OR2

Notebook

First Semester 2023/2024

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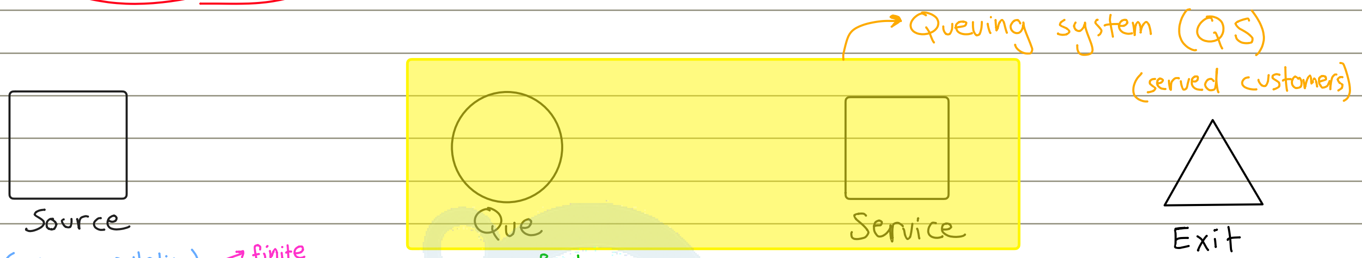
Queuing Theory :-



Source: (calling population) → finite / infinite
to generate customers

Que size → finite / infinite

→ single / multiple

Service → one server / multiple servers

→ series / parralle l

#of customers → Discrete

* التصرفات الممكنة
- Balk ← ما يدخل الدور (doesn't enter)
- Jockey ← ينط (moves from one to another)
- Reneage ← يدخل و يغير رأيه (enters then leaves)

* (μ) : customer / time
- $\left(\frac{1}{\mu}\right)$: time / customers to serve

* بصير عنا Que
لما آل service تكون busy

* Poisson Distribution :
* Arrival Rate
- (λ) customer / time

* Exponential Distribution :
* Time between Arrivals (TBA)
- $\left(\frac{1}{\lambda}\right)$ interarrival time

* Que Discipline :- order in which I will call customer to serve.
- FIFO → first in first out or (first come first served) FCFS
- FILO → first in Last out or (Last come first served) LCFS
- SRO → Service in Random Order
- P → Priority

* Symbols :
- → λ : mean arrival rate
- → μ : mean service rate per busy server
- → ρ : $\frac{\lambda}{\mu}$: utilization factor

* Common Reactions in the Que :-
- Jockeying : when the customer enteres one line and then switches to a different one in an effort to reduce the waiting time.
- Balking : The customer decides not to enter the waiting line.
- Reneging : The customers enters the line but decides to leave before being served.

SYMBOLS

λ = mean arrival rate

μ = mean service rate per busy server

ρ = λ / μ = utilization factor

n = number of units in the system

$P_n(t)$= probability of exactly *n* customers in the system at time *t*

P_n = probability of exactly *n* customers in the system

c = number of parallel servers

W_s = expected waiting time per customer in the system

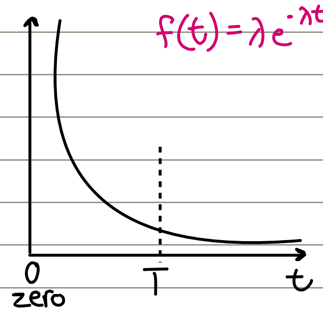
W_q = expected waiting time per customer in the queue

L_s = expected number of customers in the system

L_q = expected number of customers in the queue

تابعلا
Discrete

* Exponential Distribution



$f(t) = \lambda e^{-\lambda t}$ $\qquad t \geq 0$

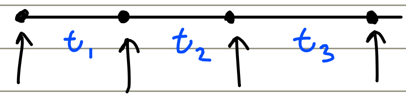
$P(t<T) = 1 - e^{-\lambda t}$

$P(T_1 < t < T_2) = e^{-\lambda T_1} - e^{-\lambda T_2}$

$P(t>T) = e^{-\lambda t}$

* Poisson Distribution

t = time between two sucesses



λ = # of arrival /time

$\lambda = 5$ customers/hour "مثلا"

$P(X=x) = \dfrac{(\lambda t)^x e^{-\lambda t}}{x!}$, $x = 0, 1, 2, 3, \dots\dots$

* Exponential vs Poisson

what is a poisson process?

