Industrial Control Systems

Chapter Two:
Mathematical Modeling

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Types of Systems

- Static System: If a system does not change with time, it is called a static system.
- Dynamic System: If a system changes with time, it is called a dynamic system.

Dynamic Systems

- A system is said to be dynamic if its current output may depend on the past history as well as the present values of the input variables.
- Mathematically:

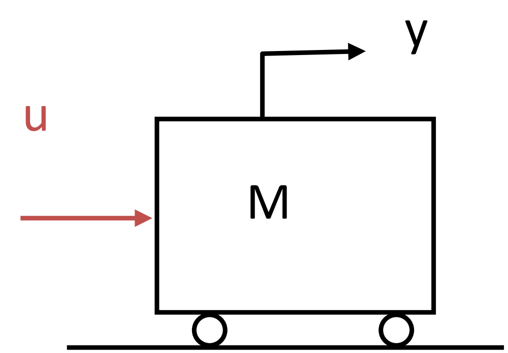
$$y(t) = \varphi[u(\tau), 0 \le \tau \le t]$$

$$u : \text{Input}, \; t : \text{Time}$$

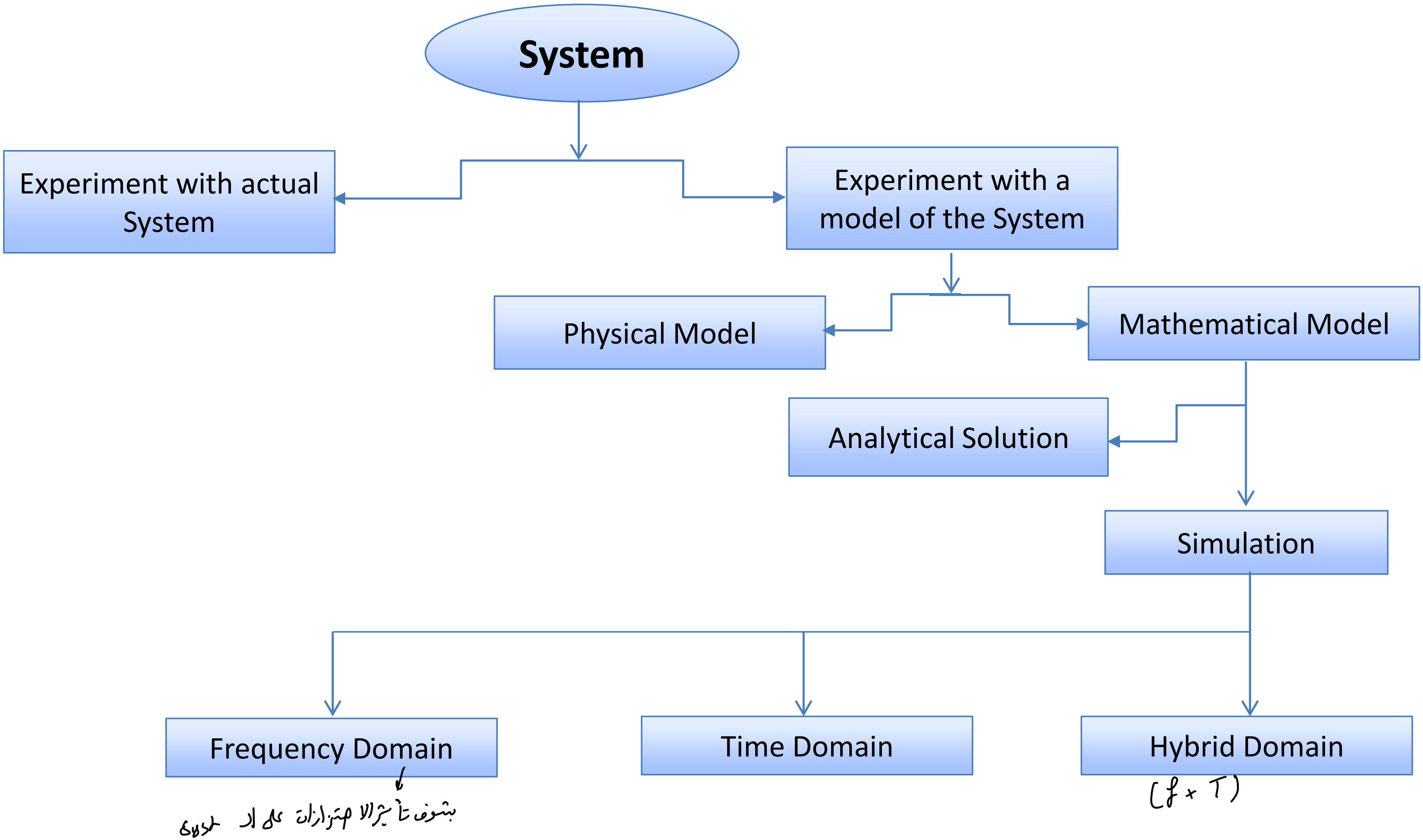
Example: A moving mass

Model: Force=Mass x Acceleration

$$M\ddot{y} = u$$



Ways to Study a System



Model

- "Dictionary.com" defines a model as *"A systematic description of an object or phenomenon that shares important characteristics with the object or phenomenon."*

  not all characteristics.

- So, models present a systematic, and most often simplified description of what they represent.

  and abstract description

- Such a description is a helpful instrument to study the characteristics of what the model represents.

Model

Connect the output with the inputs of the system

$$Model = \frac{output}{Input}$$

- A model is a simplified representation or abstraction of reality.
- Reality is generally too complex to copy exactly.
- Much of the complexity is actually irrelevant in problem solving.

Mathematical Model

- A set of mathematical equations (e.g., differential eqs.) that describes the input-output behaviour of a system.

- Mathematical models of physical systems are key elements in the design and analysis of control systems.
- The dynamic behaviour is generally described by ordinary differential equations.
- The differential equations describing the dynamic performance of a physical system are obtained by utilizing the physical laws of the process

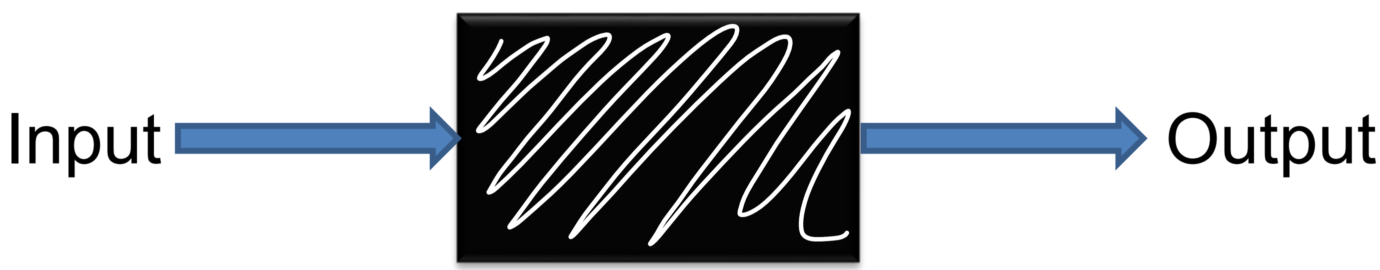
Mathematical Model

- What is a model used for?

- Simulation
- Prediction/Forecasting
- Prognostics/Diagnostics
- Design/Performance Evaluation
- Control System Design

Black Box Model

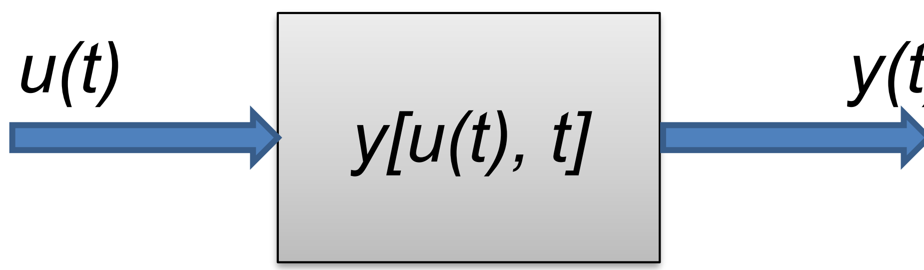
- When only input and output are known.
- Internal dynamics are either too complex or unknown.
- Easy to Model



black box = can't creat a mathematical modle for the syst.

Grey Box Model

- When input and output and some information about the internal dynamics of the system is known.
- Easier than white box Modelling.



grey box if some relation can be made into models

only the others can only be predicted.

White Box Model

- When input and output and internal dynamics of the system is known.
- One should know have complete knowledge of the system to derive a white box model.

$u(t)$ → $$\frac{dy(t)}{dt} = 3\frac{du(t)}{dt} - \frac{d^2y(t)}{dt^2}$$ → $y(t)$

white boxe can be rep all in a node

Six Step Approach to Dynamic System Modeling

- Define the system and its components.
- Formulate the mathematical model and list the necessary assumptions.
- Write the differential equations describing the model.
- Solve the equations for the desired output variables.
- Examine the solutions and the assumptions.
- If necessary, reanalyse or redesign the system.

