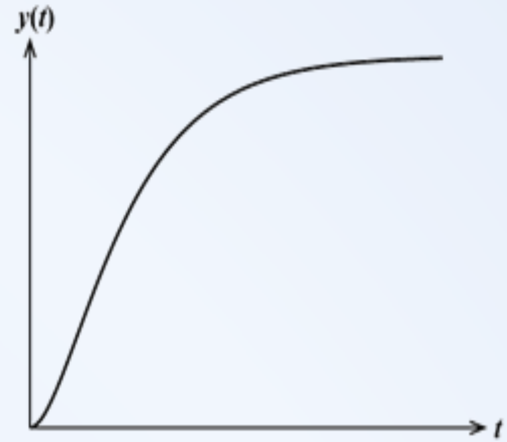
Second-Order Systems

Mid



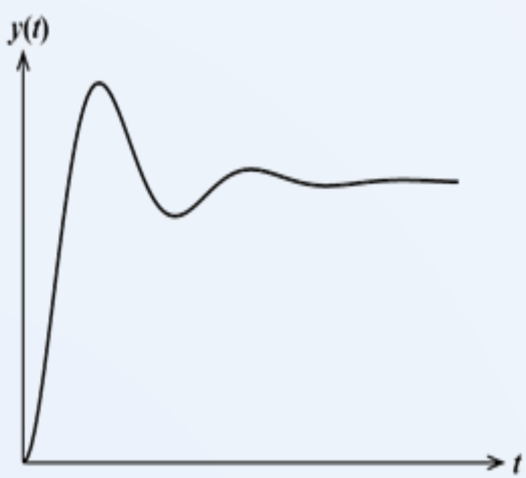
($\zeta = 0$) undamped

fluctuation بيضل للأبد

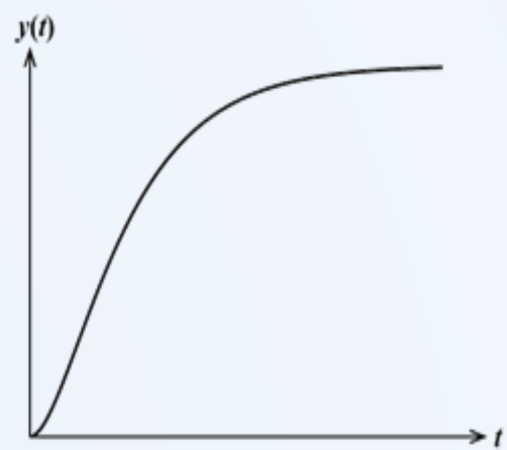


($\zeta = 1$) crit damped

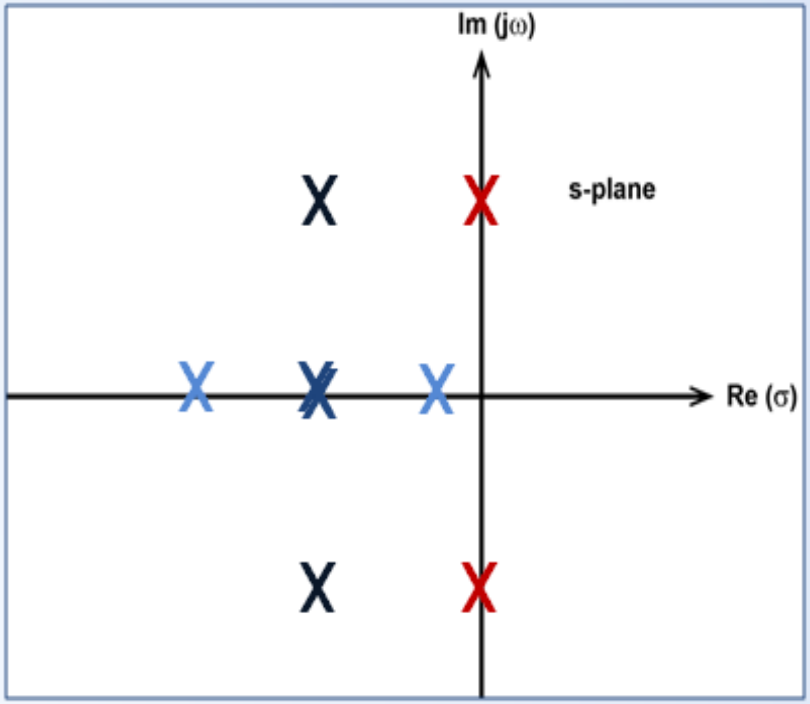
-ve poles same values (the real)



($0 < \zeta < 1$) underdamped



($\zeta > 1$) overdamped → -ve poles (real) diff values



second order syst. اذا ما دخلت step responce كان ال

$$G(s) = \frac{K\,\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$\lim_{s \to 0} G(s) = K$$