1) Average time between events is known $E(x) = \lambda$ and constant
2) events are independent of each other
3) events cannot occur simultaneously

* Arrival ⟶ Poisson
* interarrival ⟶ exponential

	Exponential	Poisson
Q	How much time between a given # of events	How many events occur in a time interval
Random variable	time continuous variable	# of events - discrete variable
Parameter	λ = Rate and occurance unit $\frac{1}{\text{time}}$	λ = expected # of occurances unitless $\lambda = E(X)$

↓
time until the next event occurs, the amount of time passed by.

Example 1: Example 18.3-1

A service machine always has a standby unit for immediate replacement upon failure.

The time to failure of the machine (or its standby unit) is exponential and occurs every 5 hours, on the average. $\lambda = \frac{1}{5} = 0.2$ failure per hour

The machine operator claims that the machine is "in the habit" of breaking down every night around 8:30 p.m.

Analyze the operator's claim

The Solution :

The average failure rate of the machine is $\lambda = \frac{1}{5} = .2$ failure per hour. Thus, the exponential distribution of the time to failure is

$$f(t) = .2e^{-.2t}, t > 0$$

Regarding the operator's claim, we know offhand that it cannot be true because it conflicts with the fact that the time between breakdowns is exponential and, hence, totally random. The probability that a failure will occur by 8:30 P.M. cannot be used to support or refute the operator's claim, because the value of such probability depends on the time (relative to 8:30 P.M.) at which it is computed. For example, if the time now is 8:20 P.M., then there is a low probability that the operator's claim is right—namely,

$$p\left\{t < \tfrac{10}{60}\right\} = 1 - e^{-.2\left(\frac{10}{60}\right)} = .03278$$

If the time now is 1:00 P.M., then the probability that a failure will occur by 8:30 P.M. increases to approximately .777 (verify!). These two extreme values show that the operator's claim is not true.

Example 2

Unique visitors arrive at JU.edu.jo by a Poisson distribution at an average rate of 3 visitors per hour.

Find the probability that the next visitor arrives :-

a) Within 10 minutes

أقل من 10

b) After 30 minutes passes

c) In exactly 15 minutes time

d) Within the first minute

* e) Within the second minute

f) Within the first two minutes

$\lambda = 3$ arrival/hour

يجب التحويل

$\lambda = 3 \frac{\text{arrival}}{\text{hour}} \times \frac{1\text{hour}}{60\text{min}} = 0.05 \frac{\text{arrival}}{\text{min}}$

ⓐ within 10 minutes

أقل من عشر دقائق

Method (1) by exponential

$P(t<10) = 1 - e^{-(\frac{3}{60}\times 10)} = 1 - e^{-0.5} = 0.393$

Method (2) by Poisson

$P(x \geq 1) = 1 - P(x=0) = 1 - \left(\frac{(0.5)^{x=0}}{0!} \times e^{-0.5}\right) = 0.393$

بدي احسب عدد الـcustomers الاكبر من صفر

if we want to solve using (poisson) we cant assume $x=1$ since we have no idea how many arrivals are in the given period so its $P(x=1)+P(x=2)+\dots$ $= P(x \geq 1) = 1 - P(x=0)$

ⓑ after 30 minutes passes

$P(t>30) = e^{-\lambda t} = e^{-30(0.05)} = e^{-1.5} = 0.223$ → $P(x=0)$

يعني على فرض انه ما اجى ولا customer 30min

ⓒ in exactly 15 minutes time

$= \text{Zero}$ → Area in exact point is zero

ⓓ within the first minute

$P(t<1) = 1 - e^{-0.05} = 0.048$

ⓔ within the second minute

1→2 as if we had zero arrivals in the first minute

ⓕ within the first two minutes

0→2 or <2

أقل من دقيقتين

حلي → $P(t<2) = 1 - e^{-.05\times 2} = 0.0951$

Example 3

- The number of cups of coffee ordered per hour at JU Cafee follows a Poisson distribution, with an average of 30 cups per hour being ordered.
- ⓐ Find the probability that exactly 60 cups are ordered between 10 P.M. and 12 midnight.
- ⓑ Find the mean and standard deviation of the number of cups ordered between 9 P.M. and 1 A.M. $t = 4$ hours
- ⓒ Find the probability that the time between two consecutive orders is between 1 and 3 minutes.

