Industrial Control Systems

Review 1:
Complex Numbers

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Complex Numbers

- In the set of real numbers, negative numbers do not have square roots.
- Imaginary numbers were invented so that negative numbers would have square roots and certain equations would have solutions.
- These numbers were devised using an **imaginary unit** named ***j***.
- Mathematicians use the symbol ***i*** for this number, but electrical engineers use ***j***:

$$i = \sqrt{-1} \quad \text{or} \quad j = \sqrt{-1}$$

The letter j represents the numbers whose square is –1.

$$j^2 = -1 \qquad\qquad j = \sqrt{-1}$$

If ***a*** is a positive real number, then the square root of negative ***a*** is the imaginary number $j\sqrt{a}$.

$$\sqrt{-a} = \sqrt{-1 * a} = j\sqrt{a}$$

Examples: $\sqrt{-4} = j\sqrt{4} = j2$

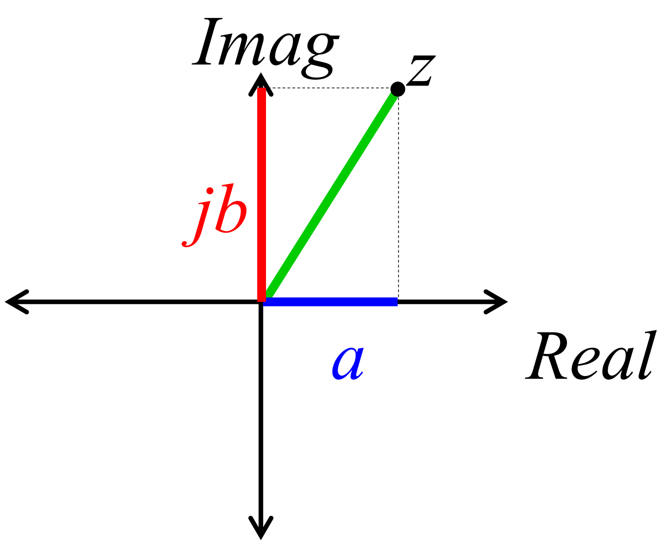
$$\sqrt{-36} = j\sqrt{36} = j6$$

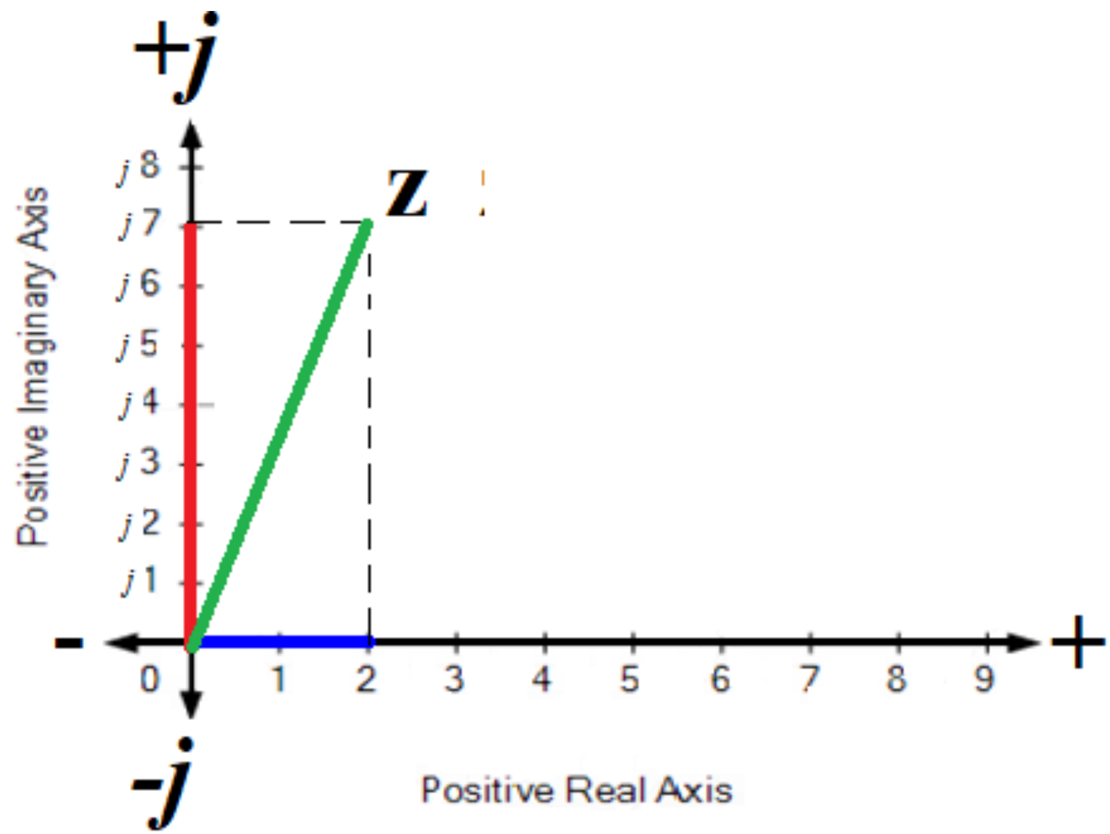
A *complex number* is a number of the form $a + jb$, where a and b are real numbers and $j = \sqrt{-1}$.