Mathematical modeling of Physical Systems (Electrical Systems)

Basic Elements of Electrical Systems

Symbol →

- The time domain expression relating voltage and current for the resistor is given by Ohm's law i-e

$$v_R(t) = i_R(t)R$$

- The Laplace transform of the above equation is

$$V_R(s) = I_R(s)R$$

Basic Elements of Electrical Systems

- The time domain expression relating voltage and current for the Capacitor is given as:

$$v_c(t) = \frac{1}{C}\int i_c(t)dt$$

$V_c(t) = \frac{1}{C}\int i_c(t)dt$

$V_c(s) = \frac{1}{C} I_c(s)$

- The Laplace transform of the above equation (assuming there is no charge stored in the capacitor) is

$$V_c(s) = \frac{1}{Cs} I_c(s)$$

Basic Elements of Electrical Systems

- The time domain expression relating voltage and current for the inductor is given as:

$$v_L(t) = L\frac{di_L(t)}{dt}$$

- The Laplace transform of the above equation (assuming there is no energy stored in inductor) is

$$V_L(s) = LsI_L(s)$$

V-I and I-V relations

Component	Symbol	V-I Relation	I-V Relation
Resistor		$v_R(t) = i_R(t)R$	$i_R(t) = \frac{v_R(t)}{R}$
Capacitor		$v_c(t) = \frac{1}{C}\int i_c(t)dt$	$i_c(t) = C\frac{dv_c(t)}{dt}$
Inductor		$v_L(t) = L\frac{di_L(t)}{dt}$	$i_L(t) = \frac{1}{L}\int v_L(t)dt$

RC Circuit

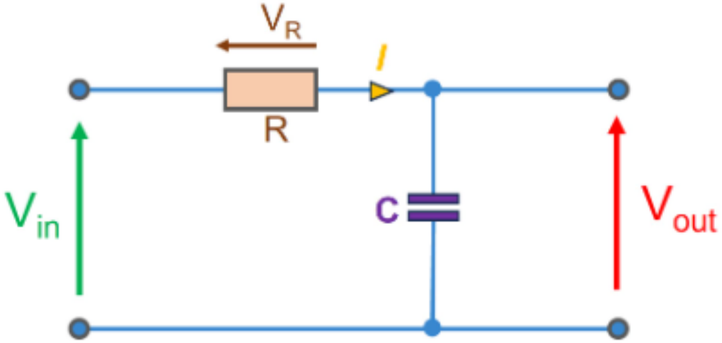


Figure 1: Schematic Diagram of the RC Circuit Configuration

Step 1: Applying Kirchhoff's voltage law (KVL) to the series RC circuit, we can write:

$$V_{in} = V_R + V_{out}$$

Step 2: Expressing the voltages using Ohm's law :

$$V_{in} = I * R + V_{out}$$

Step 3: Substituting I with $I = C * \frac{dV_{out}}{dt}$:

$$V_{in} = (C * \frac{dV_{out}}{dt}) * R + V_{out}$$

Step 4: Rearranging the equation and dividing both side with RC

$$\frac{dV_{out}}{dt} + \frac{1}{RC} * V_{out} = \frac{1}{RC} * V_{in}$$

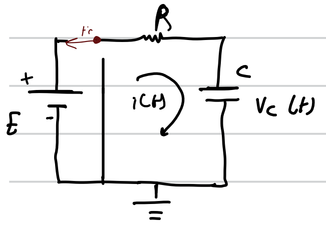
Step 5: Defining the time constant $\tau = RC$

$$\frac{dV_{out}}{dt} + \frac{1}{\tau} * V_{out} = \frac{1}{\tau} * V_{in}$$

Step 6: Differential equation (Time domain)

$$\tau * \frac{dV_{out}}{dt} = V_{in} - V_{out}$$

How to create a model from an electrical circuit.



in this model connect E with $V_c(t)$

input (المدخلات) — output (الناتج هو الجهد الكهربائي / ال volt اللي في ال c)

بدي معادلة رياضية تربط بين ال input , output

Assumptions : $V_C(0) = 0$

physics : القوانين اللي راح استخدمها

$V_R = R\, i(t)$ — $i(t)$: التيار عند زمن t

$V_c = \frac{q}{c} \Rightarrow dq = dV_c\, C$ (const.)

$i(t) = \frac{dq}{dt}$ ← $dq = C\, dV_c(t)$

$$i(t) = C \frac{dV_c(t)}{dt}$$

Kirchoff's voltage law (KVL) :

$$-E + \underbrace{R\, i(t)}_{V_R} + V_c(t) = 0$$

$$-E + RC \frac{dV_c(t)}{d(t)} + V_c(t) = 0$$

- first order diffrintial equ.
- describes a first order syst.

RC

$$\frac{dV_c(t)}{dt} + \frac{1}{RC} V_c(t) = \frac{E}{RC}$$

⇒ this eq is in time domain

first order linear diff equation :

① $\dot{Y} + P(x)\, Y = g(x)$ ⟶ رتب المعادلة جيدا

② $Y(x) = \frac{\int \mu(x) g(x)}{\int \mu(x)}$ ⟶ بحلها باستخدام

integral factor : $\mu(x) = e^{\int P(x) dx}$

Symbol	Meaning
y	The dependent variable (the unknown function we want to find)
x	The independent variable
$\frac{dy}{dx}$ or $\dot{y}$	Derivative of y with respect to x
$P(x)$	A known function of x, the coefficient of y
$g(x)$	A known function of x, the non-homogeneous term (right-hand side)
$\mu(x)$	**Integrating factor** used to simplify the equation
C	Constant of integration

$Y = V_c(t)$

$p(t) = \frac{1}{RC}$

$g(t) = \frac{E}{RC}$

$\mu(t) = e^{\int \frac{1}{RC} dt} = e^{t/RC}$

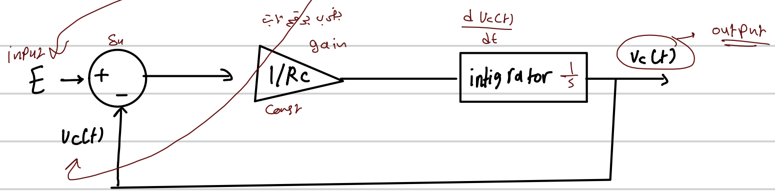
$$\Rightarrow V_c(t) = \frac{\int^t e^{t/RC} (E/RC)}{e^{t/RC}}$$

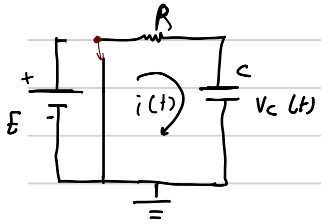
$$V_c(t) = E\,(1 - e^{-t/RC})$$

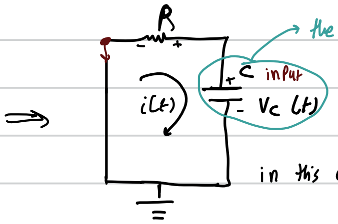
$$\frac{dV_c(t)}{dt} = \frac{E - V_c(t)}{RC} = \frac{1}{RC}\left(E - V_c(t)\right)$$

How to put the mathmatical equation in a block diagram ?

$$\frac{dV_c(t)}{dt} = \frac{1}{RC}\left(E - V_c(t)\right)$$







→ the Capasitar is full

$V_c(0) = E$

where the capcitar is charged full $= E$

in this circuit C is the energy source.

KVL:

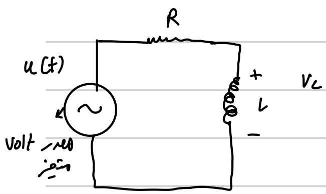
$$-V_c(t) + i(t)\,R = 0$$

$$-V_c(t) + CR\,\frac{d\,V_c(t)}{dt} = 0$$

$$\frac{d\,V_c(t)}{d(t)} - \frac{1}{CR}\,V_c(t) = 0$$

Sol → $\boxed{V_c(t) = E\,e^{-t/RC}}$

RL Circuit → RL ايه مجال مغناطيسي بره circuit.



$$V_L(t) = L\,\frac{d\,i(t)}{dt}$$

KVL:

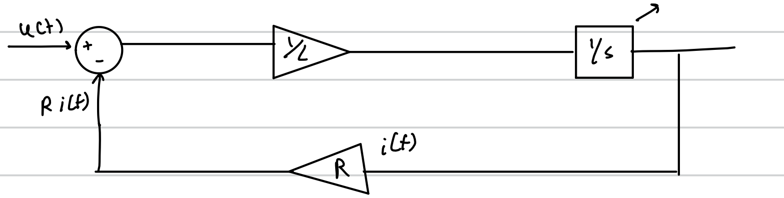
$$-u(t) + R\,i(t) + L\,\frac{di(t)}{dt} = 0$$

$$\frac{L\,di(t)}{dt} = u(t) - R\,i(t)$$

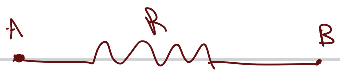
$$\frac{di(t)}{dt} = \frac{1}{L}\,(u(t) - R\,i(t))$$

دائماً خلي ال diff
على جهة لحاله

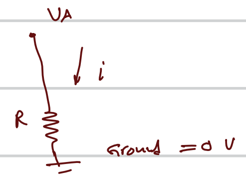
$$\frac{di(t)}{d(t)} = \frac{1}{L}\,(u(t) - R\,i(t))$$



⇐ كل القوانين تعتمد على مبدأ حفظ الطاقة

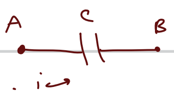


$$i = \frac{(V_A - V_B)}{R}$$



$$i = \frac{V_A - 0}{R}$$

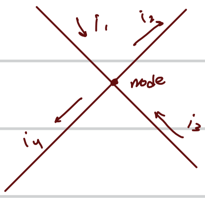
$$i = V_A / R$$



$$i_c(t) = C \frac{d\,(V_A^{(t)} - V_B^{(t)})}{dt}$$

$$i_L(t) = \frac{1}{L} \int (V_A^{(t)} - V_B^{(t)})\, dt$$

$$V_A^{(t)} - V_B^{(t)} = L \frac{d\, i_L(t)}{dt}$$



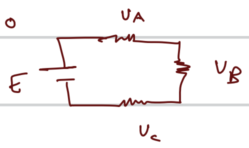
Kirchoff's Current Law (KCL)

$$i_1 + i_3 - i_2 - i_4 = 0$$

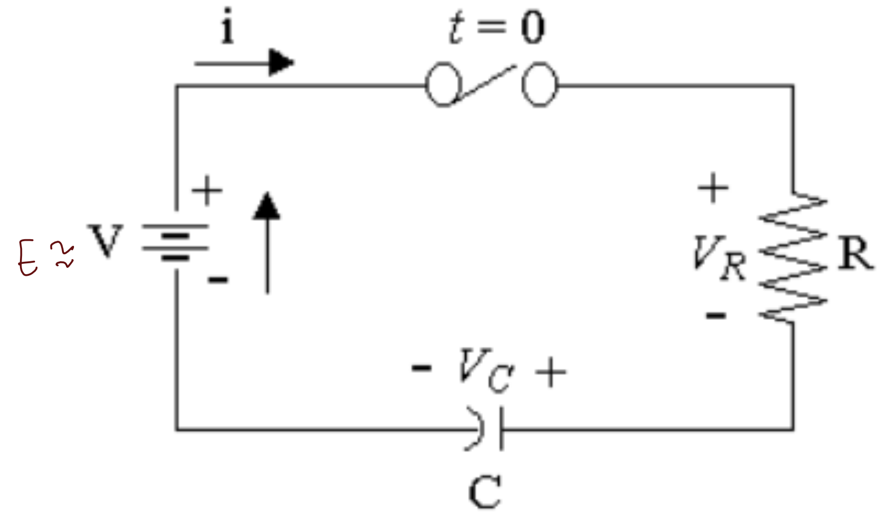
$$i_1 + i_3 = i_2 + i_4$$

Kirchoff's voltage low (KVL)