$\lambda = \frac{30 \text{ cup}}{\text{hour}} = \frac{30 \text{ cup}}{60 \text{ min}} = 0.5$

*poisson since we want a number not time

ⓐ $P(X=60) = \frac{(\lambda t)^x e^{-\lambda t}}{x!} = \frac{(30 \times 2)^{60} \cdot e^{-30 \times 2}}{60!}$

$\lambda = 30$ cup/hour
$t = 2$ hours

*if the question asks if it is poisson or not
→ we calculate mean and variance if they are close then yes ✓

Note:-

ⓑ in Poisson distribution
→ expected value $= E(X) = \lambda$ → $E(X) = 30 \times 4 = 120$ cup
variance $= V(X) = \lambda$
→ Standard deviation $= \sqrt{\lambda t}$ → $S.D = \sqrt{30 \times 4} = 10.95$

Note:-
in Exponetial distribution
expected value $= E(T) = \frac{1}{\lambda}$
variance $= V(T) = \frac{1}{\lambda^2}$

ⓒ it is exponential when its time between

$\lambda = \frac{30 \text{ cup}}{60 \text{ min}}$ ← تحويل

$P(1 < t < 3) = e^{-\lambda T_1} - e^{-\lambda T_2} = e^{\left(-\frac{30}{60} \times 1\right)} - e^{\left(-\frac{30}{60} \times 3\right)} = e^{-0.5} - e^{-1.5} = 0.383$

($T_1 = 1$, $T_2 = 3$)

λ → is always # of arrivals per time

→ when time of arrival is given its $\left(\frac{1}{\lambda}\right)$

(18.4) Pure birth Model - Pure Death Model

Example: 4 example on Pure birth Example 18.4-1

Babies are born in a large city at the rate of one birth every 12 minutes. The time between births follows an exponential distribution. Find the following:

(λ) is always # of arrivals per time but when time is given it is $\left(\frac{1}{\lambda}\right)$

(a) The average number of births per year.

a) $\lambda = \frac{1 \text{ birth}}{12\ \cancel{\text{min}}} \times \frac{60\ \cancel{\text{min}}}{1\ \cancel{\text{hour}}} \times \frac{24\ \cancel{\text{hour}}}{1\ \cancel{\text{day}}} \times \frac{365\ \cancel{\text{day}}}{1 \text{ year}} = 43800$ birth / year

(b) The probability that no births will occur during 1 day.

$\frac{1 \text{ birth}}{12 \text{ min}} \times \frac{60 \text{ min}}{1 \text{ hour}} \times \frac{24 \text{ hour}}{1 \text{ day}} = 120$ birth/day

$\rightarrow P(t>1) = e^{-\lambda t} = e^{-120}$

or

$\rightarrow P(X=x) = \frac{(\lambda t)^x}{x!} \times e^{-\lambda t} \Longrightarrow P(X=0) = \frac{(120)^0}{0!} \times e^{-120} = e^{-120} = 7.667 \times 10^{-53}$ تقريبا zero

(c) The probability of issuing 50 birth certificates in 3 hours, given that 40 certificates were issued during the first 2 hours of the 3-hr period.

due to the lack of memory we only care about the 10 left in 1 hour

$P(X=10) = \frac{(5 \times 1)^{10}}{10!} \times e^{-5} = 0.0181$

$\lambda t = \frac{1 \text{ births}}{12 \text{ min}} \times \frac{60 \text{ min}}{1 \text{ hour}}$

$\lambda t = 5$ birth / hour

* Pure Death :- only departures occur

N : # of customers at time $= 0$

this will decrease by time $(N-1, N-2, \dots\dots)$

Trancated poisson distribution since $(x = 0, \dots, N)$

$n \rightarrow$ # of customer in the time t in the system

n
⋮
3
2
1
0

$P_n(t)$: probability that we have n customers in the system

$$P_n = \frac{(\mu t)^{N-n}}{(N-n)!} \times e^{-\mu t}, \quad n = 0, 1, 2, \dots\dots N$$

Example 5: Example 18.4-2

The florist section in a grocery store stocks 18 dozen roses at the beginning of each week. On the average, the florist sells 3 dozens a day (one dozen at a time), but the actual demand follows a Poisson distribution.

Whenever the stock level reaches 5 dozens, a new order of 18 new dozens is placed for delivery at the beginning of the following week.

Because of the nature of the item, all roses left at the end of the week are disposed of. Determine the following:

(a) The probability of placing an order in any one day of the week.

(b) The average number of dozen roses discarded at the end of the week.