The number ***a*** is the *real part* of $a + jb$, and b is its *imaginary part*.

Examples of complex numbers: $z = a + j\,b$

Real Part		Imaginary Part
a	+	jb
2	+	$j7$

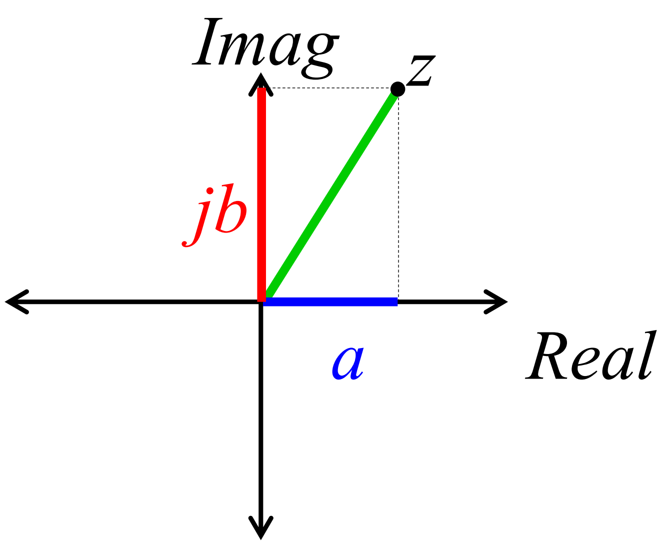


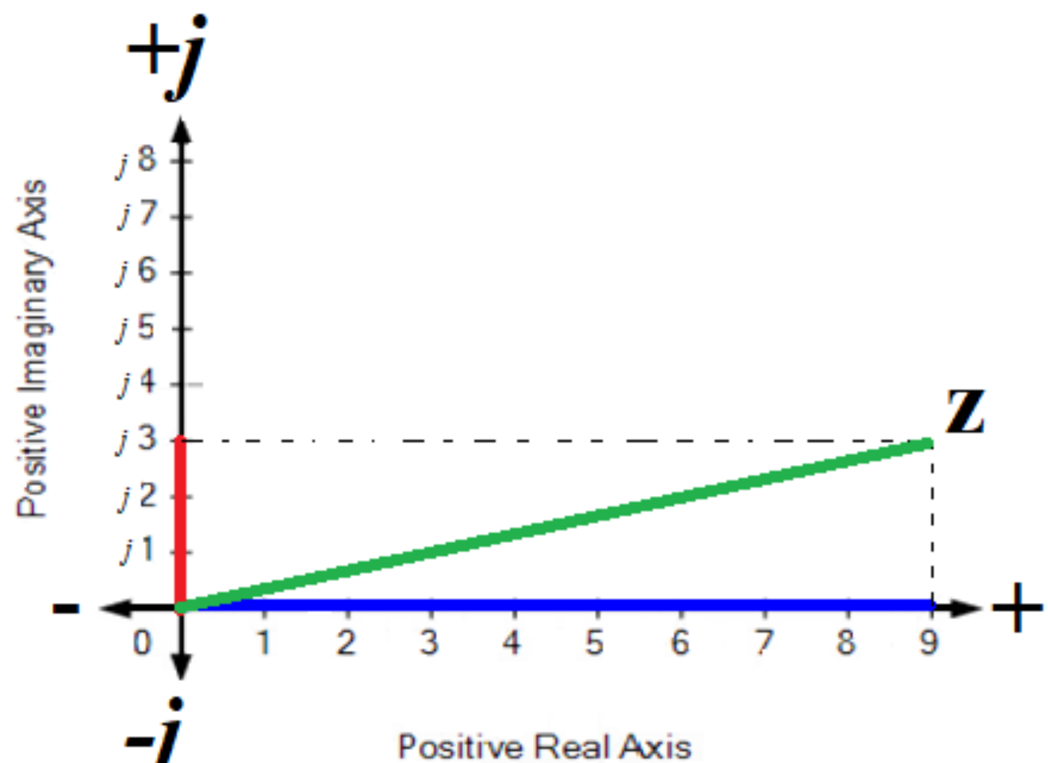


Examples of complex numbers: $z = a + j\,b$

Real Part		Imaginary Part
a	+	jb
9	+	$j3$

magnetud of z $|z| = \sqrt{a^2 + b^2}$





Real Numbers: $a + j0$ Imaginary Numbers: $0 + jb$

Simplify: 1. $\sqrt{-90} = j\sqrt{90} = j\sqrt{9 \cdot 10} = j\,3\sqrt{10}$

2. $\sqrt{-64} = j\sqrt{64} = j8$

3. $\sqrt{16} + \sqrt{-50}$

$= \sqrt{16} + j\sqrt{-50}$