$$-E + V_A + V_B + V_c = 0$$



RC Circuit



An RC series circuit

$$V_R = Ri$$

and

$$V_C = \frac{1}{C}\int i\,dt$$

Kirchhoff's voltage law says the total voltages must be zero. So applying this law to a series RC circuit results in the equation:

$$Ri + \frac{1}{C}\int i\,dt = V$$

One way to solve this equation is to turn it into a **differential equation**, by differentiating throughout with respect to t:

$$R\frac{di}{dt} + \frac{i}{C} = 0$$

Solving the equation gives us:

$$i = \frac{V}{R}e^{-t/RC}$$

RC Circuit

If $P = P(x)$ and $Q = Q(x)$ are functions of x only, then

$$\frac{dy}{dx} + Py = Q$$

is called a **linear differential equation order 1.**

We can solve these **linear** DEs using an **integrating factor.**

For linear DEs of order 1, the integrating factor is:

$$e^{\int P\,dx}$$

The **solution** for the DE is given by multiplying y by the integrating factor (on the left) and multiplying Q by the integrating factor (on the right) and integrating the right side with respect to x, as follows:

$$ye^{\int P\,dx} = \int \left(Qe^{\int P\,dx}\right) dx + K$$

We start with:

$$R\frac{di}{dt}+\frac{i}{C}=0$$

Divide through by R:

$$\frac{di}{dt}+\left(\frac{1}{RC}\right)i=0$$

Identify P and Q:

$$P=\frac{1}{RC}$$

Q = 0

Find the integrating factor (our independent variable is t and the dependent variable is i):

$$\int P\,dt=\int\frac{1}{RC}\,dt=\frac{1}{RC}t$$

So

$$IF=e^{t/RC}$$

Now for the right hand integral of the 1st order linear solution:

$$\int Qe^{\int P\,dt}\,dt=\int 0\,dt=K$$

RC Circuit

Applying the linear first order formula:

$$ie^{t/RC} = K$$

Since $i = \dfrac{V}{R}$ when $t = 0$:

$$K = \frac{V}{R}$$

Substituting this back in:

$$ie^{t/RC} = \frac{V}{R}$$

Solving for i gives us the required expression:

$$i = \frac{V}{R}e^{-t/RC}$$

RC Circuit

The **time constant** in the case of an RC circuit is:

$$\tau = RC$$

The function

$$i = \frac{V}{R}e^{-t/RC}$$

has an **exponential decay** shape as shown in the graph. The current stops flowing as the capacitor becomes fully charged.

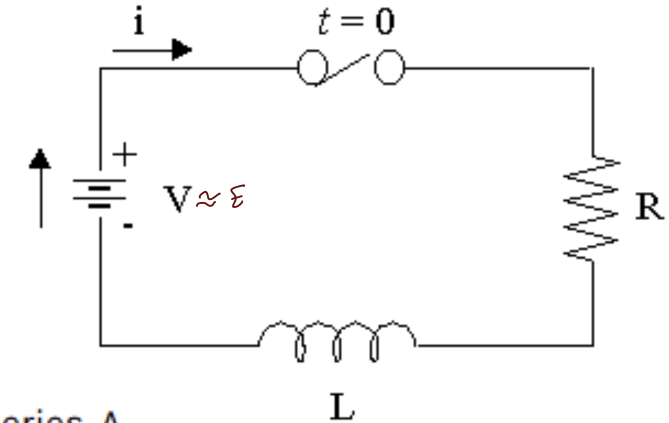
Graph of $i = \frac{V}{R}e^{-(t/RC)}$, an exponential decay curve.

Applying our expressions from above, we have the following expressions for the voltage across the resistor and the capacitor:

$$V_R = Ri = Ve^{-t/RC}$$

$$V_C = \frac{1}{C}\int i\,dt = V\left(1 - e^{-t/RC}\right)$$

RL Circuit



RL circuit diagram

The RL circuit shown above has a resistor and an inductor connected in series. A constant voltage V is applied when the switch is closed.

diff eq
↳ first order
↳ second order
integration

The (variable) voltage across the **resistor** is given by:

$$V_R = iR$$

The (variable) voltage across the **inductor** is given by:

$$V_L = L\frac{di}{dt}$$

Kirchhoff's voltage law says that the directed sum of the voltages around a circuit must be zero. This results in the following differential equation:

$$Ri + L\frac{di}{dt} = V$$

The solution of the differential equation $Ri + L\frac{di}{dt} = V$ is:

$$i = \frac{V}{R}\left(1 - e^{-(R/L)t}\right)$$

Once the switch is closed, the current in the circuit is not constant. Instead, it will build up from zero to some steady state.

We start with:

$$Ri + L\frac{di}{dt} = V$$

Subtracting Ri from both sides:

$$L\frac{di}{dt} = V - Ri$$

Divide both sides by L:

$$\frac{di}{dt} = \frac{V - Ri}{L}$$

Multiply both sides by dt and divide both by (V - Ri):

$$\frac{di}{V - Ri} = \frac{dt}{L}$$

$$\int \frac{du}{u} = \ln |u| + K$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln |f(x)| + K$$

$$\int \frac{di}{V - Ri} = \int \frac{dt}{L}$$

$$-\frac{\ln (V - Ri)}{R} = \frac{1}{L}t + K$$

Now, since $i = 0$ when $t = 0$, we have:

$$K = -\frac{\ln V}{R}$$

Substituting K back into our expression:

$$-\frac{\ln(V-Ri)}{R}=\frac{1}{L}t-\frac{\ln V}{R}$$

Rearranging:

$$\frac{\ln\ V}{R}-\frac{\ln(V-Ri)}{R}=\frac{1}{L}t$$

Multiplying throughout by -R:

$$-\ln\ V+\ln(V-Ri)=-\frac{R}{L}t$$

Collecting the logarithm parts together:

$$\ln\left(\frac{V-Ri}{V}\right)=-\frac{R}{L}t$$

Taking "e to both sides":

$$\frac{V-Ri}{V}=e^{-(R/L)t}$$

$$1-\frac{R}{V}i=e^{-(R/L)t}$$

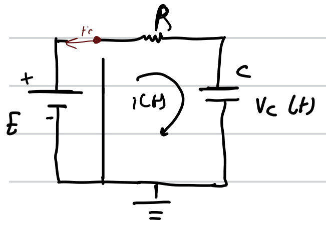
Subtracting 1 from both sides:

$$-\frac{R}{V}i=-1+e^{-(R/L)t}$$

Multiplying both sides by $-\left(\frac{V}{R}\right)$:

$$i=\frac{V}{R}\left(1-e^{-(R/L)t}\right)$$

How to create a model from an electrical circuit.



in this model connect E with $V_c(t)$

input (مصدر الطاقة) out put (الناتج عن مصدر الطاقة هي ال Volt المخزنة في c)

بدي معادلة رياضية تربط بين ال input , output

Assumptions : $V_c(0) = 0$

physics : القوانين الي ممكن استخدمها

$V_R = R\, i(t)$ — التيار عند الزمن t : i(t)

$V_c = \frac{q}{c}$ ⇒ $dq = dV_c\, C$ (C → const.)

$i(t) = \frac{dq}{dt}$ ⇐ $dq = C\, dV_c(t)$

$$i(t) = C \frac{dV_c(t)}{dt}$$

Kirchoff's voltage law (KVL) :

$$-E + \underbrace{R\, i(t)}_{V_R} + V_c(t) = 0$$

$$-E + RC\frac{dV_c(t)}{d(t)} + V_c(t) = 0$$

↳ first order differential equ.

↳ describes a first order syst.

RC

$$\frac{dV_c(t)}{} + \frac{1}{RC} V_c(t) = \frac{E}{RC}$$

⇒ this eq is in time domain

$$\bar{Y} + P(x)\, Y = g(x)$$

$$Y(x) = \frac{\int \mu(x) g(x)}{\int \mu(x)}$$

integral factor : $\mu(x) = e^{\int p(x)dx}$

$Y = V_c(t)$

$p(t) = \frac{1}{RC}$

$g(t) = \frac{E}{RC}$

$$\mu(t) = e^{\int \frac{1}{RC} dt} = e^{t/RC}$$

⇒

$$V_c(t) = \frac{\int^t e^{t/RC}\, (E/RC)}{e^{t/RC}}$$

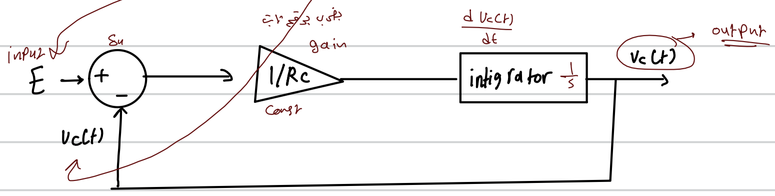
$$V_c(t) = E\left(1 - e^{-t/RC}\right)$$

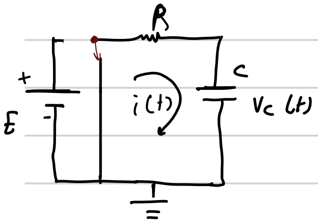
$$\frac{dV_c(t)}{dt} = \frac{E - V_c(t)}{RC}$$

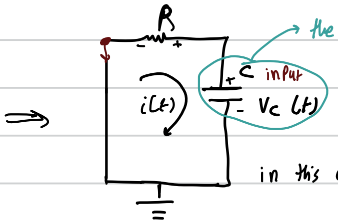
$$= \frac{1}{RC}\left(E - V_c(t)\right)$$

How to put the mathmatical equation in a block diagram ?

$$\frac{dV_c(t)}{dt} = \frac{1}{RC}\left(E - V_c(t)\right)$$







the capasitor is full

$V_C(0) = E$

where the capcitor is charged full $= E$

in this circuit C is the energy soura.