*Death Models :-

N : # of customers in the system at $t=0$

$P_n(t)$: probability that we have n customers in the system

$$P_n(t) = \frac{(\mu t)^{N-n}}{(N-n)!} \times e^{-\mu t}, \quad n = 1, 2, \ldots N$$

بدأ من N و بنزل

$$P_0 = 1 - \sum_{n=1}^{N} P_n = 1 - \sum_{n=1}^{n=18} P_n$$

*The Solution :-

$P_n(n \leq 5) = P_0 + P_1 + P_2 + P_3 + P_4 + P_5$

For day
1 $= \mu t = 3 \times 1$
2 $= \mu t = 3 \times 2$
3 $= \mu t = 3 \times 3$ ← assume we are dealing with day (3)

* for day (3)

$P_1 = \frac{9^1}{1!} \times e^{-9} = 1.110 \times 10^{-3}$

$P_2 = \frac{9^2}{2!} \times e^{-9} = 4.998 \times 10^{-3}$

$P_3 = \frac{9^3}{3!} \times e^{-9} = 0.0149$

(Sold) $N-n$	(stock) n	$P_n = \frac{(\mu t)^{(N-n)}}{(N-n)!} \cdot e^{-\mu t}$
0	18	P_0 ← آخر اشي حسابها
1	17	.0011106
2	16	.00499
3	15	.01499
4	14	.03373
5	13	.06072
6	12	.09109
7	11	.1171
8	10	.13175
9	9	.13175
10	8	.11858
11	7	.09702
12	6	.07276
13	5	.050375 بدي مجموعهم
14	4	.03238
15	3	.01973
16	2	.010929
17	1	.005786
18	0	.00289

$\mu = 3$

day 3 so $t = 3$

(a) Probability of placing an order in any one day of the week

$P_0 = 1 - 0.994578 = 0.005422$

$P_n(n \leq 5) = P_0 + P_1 + P_2 + P_3 + P_4 + P_5 = 0.12433$ for day (3)

→ we have to repeat this for the (7) days

(b) The average number of dozen roses discarded at the end of the weak.

expected value يعني بده

*Remember in Stat (1) :-
$E(X) = \sum x . f(x)$
$= \sum x . P(x)$

$$E(n) = \sum_{n=0}^{n=18} n . P_n$$

$= 0 \times P_0 + 1 \times P_1 + 2 \times P_2 + \cdots\cdots + 18 \times P_{18} = .664$

(18.5) General Poisson Model

* State of the system (n) $\longrightarrow$ that means # of customers in the system
  - Que System (Que + Service)

* (P_n) $\rightarrow$ Probability that we have n customers in the system

* Steady State
  - P_n
  - n
  - Transition Rate
  - λ_n
  - μ_n

(Steady State)

يعني ال system تبعي بطل
الا علاقة بالزمن
Not a function of time
it is independent

* Probability of more than 1 event during small time interval $h = \text{zero}$ as $h \rightarrow 0$

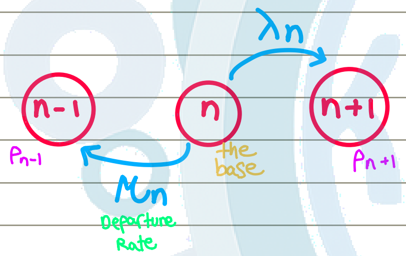
balance equation

$$\boxed{\text{Total Transitions into } n = \text{Total transition out of } n}$$

Flow Rate in = Flow Rate out

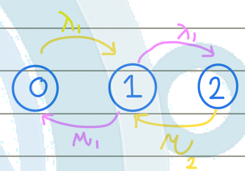
لازم يكونو
equal يا
يا زيادة عن
الثاني بواحد

$$\lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1} = \lambda_n P_n + \mu_n P_n$$
$$= (\lambda_n + \mu_n) P_n$$



State one

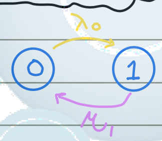
* مثال :-



$$\lambda_0 P_0 + \mu_2 P_2 = P_1 \lambda_1 + P_1 \mu_1$$
$$\lambda_0 P_0 + \mu_2 P_2 = P_1 (\lambda_1 + \mu_1)$$