$= \sqrt{16} + j\sqrt{25 \cdot 2}$

$= 4 + j5\sqrt{2}$

Simplify.

4.) $\sqrt{-5} = \sqrt{-1 * 5} = \sqrt{-1}\sqrt{5} = i\sqrt{5}$

5.) $-\sqrt{-7} = -\sqrt{-1 * 7} = -\sqrt{-1}\sqrt{7} = -i\sqrt{7}$

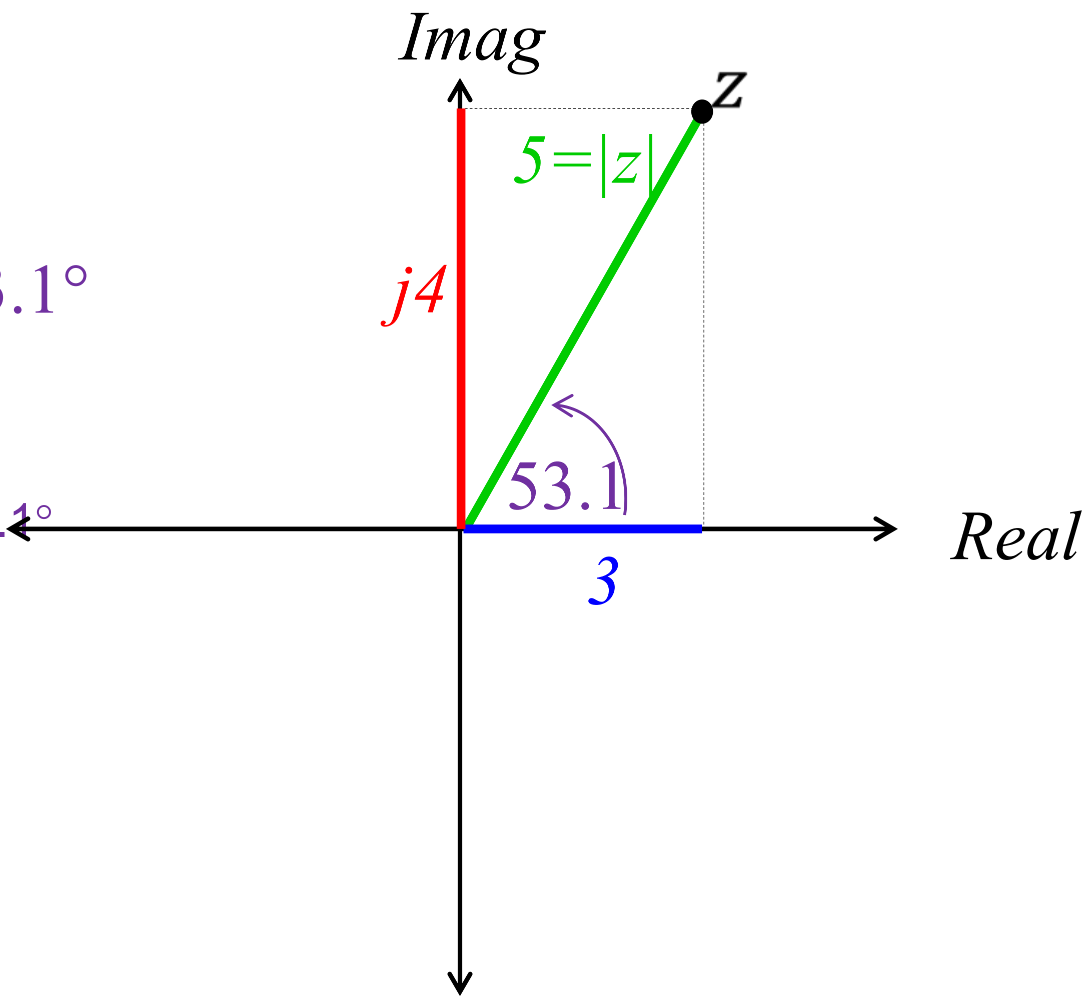
6.) $\sqrt{-99} = \sqrt{-1 * 99} = \sqrt{-1}\sqrt{99}$

$$= i\sqrt{3 \cdot 3 \cdot 11}$$

$$= 3i\sqrt{11}$$

Rectangular versus Polar Form

- Any complex number can be expressed in three forms:
  - Rectangular form
    - Example: $z = 3 + j\,4$
  - Polar form / phasor form
    - Example: $z = 5 \angle 53.1°$
  - Exponential form
    - Example: $z = 5\,e^{j\,53.1°}$