KVL:

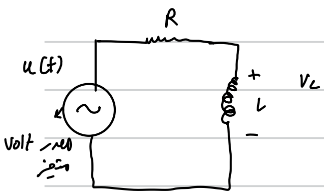
$$-V_C(t) + i(t)\,R = 0$$

$$-V_C(t) + CR\,\frac{dV_C(t)}{dt} = 0$$

$$\frac{dV_C(t)}{d(t)} - \frac{1}{CR}V_C(t) = 0$$

Sol → $V_C(t) = E\,e^{-t/RC}$

RL circuit ⇒ RL ايه مجال مغناطيسي بره circuit.



$$V_L(t) = L\,\frac{di(t)}{dt}$$

KVL:

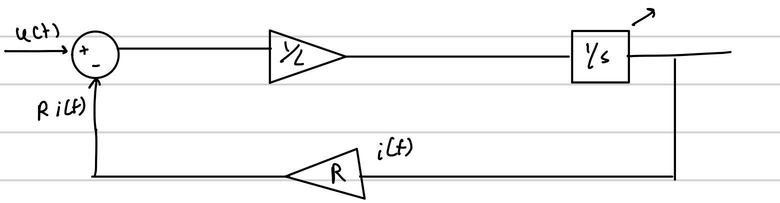
$$-u(t) + R\,i(t) + L\frac{di(t)}{dt} = 0$$

$$\frac{L\,di(t)}{dt} = u(t) - R\,i(t)$$

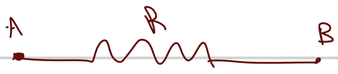
$$\frac{di(t)}{dt} = \frac{1}{L}\left(u(t) - R\,i(t)\right)$$

دايمًا بخلي ال diff على جهة لحاله

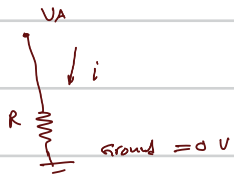
$$\frac{di(t)}{d(t)} = \frac{1}{L}\left(u(t) - R\,i(t)\right)$$



⇐ كل القوانين تعتمد على مبدأ حفظ الطاقة



$$i = \frac{(V_A - V_B)}{R}$$



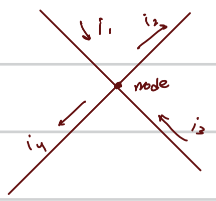
$$i = \frac{V_A - 0}{R}$$

$$i = V_A / R$$

$$i_c(t) = C \frac{d\,(V_A^{(t)} - V_B^{(t)})}{dt}$$

$$i_L(t) = \frac{1}{L} \int (V_A^{(t)} - V_B^{(t)})\, dt$$

$$V_A^{(t)} - V_B^{(t)} = L \frac{d\, i_L(t)}{dt}$$



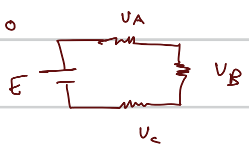
Kirchoff's Current Law (KCL)

$$i_1 + i_3 - i_2 - i_4 = 0$$

$$i_1 + i_3 = i_2 + i_4$$

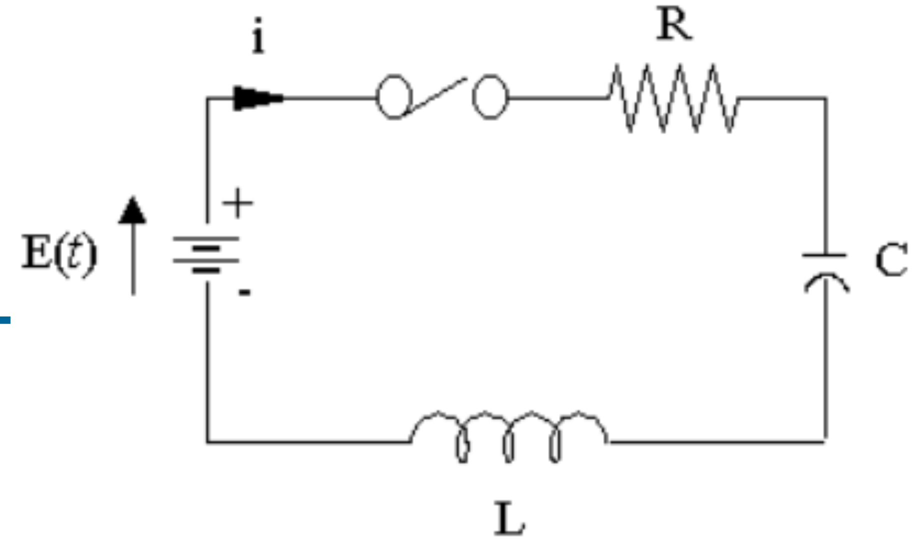
Kirchoff's voltage low (KVL)

$$-E + V_A + V_B + V_C = 0$$



RLC Circuit

second order diff equ.



Consider a series RLC circuit (one that has a resistor, an inductor and a capacitor) with a **constant** driving electro-motive force (emf) E. The current equation for the circuit is

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\ dt = E$$

This is equivalent: $L\frac{di}{dt} + Ri + \frac{1}{C}q = E$

Differentiating, we have

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = 0$$

$$-E + V_R(t) + V_C(t) + V_L(t) = 0$$

$$-E + R\,i(t) + \frac{1}{C}\int i(t)\,dt + L\frac{di(t)}{dt} = 0$$

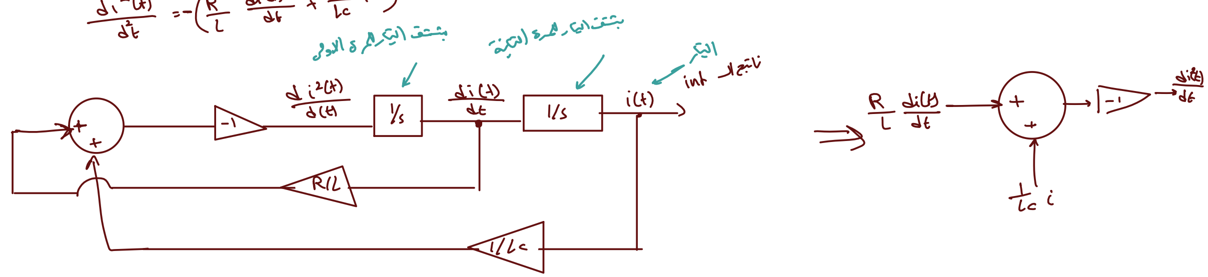
$$R\,i(t) + \frac{1}{C}\int i(t)\,dt + L\frac{di(t)}{dt} = E$$

$$L\frac{d^2i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i = 0$$

second order

$$L\frac{d^2i(t)}{dt^2} = -R\frac{di(t)}{dt} - \frac{1}{C}i$$

$$\frac{d^2i(t)}{dt^2} = -\left(\frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i\right)$$



RLC Circuit

The general form of the second order differential equation with constant coefficients is

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = Q(x)$$

where $a,\ b,\ c$ are constants with $a > 0$ and $Q(x)$ is a function of x only.

Homogeneous Equation

In this section, most of our examples are **homogeneous** 2nd order linear DEs (that is, with $Q(x)$ = 0):

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

RLC Circuit

Method of Solution

The equation

$$am^2 + bm + c = 0$$

is called the **Auxiliary Equation** (A.E.)

The **general solution** of the differential equation depends on the solution of the A.E. To find the general solution, we must determine the roots of the A.E. The roots of the A.E. are given by the well-known quadratic formula:

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Nature of roots	Condition	General Solution
1. Real and distinct roots, m_1, m_2	$b^2 - 4ac > 0$	$y = Ae^{m_1 x} + Be^{m_2 x}$
2. Real and equal roots, m	$b^2 - 4ac = 0$	$y = e^{mx}(A + Bx)$
3. Complex roots $m_1 = \alpha + j\omega$, $m_2 = \alpha - j\omega$	$b^2 - 4ac < 0$	$y = e^{\alpha x}(A \cos \omega x + B \sin \omega x)$

RLC Circuit

Its corresponding auxiliary equation is

$$Lm^2 + Rm + \frac{1}{C} = 0$$

with roots:

$$m_1 = \frac{-R}{2L} + \frac{\sqrt{(R^2 - 4L/C)}}{2L}$$

$$= -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$m_2 = \frac{-R}{2L} - \frac{\sqrt{(R^2 - 4L/C)}}{2L}$$

$$= -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

RLC Circuit

Now

$\alpha = \dfrac{R}{2L}$ is called the **damping coefficient** of the circuit

$\omega_0 = \sqrt{\dfrac{1}{LC}}$ is the **resonant frequency** of the circuit.

m_1 and m_2 are called the **natural frequencies** of the circuit.

The nature of the current will depend on the relationship between R, L and C.

There are three possibilities:

Case 1: $R^2 > 4L/C$ (Over-Damped)

Graph of overdamped case.

Here both m_1 and m_2 are real, distinct and negative. The general solution is given by

$$\boxed{i(t) = Ae^{m_1 t} + Be^{m_2 t}}$$

The motion (current) is not oscillatory, and the vibration returns to equilibrium.

RLC Circuit

Case 2: $R^2 = 4L/C$ (Critically Damped)

Here the roots are negative, real and equal,

i.e. $m_1 = m_2 = -\dfrac{R}{2L}$.

The general solution is given by

$$\boxed{i(t) = (A + Bt)e^{-Rt/2L}}$$

The vibration (current) returns to equilibrium in the minimum time and there is just enough damping to prevent oscillation.