State zero

* مثال :-

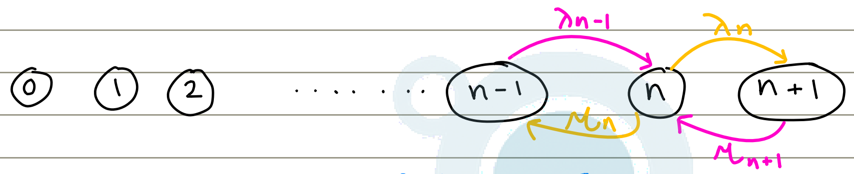


$$P_1 \mu_1 = \lambda_0 P_0$$
$$P_1 = \frac{\lambda_0 P_0}{\mu_1}$$

Tuesday
24/10/2023

(18.5) General Poisson Model :-

Arrivals + Departure



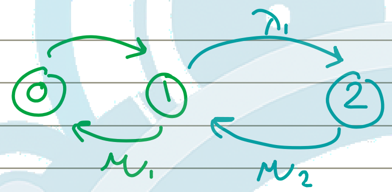
Balance Equation

In | out

$$P_{n-1} \cdot \lambda_{n-1} + P_{n+1} \cdot \mu_{n+1} = P_n \lambda_n + P_n \mu_n$$
$$= (\lambda_n + \mu_n) P_n$$

λ_n
μ_n
S.S
$h \to 0$

*الاشتقاق غير مهم
اهم اشي التطبيق على ال
(General Rule)



* For the zero

$$P_1 \mu_1 = P_0 \lambda_0$$

$$\boxed{P_1 = \frac{\lambda_0}{\mu_1} P_0}$$

* for the one

$$P_0 \lambda_0 + P_2 \mu_2 = P_1 \mu_1 + P_1 \lambda_1$$
$$= (\mu_1 + \lambda_1) P_1$$
$$= (\mu_1 + \lambda_1) \times \frac{\lambda_0}{\mu_1} P_0$$
$$= \lambda_0 P_0 + \frac{\lambda_0 \lambda_1}{\mu_1} P_0$$

$$\boxed{P_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} P_0}$$

$$\boxed{P_3 = \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} P_0}$$

* General Rule :-

$$\boxed{P_n = \frac{\lambda_0 \lambda_1 \cdots\cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots\cdots \mu_n} \times P_0}$$

$$\boxed{P_0 = 1 - \sum_{n=1}^{\infty} P_n}$$

Example 6: Example 18.5-1+problem 18.39

B&K Groceries operates with three checkout counters. The manager uses the following schedule to determine the number of counters in operation, depending on the number of customers in line:

-

Number of customers in store	Number of counters in operation
1 to 3	$\mu =$ 1 $\times 5$
4 to 6	2 $\times 5$
More than 6	3 $\times 5$

Customers arrive in the counters area according to a Poisson distribution with a mean rate of 10 customers per hour.

The average checkout time per customer is exponential with mean 12 minutes.

1. **Determine the steady-state probability *pn* of *n* customers in the checkout area.**
2. **The probability that only one counter will be open**
3. **Expected number of idle counters**
4. **The probability distribution of the number of open counters.**
5. **The average number of busy counters**

$\lambda = 10$ customer/hour

$\frac{1}{\mu} = 12$ min/per customer

للتحويل

$\mu = \frac{1 \text{ customer}}{12 \text{ min}} \times \frac{60 \text{ min}}{\text{hr}} = \boxed{\frac{5 \text{ customer}}{\text{hr}}}$ (μ)

$$\boxed{P_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} P_0}$$

$\lambda_n = \lambda = 10$ customer/hr

$$\mu_n = \begin{cases} 5 & n = 1,2,3 \\ 10 & 4,5,6 \\ 15 & 7,8,\ldots \end{cases}$$

بعد إيجاد P_0

$P_1 = \frac{\lambda_0}{\mu_1} P_0 = \frac{10}{5} \times P_0 = 2P_0 = \frac{2}{55}$

$P_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} P_0 = \frac{10 \times 10}{5 \times 5} = 4P_0 = \frac{4}{55}$

$P_3 = \quad = 8P_0 = \frac{8}{55}$

$P_4 = \frac{\lambda_0 \lambda_1 \lambda_2 \lambda_3}{\mu_1 \mu_2 \mu_3 \mu_4} \times P_0 = \frac{10^4}{5^3 \times 10} = 8P_0$