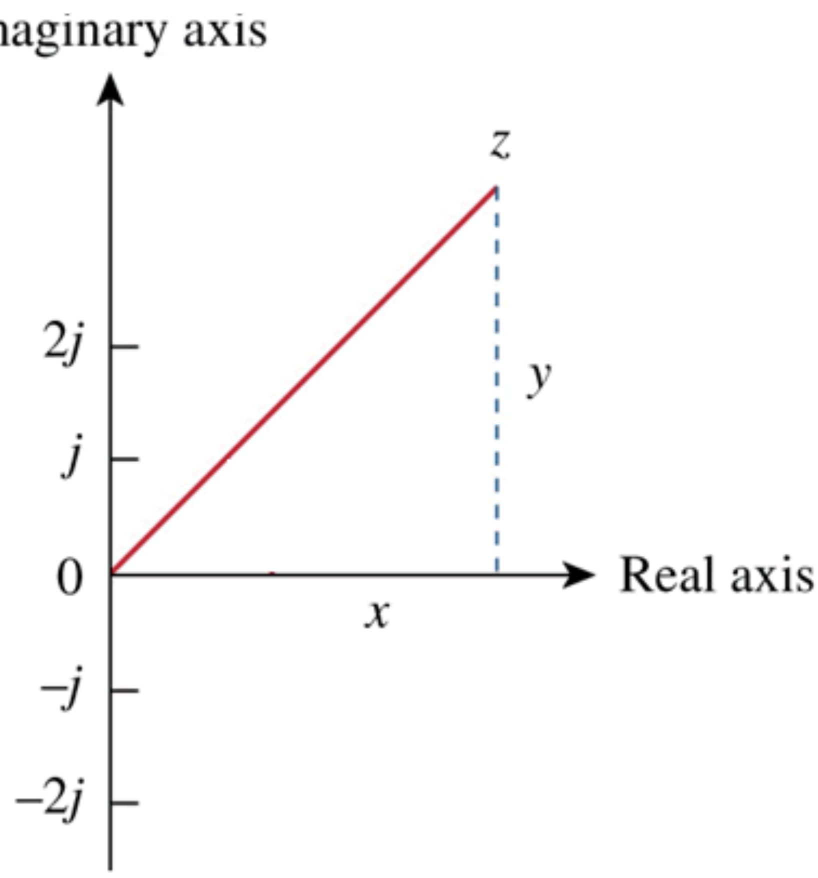
Rectangular Form

- In rectangular form, a complex number z is written as the sum of a **real part** x and an **imaginary part** y:

$$z = x + jy$$

The Complex Plane

- We often represent complex numbers as points in the complex plane, with the real part plotted along the horizontal axis (or "real axis") and the imaginary part plotted along the vertical axis (or "imaginary axis").