RLC Circuit

Case 3: $R^2 < 4L/C$ (Under-Damped)

Here the roots are complex where

$m_1 = \alpha + j\omega$, and $m_2 = \alpha - j\omega$

The general solution is given by

$$\boxed{i(t) = e^{-\alpha t}(A\ \cos\ \omega t + B\ \sin\ \omega t)}$$

where

$\alpha = \dfrac{R}{2L}$ is called the **damping coefficient**, and ω is given by:

$$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

In this case, the motion (current) is oscillatory and the amplitude decreases exponentially, bounded by

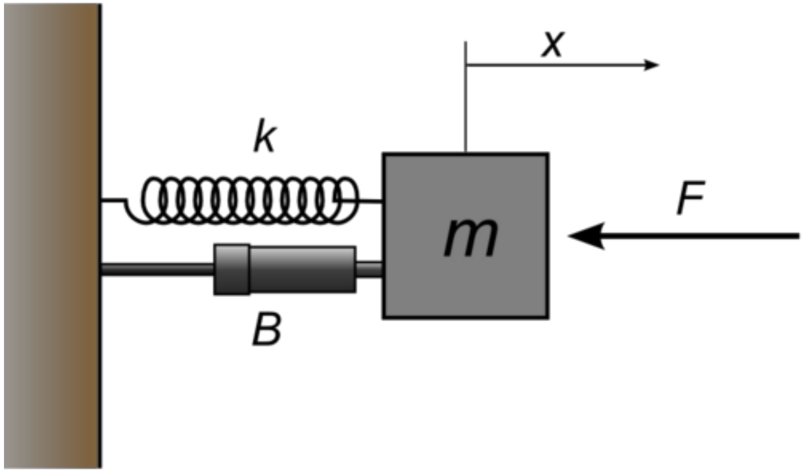
$$i = \pm\sqrt{A^2 + B^2}\, e^{-Rt/2L}$$

mass
force
displacement

Mathematical modeling of Physical Systems (Mechanical Systems)

- **Part-I:** Translational Mechanical System
- **Part-II:** Rotational Mechanical System

Basic Types of Mechanical Systems

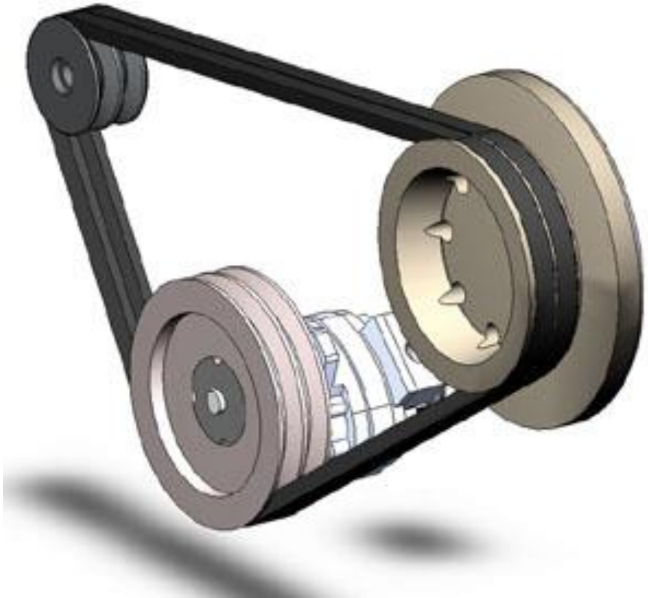


Translational

Linear Motion

Rotational

Rotational Motion



Part-I

TRANSLATIONAL MECHANICAL SYSTEMS

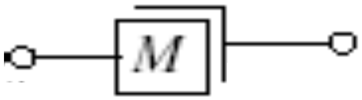
Basic Elements of Translational Mechanical Systems

Translational Spring

i)

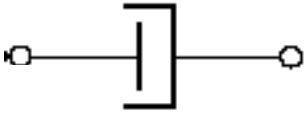
Translational Mass

ii)



iii)

Translational Damper

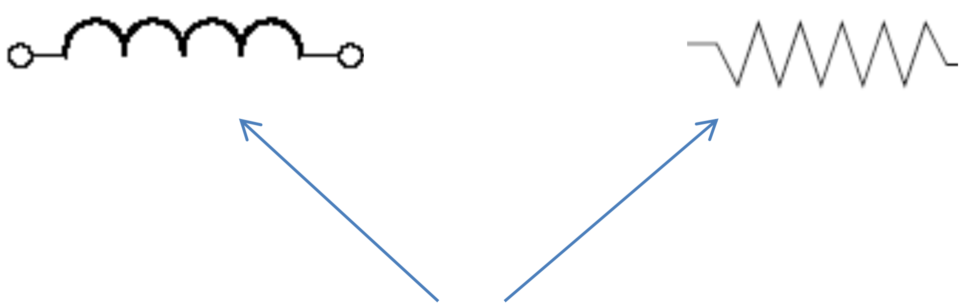


Translational Spring

A translational spring is a mechanical element that can be deformed by an external force such that the deformation is directly proportional to the force applied to it.

Translational Spring

i)



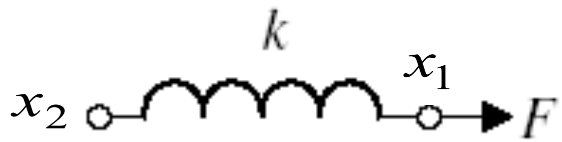
Circuit Symbols

Translational Spring

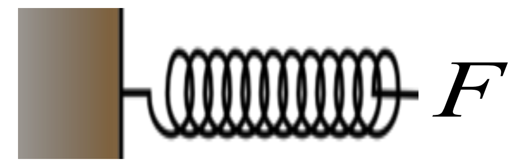
Translational Spring

If

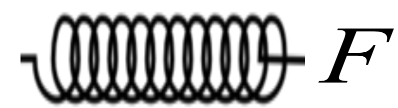
F is the applied force



Then x_1 is the deformation if $x_2 = 0$



Or $(x_1 - x_2)$ is the deformation. (or displacement)



The equation of motion is given as

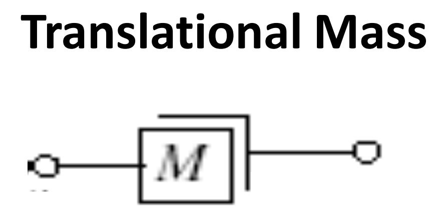
$$F = k(x_1 - x_2)$$

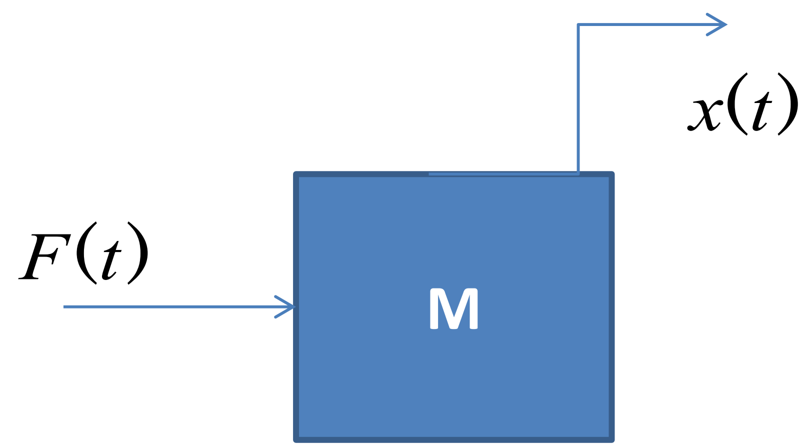
Where k is stiffness of spring expressed in N/m

Translational Mass

- Translational Mass is an inertia element.
- A mechanical system without mass does not exist.
- If a force *F* is applied to a mass and it is displaced to *x* meters then the relation b/w force and displacements is given by Newton's law.

ii)





$$F = M\ddot{x}$$

Second newton's law

المشتقة الثانية للإزاحة (X) هي التسارع (acceleration)

other forces that can effect on the applied force ← { inertia → dep on mass; frection

Translational Damper

more flexable than spring

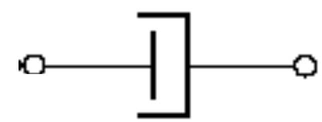
- When the viscosity or drag is not negligible in a system, we often model them with the damping force.

- All the materials exhibit the property of damping to some extent.

- If damping in the system is not enough then extra elements (e.g. Dashpot) are added to increase damping.

Translational Damper

iii)



Common Uses of Dashpots

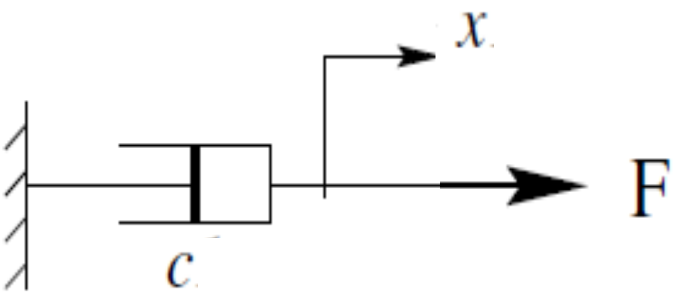
Door Stoppers

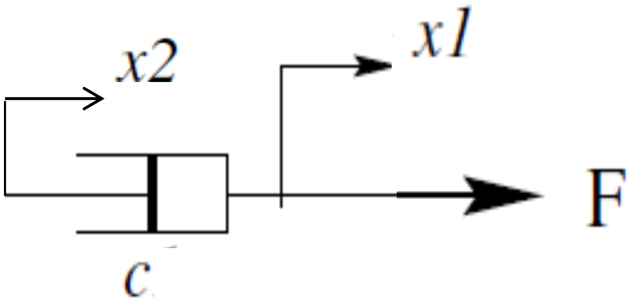
Vehicle Suspension

Bridge Suspension

Flyover Suspension

Translational Damper





$$F = C\dot{x}$$

$$F = C(\dot{x}_1 - \dot{x}_2)$$

Damping coef

- Where *C* is damping coefficient (*N/ms*$^{-1}$).