$P_5 = \quad = 8P_0$

$P_6 = \quad = 8P_0$

لأنهم نضرب بـ 10 و نقسم على 10

$P_7 = 8P_0 \times \frac{10}{15} = \frac{80}{15} P_0 = \frac{16}{3} P_0$

ضرب $\frac{2}{3}$

$P_8 = 8P_0 \times \frac{10^2}{15^2} = \frac{32}{9} P_0$

$P_9 = 8P_0 \times \frac{10^3}{15^3} = \frac{64}{27} P_0$

* بما أن السؤال ماذكر اشي إذا

$\lambda = 10$ customer/hr

ثابتة

$P_0 = 1 - \sum_{n=1}^{\infty} P_n$

$= 2P_0 + 4P_0 + 8P_0 + 8P_0 + 8P_0 + 8P_0 + 8\,\frac{2}{3}P_0 + 8\left(\frac{2}{3}\right)^2 P_0 + 8\left(\frac{2}{3}\right)^3 P_0 + \cdots$

$30P_0 + 8P_0\left[1 + \frac{2}{3} + \frac{2}{3}^2 + \frac{2}{3}^3\right) + \cdots$

$30P_0 + 8P_0\left[\frac{1}{1-\frac{2}{3}}\right]$

* Rule

$\frac{1}{1-\rho}$

$1 + \rho + \rho^2 + \rho^3$

$\rho < 1$

$P_0 + \sum_{i=1}^{\infty} P_n = 1$

$P_0 + 54P_0 = 1$

$55P_0 = 1 \longrightarrow \boxed{P_0 = \frac{1}{55}}$

Continue

$\lambda_n = \lambda = 10$ customer/hr

$$\mu_n = \begin{cases} 5 & 1,2,3 \\ 10 & 4,5,6 \\ 15 & 7,8,\ldots \end{cases}$$

$n \rightarrow$ # of customers

$P_0 = \frac{1}{55}$
$P_1 = 2P_0$
$P_2 = 4P_0$
$P_3 = 8P_0$
$P_4 = 8P_0$
$P_5 = 8P_0$
$P_6 = 8P_0$
$P_7 = \frac{16}{3}P_0$
$P_8 = \frac{32}{9}P_0$
$P_9 = \frac{64}{27}P_0$

* لما يكونو ال P متساوين بكون عندي uniform distribution

② The Probability that only one counter will open.

$P(n \leq 3) = P_0 + P_1 + P_2 + P_3 = \frac{15}{55}$

③ expected number of idle counters.

فاضيين

$$\boxed{E(n) = \sum_{n=0}^{\infty} n \cdot P_n}$$

$E(\text{idle counters}) = 0(1-(P_0+\ldots+P_6)) + 1(P_4+P_5+P_6) + 2(P_1+P_2+P_3) + 3P_0$

صفر ضرب صفر فهو مهم

$= \frac{55}{55} = 1$

n	idle	Probability distribution of idle counters	busy	E(Busy)
0	3	P_0	0	$= 0(\frac{1}{55})$
1	2		1	
2	2	$P_1+P_2+P_3 = 14P_0$	1	$+1(\frac{14}{55}) = \frac{14}{55}$
3	2		1	
4	1		2	$+2(\frac{24}{55}) = \frac{48}{55}$
5	1	$P_4+P_5+P_6 = 24P_0$	2	
6	1		2	
7	0	$P_7+P_8+P_9+\ldots = 1-[P_0+\ldots+P_6] = \frac{16}{55}$	3	$+3(\frac{16}{55}) = \frac{48}{55}$
8	0		3	
⋮				

حل فرع ⑤

بعد ما اجمعهم $= \frac{110}{55} = ②$

④ The Probability distribution of the number of open counters.

$x = 0, 1, 2, 3$

Probability Distribution

Possible values $\xrightarrow{\text{and the}}$ corresponding Probability of them

We write the event and the corresponding probability (زي الجدول)

⑤ The average number of busy counters

expected number

$E(\text{Busy}) = 0(\frac{1}{55}) + 1(\frac{14}{55}) + 2(\frac{24}{55}) + 3(\frac{16}{55})$

$= \frac{110}{55}$

$= \boxed{2}$

الجواب حيطلع $\frac{110}{55}$

$= 2$

بالمنطق عشان ال expected ال idle حسبنا = ١ واحد و عنا ٣ counters اذاً الجواب ٢

انتهى سكشن (18.5)

[Generalized Poisson Model]

الاحد
26/10/2023

Steady State Performance measures :-

① L_s : Expected numbers of customers in the system. → Que + service

$L_s = E(n) = \sum_{n=0}^{\infty} n \cdot P_n$

$L_s = L_q + \frac{\lambda_{eff}}{\mu}$ من الكتاب

② L_q : Expected numbers of customers in the Que.