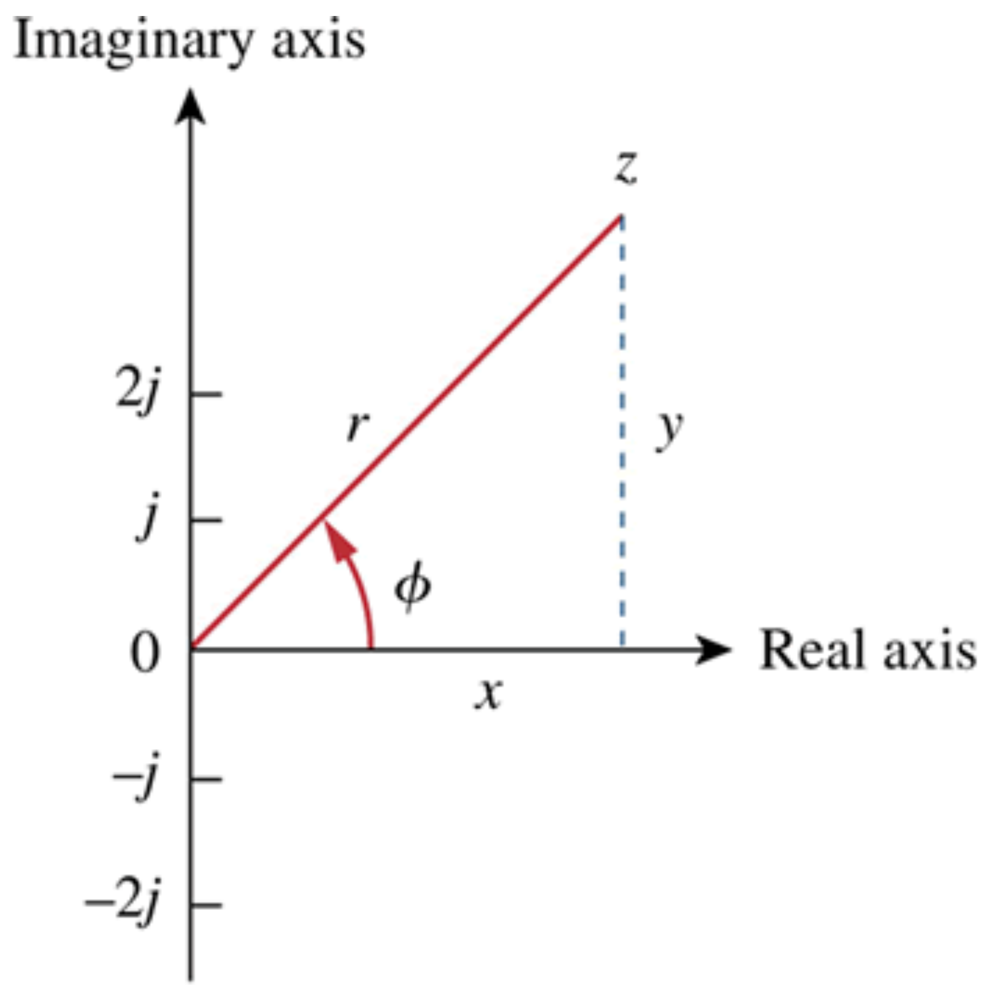


Polar Form

- In polar form, a complex number z is written as a **magnitude** *r* at an **angle** ϕ:

$$z = r\angle\phi$$

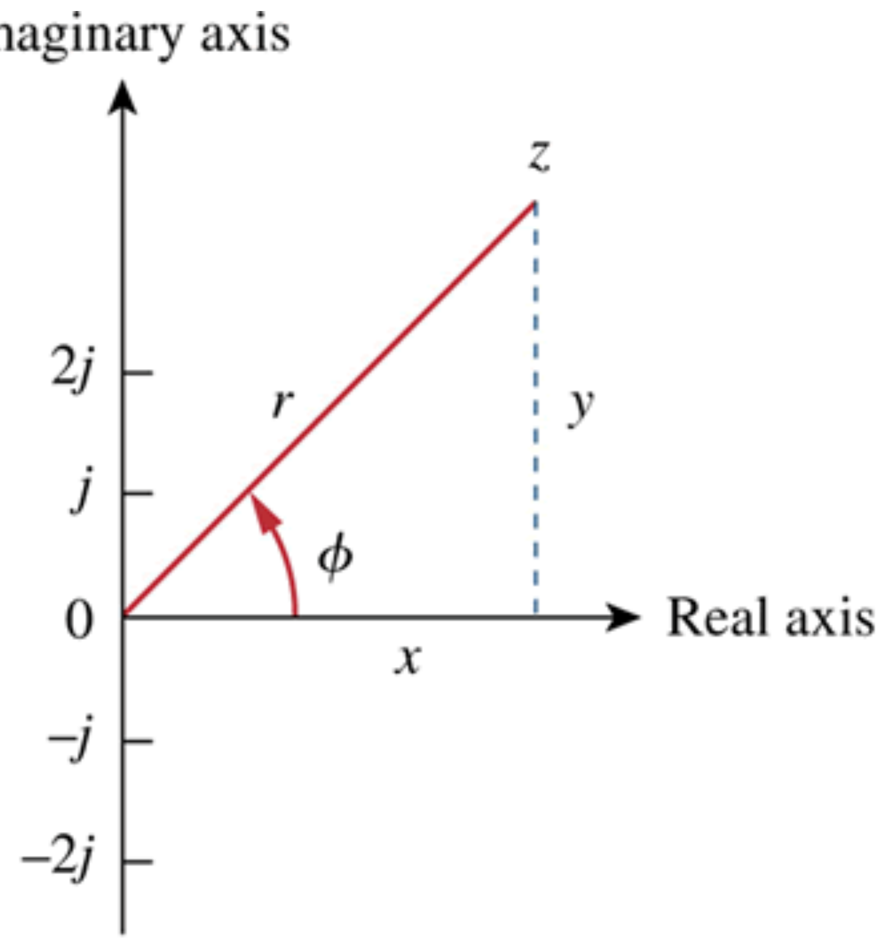
The angle ϕ is measured from the positive real axis.