Mechanical Translational Systems

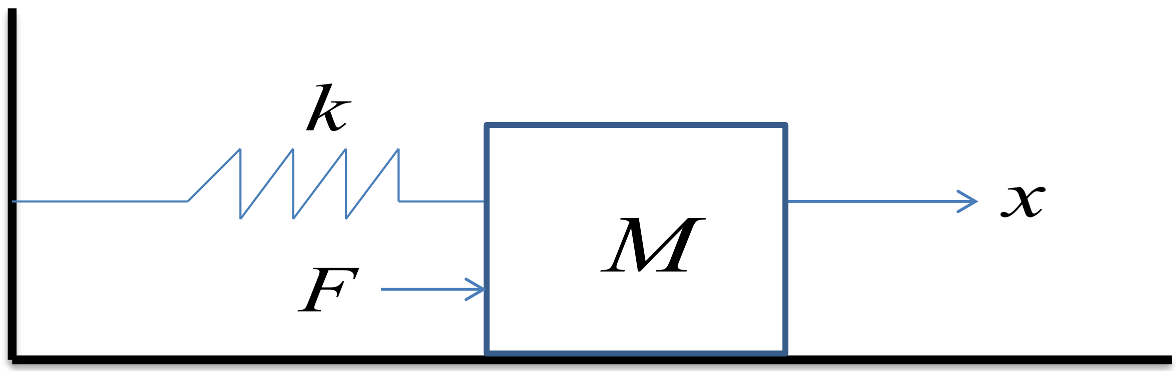
Newton's 2nd Law for Translational Systems

$$\sum_{k} F_k(t) = M\,\ddot{x}(t)$$

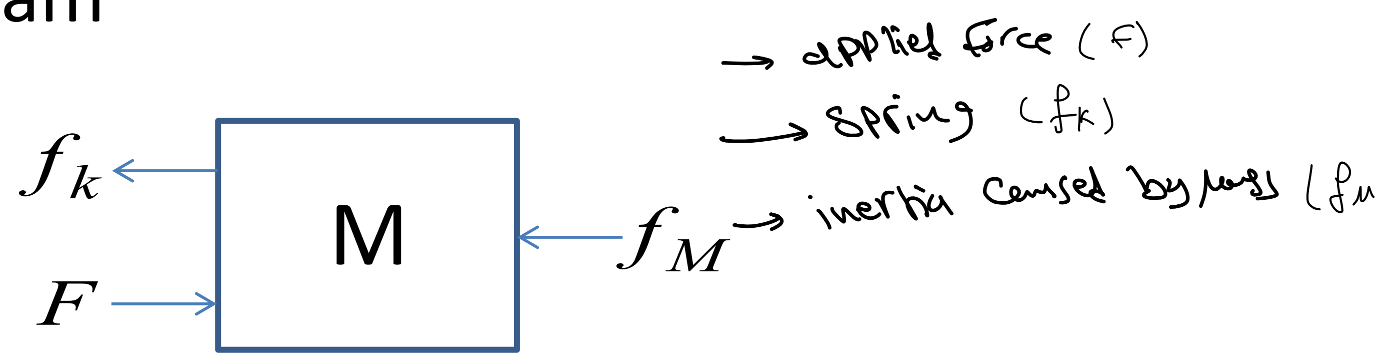
(*) free body diagram

Mass-Spring System

Consider the following system (friction is negligible)

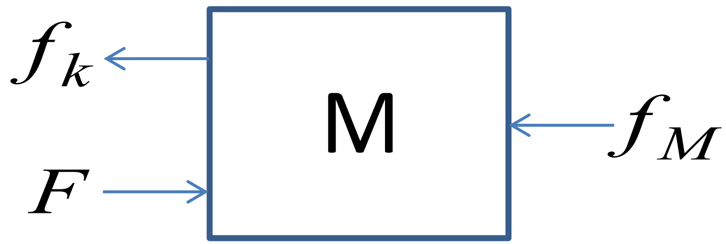


Free Body Diagram



Where f_k and f_M are forces applied by the spring and inertial force respectively.

Mass-Spring System



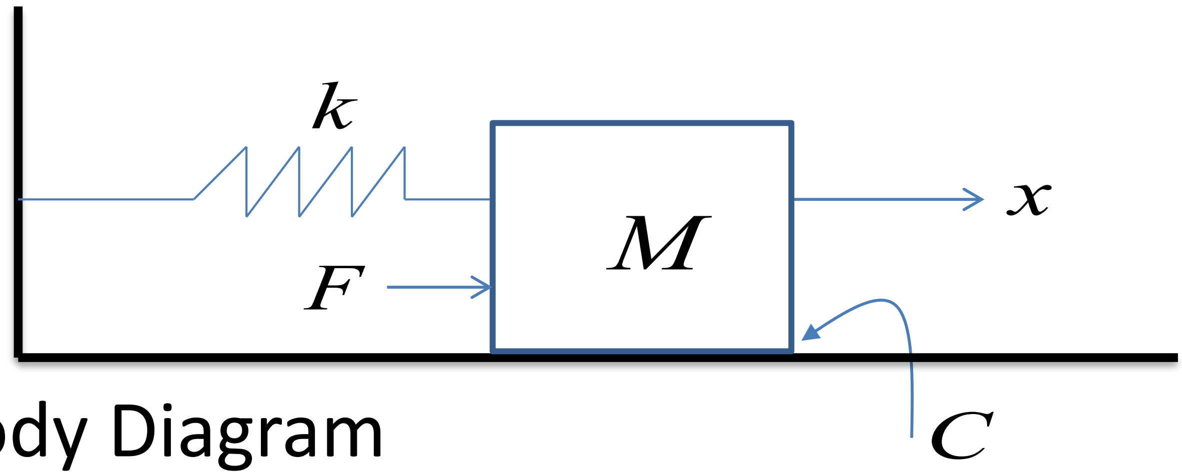
Then the differential equation of the system is:

$$F = M\ddot{x} + kx$$

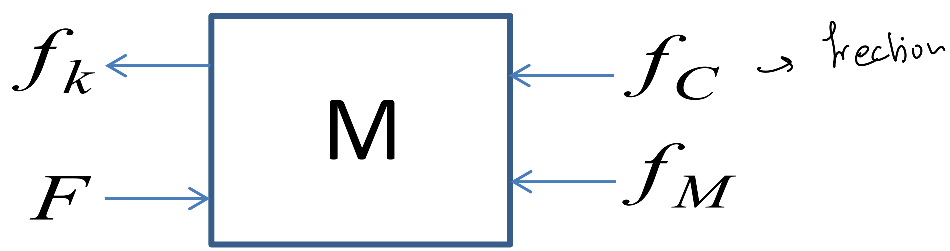
second order diff eqn. ($\frac{dx^2}{dt^2}$