↳ DC Value

Steady state Value

$$y(t) = K\left[1 - e^{-\delta t}\left(\cos(\omega_d t) + \frac{\delta}{\omega_d}\sin(\omega_d t)\right)\right]$$

TD Specs in Frequency Domain

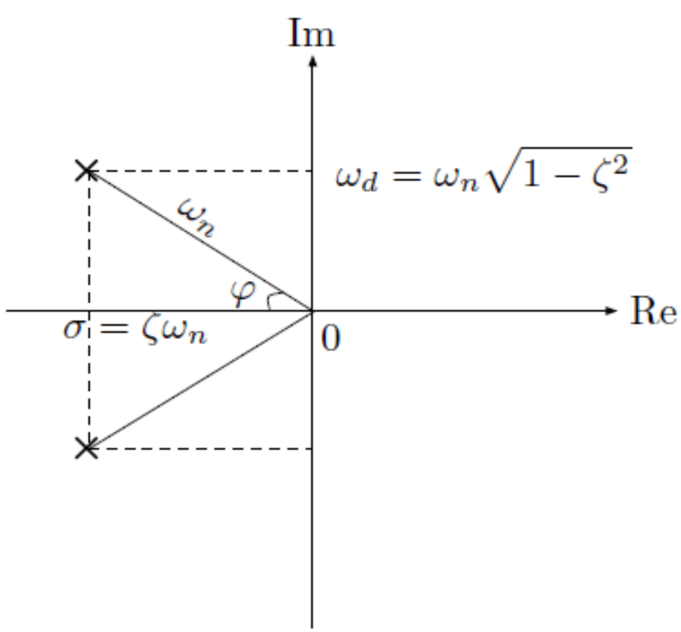
We want to *visualize* time-domain specs in terms of *admissible pole locations* for the 2nd-order system

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\sigma^2 + \omega_d^2}{(s+\sigma)^2 + \omega_d^2}$$

where $\sigma = \zeta\omega_n$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Step response: $y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$



$$\omega_n^2 = \sigma^2 + \omega_d^2$$

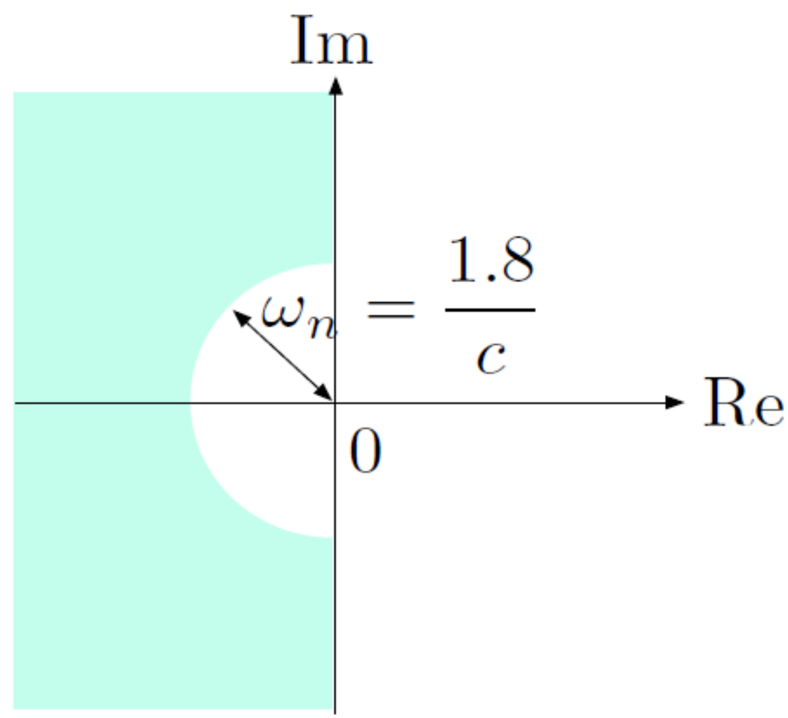
$$\zeta = \cos\varphi$$

Rise Time in Frequency Domain

Suppose we want $t_r \leq c$ (c is some desired given value)

$$t_r \approx \frac{1.8}{\omega_n} \leq c \qquad \Longrightarrow \qquad \omega_n \geq \frac{1.8}{c}$$

Geometrically, we want poles to lie in the shaded region:



(recall that ω_n is the *magnitude of the poles*)