Converting Between Rectangular and Polar Forms

- We will very often have to convert from rectangular form to polar form, or vice versa.

- This is easy to do if you remember a bit of right-angle trigonometry.



Converting from Rectangular Form to Polar Form

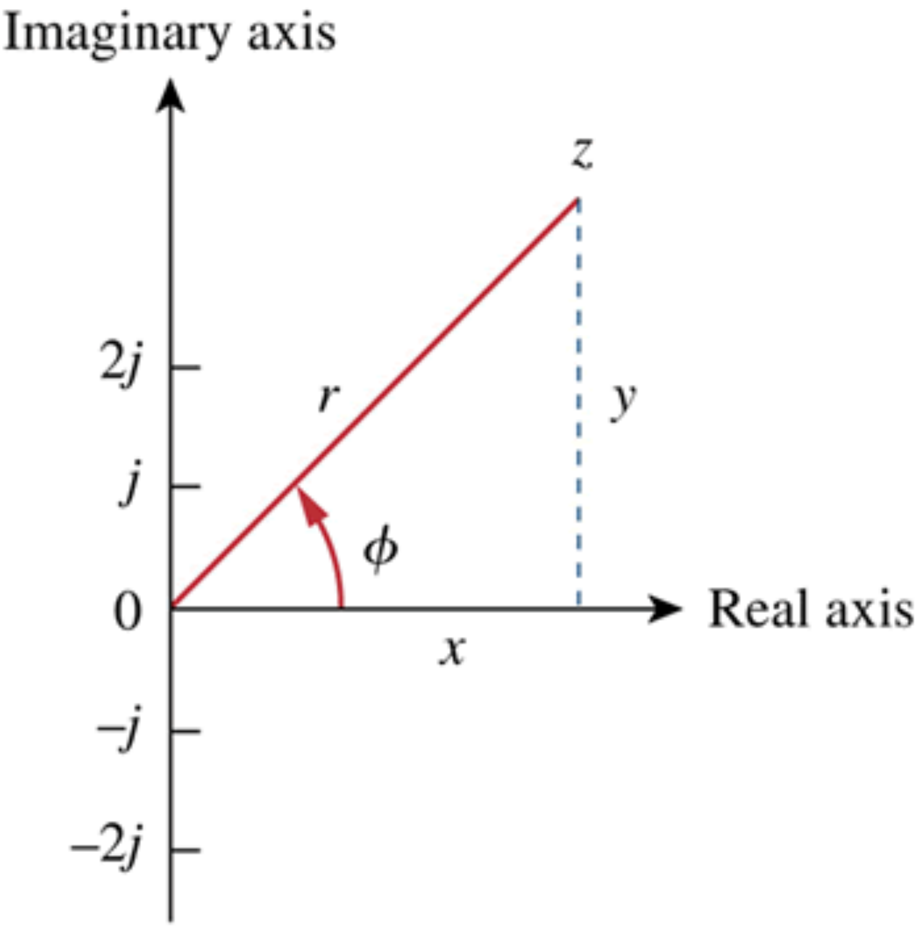
- Given a complex number z with real part x and imaginary part y, its magnitude is given by

$$r = \sqrt{x^2 + y^2}$$

and its angle is given by

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \left(\frac{Imag}{Real}\right)$$