Mass-Spring-Friction System

Consider the following system



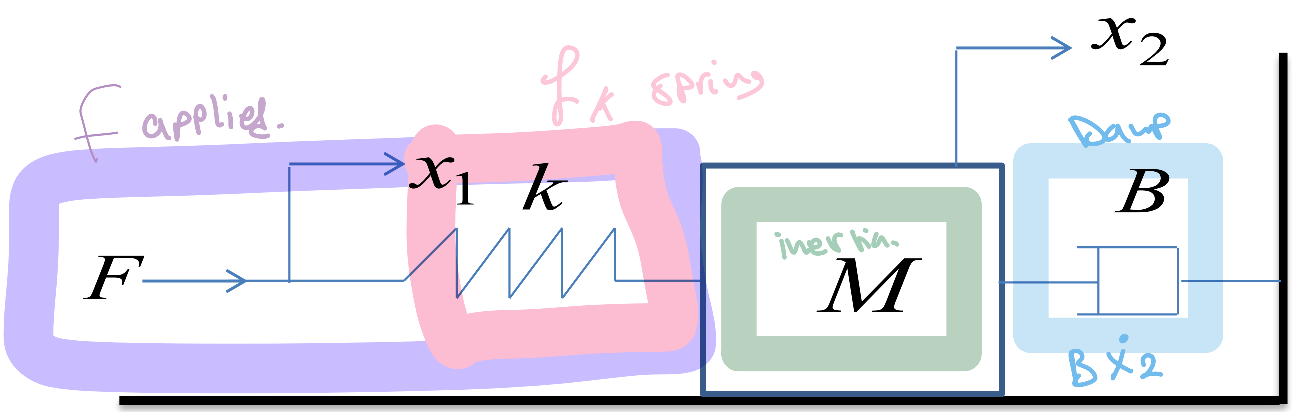
- Free Body Diagram



$$F = f_k + f_M + f_C$$

Mass-Spring- Damper System

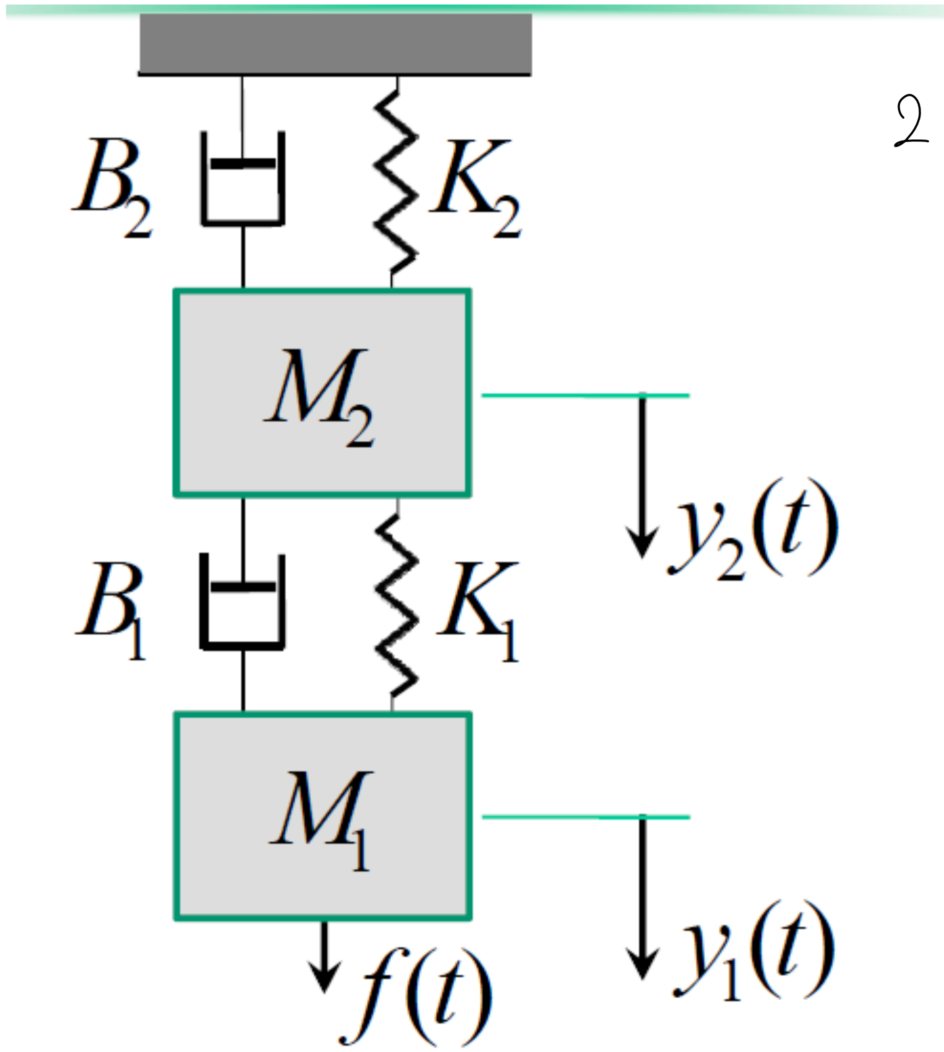
Consider the following system



$$F = k(x_1 - x_2)$$

$$0 = k(x_2 - x_1) + M\ddot{x}_2 + B\dot{x}_2$$

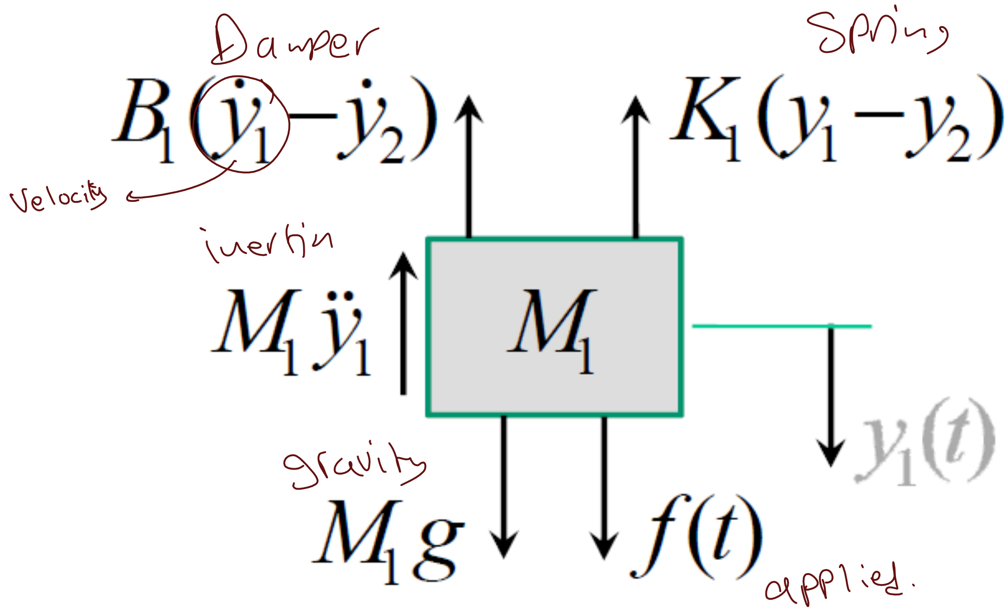
Mechanical System



2 masses ⇒ 2 equ.

Mechanical System

Free-body diagram for Mass 1:

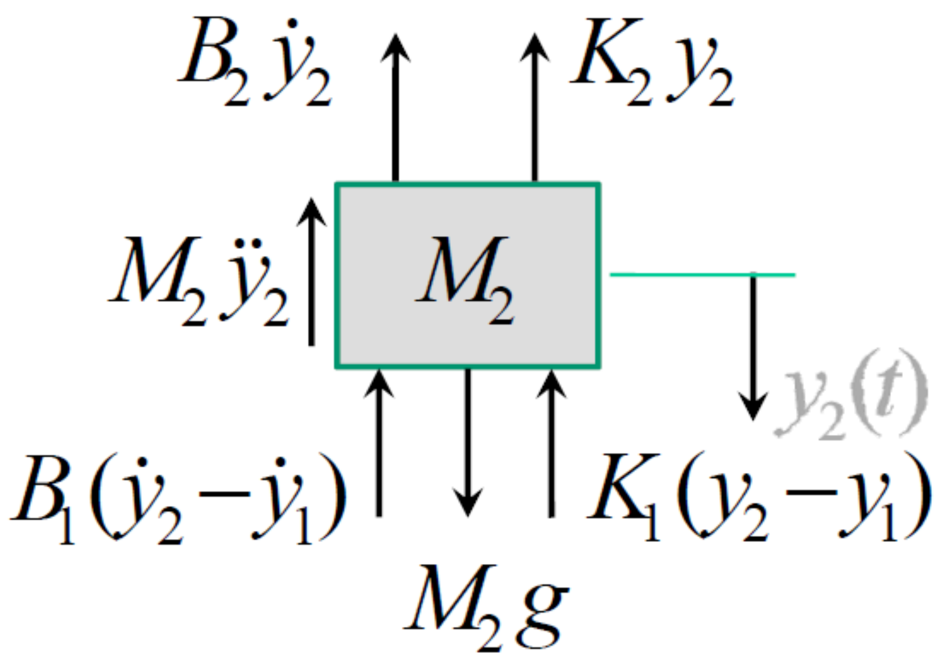


$$-M_1\ddot{y}_1 - B_1(\dot{y}_1 - \dot{y}_2) - K_1(y_1 - y_2) + M_1 g + f(t) = 0$$

$$M_1\ddot{y}_1 + B_1\dot{y}_1 + K_1 y_1 - B_1\dot{y}_2 - K_1 y_2 = M_1 g + f(t)$$

Mechanical System

Free-body diagram for Mass 2:



force with the direction of acceleration
⇒ f (+ve)
against acceleration
⇒ f (-ve)

$$-M_2\ddot{y}_2 - B_2\dot{y}_2 - K_2 y_2 - B_1(\dot{y}_2-\dot{y}_1) - K_1(y_2-y_1) + M_2 g = 0$$

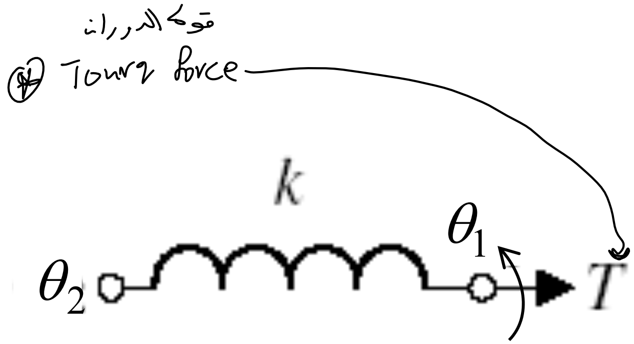
$$-B_1\dot{y}_1 - K_1 y_1 + M_2\ddot{y}_2 + (B_1+B_2)\dot{y}_2 + (K_1+K_2)y_2 = M_2 g$$

Part-II

ROTATIONAL MECHANICAL SYSTEMS

Basic Elements of Rotational Mechanical Systems

Rotational Spring

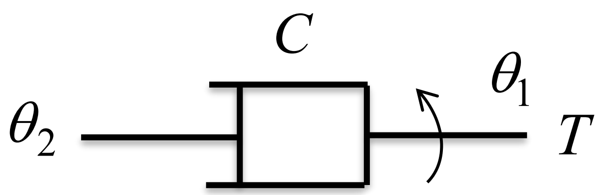


$$T = k(\theta_1 - \theta_2)$$

angulare velocity

Basic Elements of Rotational Mechanical Systems

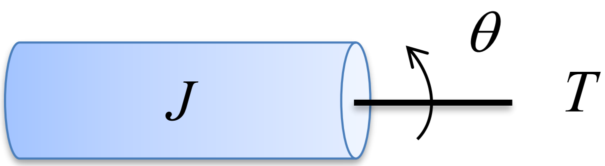
Rotational Damper



$$T = C(\dot{\theta}_1 - \dot{\theta}_2)$$

Basic Elements of Rotational Mechanical Systems

Moment of Inertia



$$T = J\ddot{\theta}$$

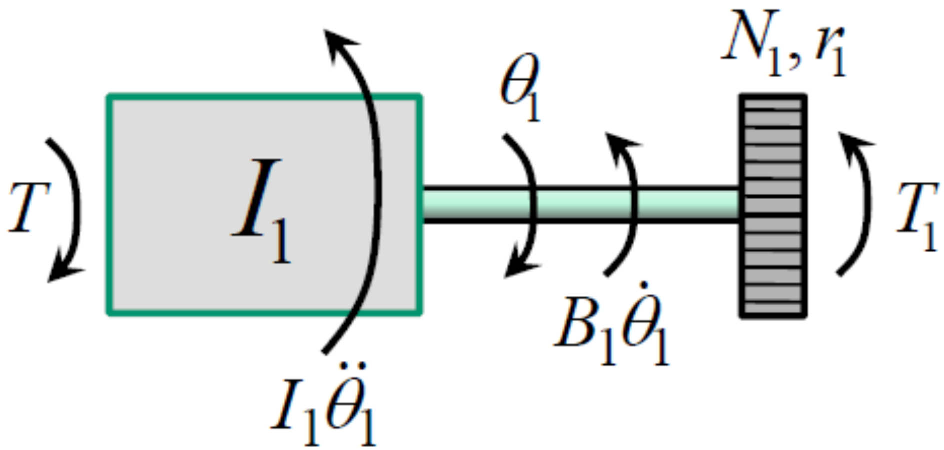
Mechanical Rotational Systems

Newton's 2^{nd} Law for Rotational Systems

$$\sum_k T_k(t) = I\,\ddot{\theta}(t)$$

Example

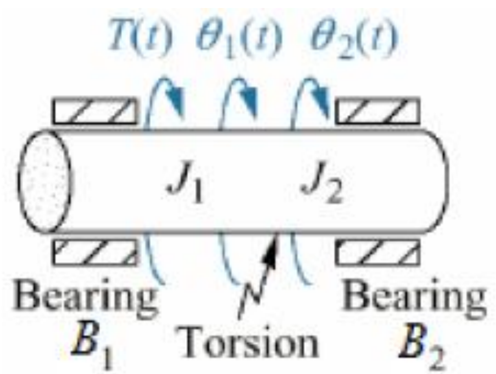
Free-body diagrams for inertia I_1:

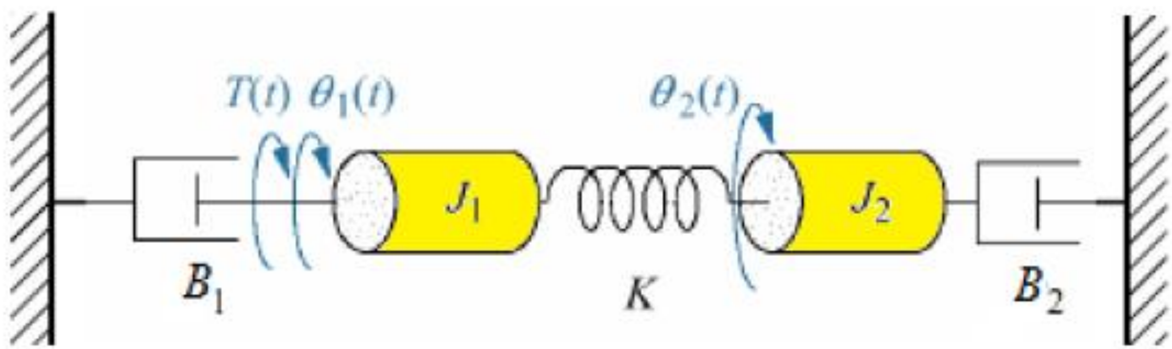


Newton's 2nd law: $T(t) - I_1\ddot{\theta}_1(t) - B_1\dot{\theta}_1(t) - T_1(t) = 0$

$$\Rightarrow \quad T(t) - I_1\ddot{\theta}_1(t) - B_1\dot{\theta}_1(t) = T_1(t)$$

Example 2





$$T(t) = J_1\ddot{\theta}_1 + B_1\dot{\theta}_1 + k(\theta_1 - \theta_2)$$

$$0 = J_2\ddot{\theta}_2 + B_2\dot{\theta}_2 + k(\theta_2 - \theta_1)$$

no torque of J

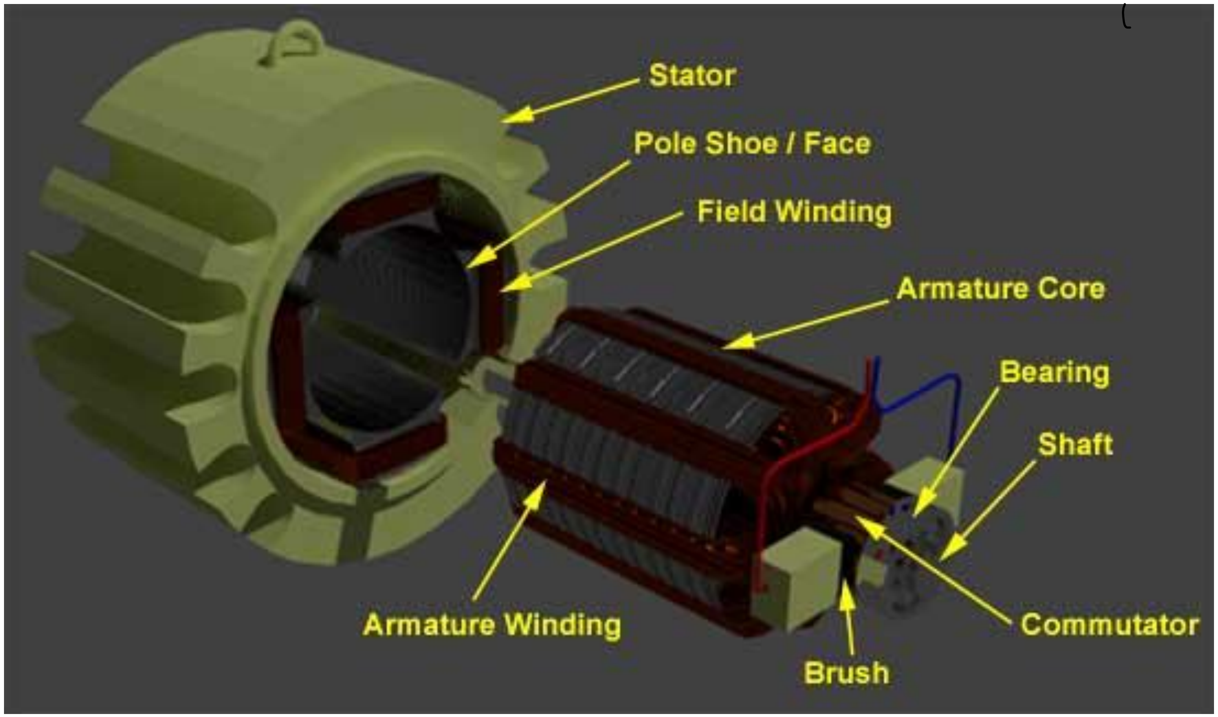
السؤال بيجي مصدر اذا في T او لا

وبيجي انا لازم اطلع القانون

Modeling of Electromechanical Systems

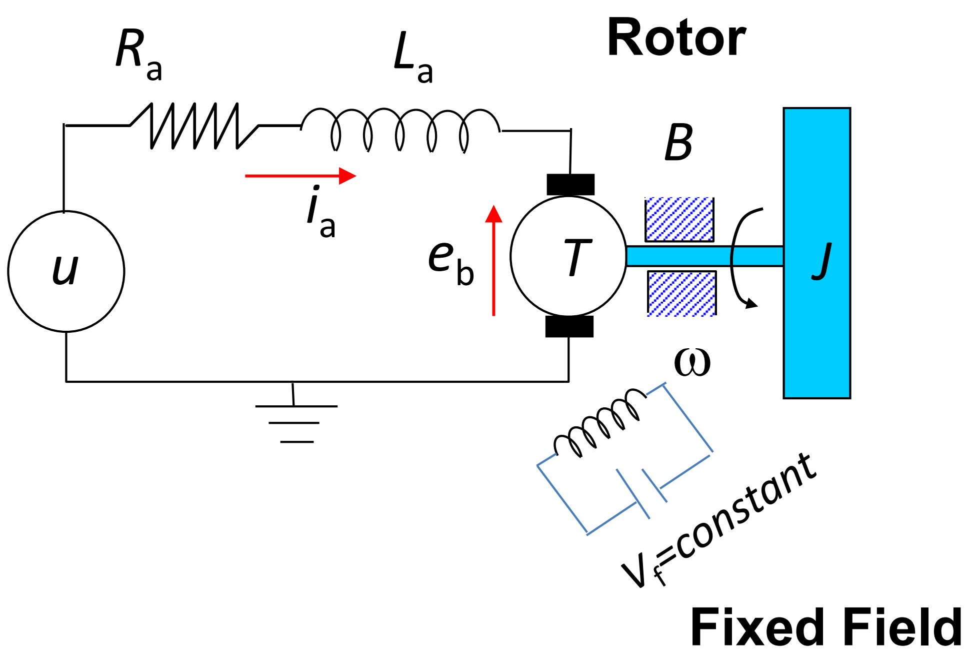
DC motor → Direct current motor

D.C Drives



- Variable Voltage can be applied to the armature terminals of the DC motor.
- Another method is to vary the flux per pole of the motor.
- The first method involves adjusting the motor's armature, while the latter method involves changing the motor field. These methods are referred to as "armature control" and "field control."

Armature Controlled D.C Motor



Input: voltage u
Output: Angular velocity ω

Electrical Subsystem (loop method):

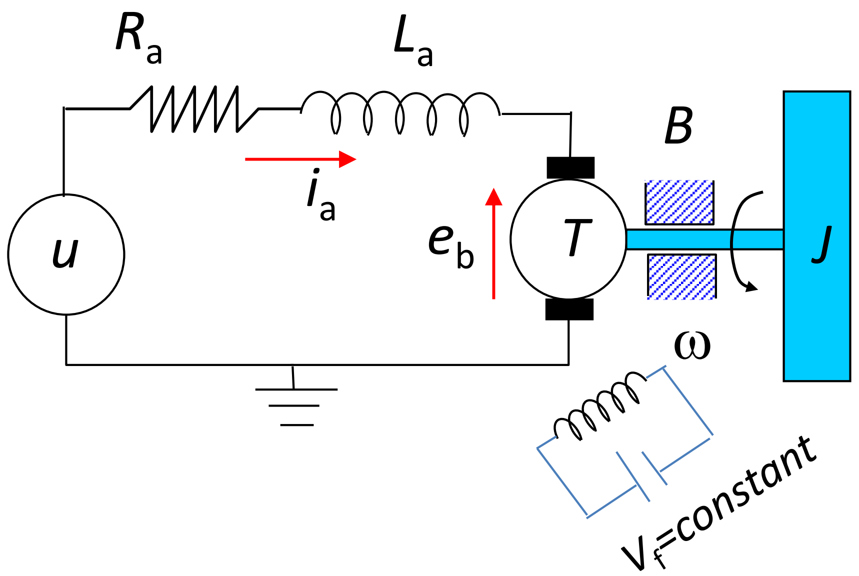
$$u = R_a i_a + L_a \frac{di_a}{dt} + e_b, \qquad \text{where } e_b = \text{back-emf voltage}$$

Mechanical Subsystem

$$T_{motor} = J\dot{\omega} + B\omega$$

Back electromotive force (back emf) is a voltage generated by a rotating DC motor that opposes the supply voltage.

Armature Controlled D.C Motor



Power Transformation:

Torque-Current: $T_{motor} = K_t i_a$

Voltage-Speed: $e_b = K_b \omega$

- Combing previous equations results in the following mathematical model:

$$\begin{cases} L_a \dfrac{di_a}{dt} + R_a i_a + K_b \omega = u \\ J\dot{\omega} + B\omega - K_t i_a = 0 \end{cases}$$