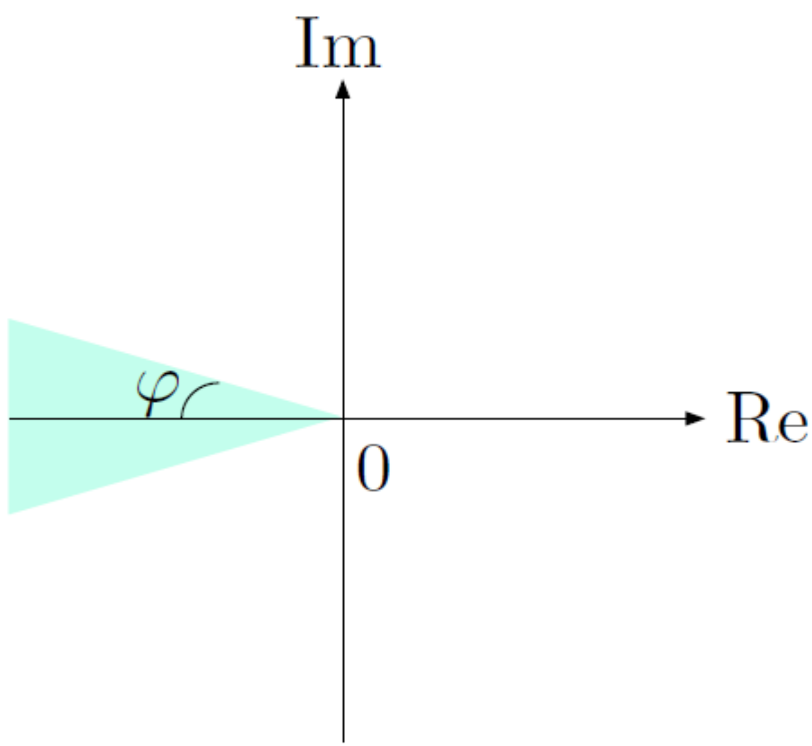
Overshoot in Frequency Domain

Suppose we want $M_p \le c$

$$M_p = \underbrace{\exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)}_{\text{decreasing function}} \le c$$

— need large damping ratio

Geometrically, we want poles to lie in the shaded region:



$$\frac{\zeta}{\sqrt{1-\zeta^2}} = \frac{\omega_n\zeta}{\omega_n\sqrt{1-\zeta^2}}$$
$$= \frac{\sigma}{\omega_d} = \cot\varphi$$

— need φ to be small

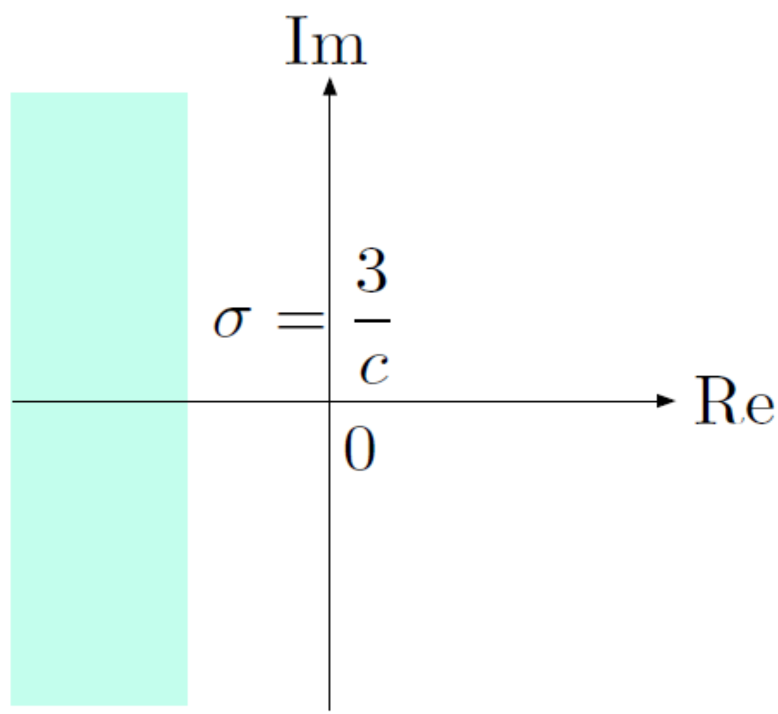
Intuition: good damping $\rightarrow$ good decay in $1/2$ period

Settling Time in Frequency Domain

Suppose we want $t_s \leq c$

$$t_s \approx \frac{3}{\sigma} \leq c \qquad \Longrightarrow \qquad \sigma \geq \frac{3}{c}$$

Want poles to be sufficiently fast (large enough magnitude of real part):



Intuition: poles far to the left $\rightarrow$ transients decay faster $\rightarrow$ smaller t_s

Questions

(a) The closed-loop transfer function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{12K}{s^2 + 12s + 12K} \ .$$

The percent overshoot specification $P.O. \leq 10\%$ implies $\zeta \geq 0.59$. From the characteristic equation we find that

$$\omega_n^2 = 12K \quad \text{and} \quad \zeta\omega_n = 6 \ .$$

Solving for K yields

$$2(0.59)\sqrt{12K} = 12 \quad \text{which implies that} \quad K = 8.6 \ .$$

So, any gain in the interval $0 < K < 8.6$ is valid. The settling time is $T_s = 4/\zeta\omega_n = 4/6$ seconds and satisfies the requirement. Notice that T_s is not a function of K.