$$z = r\angle\phi$$



Converting from Polar Form to Rectangular Form

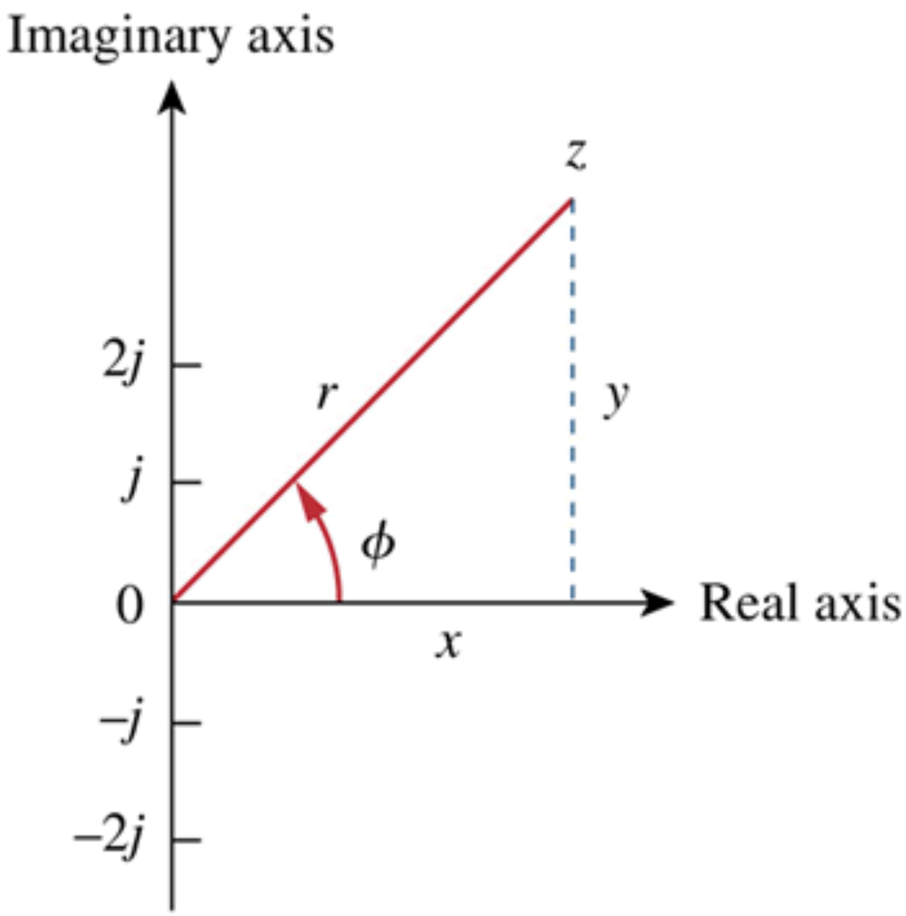
- Given a complex number z with magnitude ***r*** and angle ϕ, its real part is given by

$$x = r\cos\phi$$

and its imaginary part is given by

$$y = r\sin\phi$$

$$z = x + jy$$



Exponential Form

- Complex numbers may also be written in exponential form.

Polar form		**Exponential Form**
$r\angle\phi$	$\Leftrightarrow$	$r\,e^{j\phi}$

Example: $3\angle 30^\circ \quad \Leftrightarrow \quad 3e^{\,j30}$

Euler's Formula

- The exponential form is based on **Euler's identity**, which says that, for any ϕ,

$$e^{j\phi} = \cos\phi + j\sin\phi$$

$$r\,\mathrm{e}^{j\phi}$$

$$= r\,(cos\,\phi + j sin\,\phi)$$

$$= r\,cos\,\phi + j\,r\,sin\,\phi$$

$$= x + j\,y$$

Addition and Subtraction

- Adding and Subtracting complex numbers is easiest if the numbers are in rectangular form.

Steps:

1. Write each complex number in the form $a + jb$.

2. Add or subtract the real parts of the complex numbers.

3. Add or subtract the imaginary parts of the complex numbers.

- Add

$$(a + jb) + (c + jd) = (a + c) + (jb + jd) = (a + c) + j(b + d)$$

- Subtract

$$(a + jb) - (c + jd) = (a + jb) + (-c - jd) = (a - c) + (jb - jd)$$

$$= (a - c) + j(b - d)$$

A*ddition* Examples:

Add $(11 + j5) + (8 - j2)$

$= (11 + 8) + (j5 - j2)$

$= 19 + j3$

Add $(10 + \sqrt{-5}) + (21 - \sqrt{-5})$

$= (10 + j\sqrt{5}) + (21 - j\sqrt{5})$

$= (10 + 21) + (j\sqrt{5} - j\sqrt{5})$

$= 31$

***Subtraction* Examples**:

Subtract: $(-21 + j3) - (7 - j9)$

$= (-21 + j3) + (-7 + j9)$

$= (-21 - 7) + (j\,3 + j9)$

$= (-21 - 7) + j(3 + 9)$

$= -28 + j12$

Subtract: $(11 + \sqrt{-16}) - (6 + \sqrt{-9})$

$= (11 + j\sqrt{16}) - (6 + j\sqrt{9})$

$= (11 + j\sqrt{16}) + (-6 - j\sqrt{9})$

$= (11 - 6) + (j\sqrt{16} - j\sqrt{9})$

$= (11 - 6) + (j4 - j3)$

$= 5 + j$

Multiplication in Polar Form

- Multiplying complex numbers is easiest if the numbers are in polar form.
- Suppose $z_1 = r_1 \angle \phi_1$ and $z_2 = r_2 \angle \phi_2$

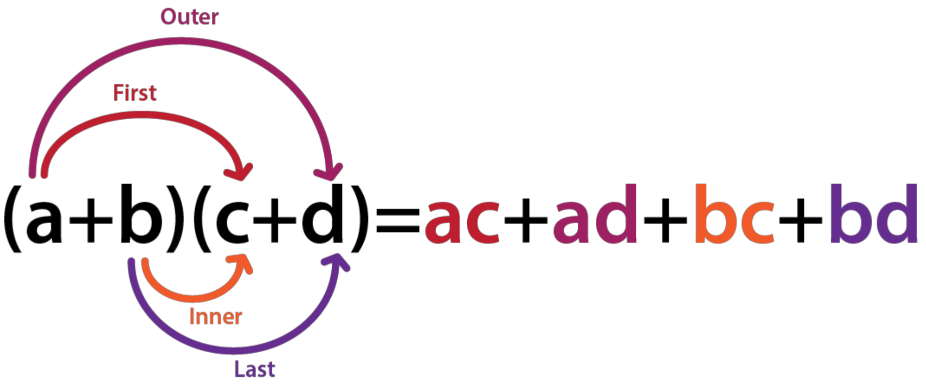
  Then $z_1 \times z_2 = (r_1 \times r_2) \angle (\phi_1 + \phi_2)$

- In words: to multiply two complex numbers in polar form, multiply their magnitudes to get the magnitude of the result, and add their angles to get the angle of the result.

Multiplication in Rectangular Form

$$(a + jb)(c + jd\,) = ac + jad + jbc + j^2bd$$
$$= ac + jad + jbc + (-1)bd$$
$$= (ac - bd\,) + j(ad + bc)$$

1. Use the FOIL method to find the product.



2. Replace j^2 by -1.

3. Write the answer in the form $a + jb$.

Multiplication Examples:

1) $\sqrt{-25}\ \sqrt{-5} = j\sqrt{25}\ \ j\sqrt{5}$

$= j5\ \ j\sqrt{5}$

$= 5j^2\sqrt{5}$

$= 5\,(-1)\sqrt{5}$

$= -5\sqrt{5}$

2) $j7\,(11 - j5)$

$= j77 - j^2 35$

$= j77 - (-1)35$

$= 35 + j77$

3) $(2 + j3)(6 - j7)$

$= 12 - j14 + j18 - j^2 21$

$= 12 + j4 - j^2 21$

$= 12 + j4 - (-1)21$

$= 12 + j4 + 21$

$= 33 + j4$

4) $z_1 = 2\angle 10$

$z_2 = 4\angle 40$

$z_1 {}^* z_2$

$(2\angle 10) * (4\angle 40) = 8\angle 50$

5) $z_1 = 12.81\angle 51.34$

$z_2 = 6.4\angle -38.66$

$z_1 {}^* z_2$

$(12.81\angle 51.34) * (6.4\angle -38.66) =$

$81.984\angle 12.68$

Complex Conjugate

حاصل ضرب به يطفين magnitude

- Given a complex number in rectangular form,

  $$z = x + jy$$

  its **complex conjugate** is simply

  $$z^* = x - jy$$

- Given a complex number in polar form,

  $$z = r\angle\phi$$

  its **complex conjugate** is simply

  $$z^* = r\angle -\phi$$

Complex Conjugate

The product of conjugates is the real number $a^2 + b^2$.

$$(a + jb)(a - jb) = a^2 - j^2b^2$$

$$= a^2 - (-1)b^2$$

$$= a^2 + b^2$$

$$(r\angle \phi)(r\angle -\phi) = r^2$$

Example: $(5 + j2)(5 - j2) = (5^2 - j^2 4)$

$$= 25 - (-1)\,4$$

$$= 29$$

Example: $(5\angle 45)(5\angle -45) = 25$

Division in Polar Form

- Dividing complex numbers is also easiest if the numbers are in polar form.
- Suppose $z_1 = r_1 \angle \phi_1$ and $z_2 = r_2 \angle \phi_2$

  Then $z_1 \div z_2 = (r_1 \div r_2) \angle (\phi_1 - \phi_2)$

- In words: to divide two complex numbers in polar form, divide their magnitudes to get the magnitude of the result, and subtract their angles to get the angle of the result.

Division in Rectangular Form

A rational expression, containing one or more complex numbers, is in simplest form when there are no imaginary numbers remaining in the denominator.

Example:

$$\frac{5+3i}{2+i}$$

$$= \frac{5+3i}{2+i} \cdot \frac{2-i}{2-i} = \frac{10-5i+6i-3i^2}{2^2+1^2}$$

$$= \frac{10+i-3(-1)}{4+1} = \frac{13+i}{5} = \frac{13}{5} + \frac{1}{5}i$$

Example:

$z_1 = 6 \angle 26.57$

$z_2 = 3 \angle -18.43$

$$\frac{z_1}{z_2}$$

$$\frac{(6 \angle 26)}{(3 \angle -18)} = 2 \angle 45$$