$L_q = \sum_{n=c+1}^{\infty} (n-c) P_n$

C: Parallel Servers.

$n < c \Rightarrow$ No Que
$n = c \Rightarrow$ No Que
$n > c \Rightarrow$ Que ✓

③ $\bar{c}$: Expected numbers of busy servers $= (\rho = \bar{c})$

$\bar{c} = L_s - L_q$

$\bar{c} = \frac{\lambda eff}{\mu}$ من الكتاب

Little's Law

$L_s = \lambda_{eff} W_s$

$L_q = \lambda_{eff} W_q$

λ , $\lambda_{effective}$

④ $W_s = W_q$ + service time

$W_s = W_q + \frac{1}{\mu}$

* flow Rate of residance time
→ Sometimes we use λ or $\lambda_{effective}$

Example

Arrival Rate $\lambda = 8 \frac{customer}{hour}$

$c = 2$

The state probabilities for a two-server queue, with arrival rate 8 customers/hour, were found to be $P_0 = 0.4$, P_1, $= 0.3$, $P_2 = 0.2$, $P_3 = 0.1$. All other state probabilities equal zero. Find

$P_0 = 0.4$
$P_1 = 0.3$
$P_2 = 0.2$
$P_3 = 0.1$

1. $L_s = \sum_{n=0}^{\infty} n P_n = 1 \Rightarrow (0 \times 0.4 + 1 \times 0.3 + 2 \times 0.2 + 3 \times 0.1) = 1$; $0 \times P_0 + 1 \times P_1 + 2 \times P_2 + 3 \times P_3$
2. $L_q = \sum_{n=c+1}^{\infty} (n-c) P_n = 1 \times 0.1 = 0.1$; $(3-2) P_3$
3. $\rho = \bar{c} = L_s - L_q = 0.9$
4. $Ws = \frac{L_s}{\lambda} = \frac{1}{8} = 0.125$ hour
5. $Wq = 0.0125\,hr = \frac{L_q}{\lambda} = \frac{0.1}{8}$
6. $I/\mu = 0.125\,hr - 0.0125 = 0.1125\,hr$; $W_s - W_q$ $\xrightarrow{min}$ 6.75 min

Example 18.6-1

Visitors' parking at Ozark College is limited to five spaces only. Cars making use of this space arrive according to a Poisson distribution at the rate of six cars per hour. Parking time is exponentially distributed with a mean of 30 minutes. Visitors who cannot find an empty space on arrival may temporarily wait inside the lot until a parked car leaves. That temporary space can hold only three cars. Other cars that cannot park or find a temporary waiting space must go elsewhere. Determine the following:

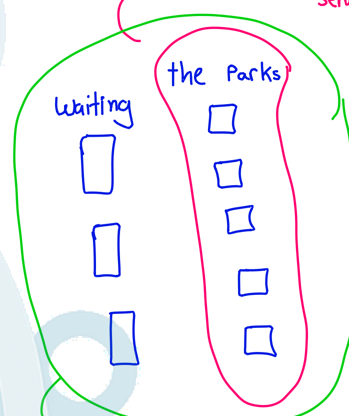
1. The probability, p_n, of n cars in the system.
2. The effective arrival rate for cars that actually use the lot.
3. The average number of cars in the lot.
4. The average time a car waits for a parking space inside the lot.
5. The average number of *occupied* parking spaces.
6. The average utilization of the parking lot.

$\lambda = 6$ cars/hour

$\mu = \frac{1}{30\text{ min}} \times \frac{60\text{ min}}{1\text{ hour}} = 2$ cars/hour **but** it depends on n, so it is not constant

للتحويل

acts as a server so ($c=5$) parrallel servers



the Maximum Capacity of the system is $(5+3=8)$

$$\Rightarrow \mu_n = \begin{cases} 2 \times n = 2n \text{ cars/hour} & n = 1,2,3,4,5 \\ \underset{c_n}{5} \times 2 = 10 \text{ cars/hour} & n = 6,7,8 \end{cases}$$

لأنه لو عندي سيارة وحدة إذا park واحد شغال ف بالساعة بطلع 2
ولو 2parks شغالين ف بطلع 4 و هكذا لحد ما يتعبوا ال 5

1. The probability, p_n, of n cars in the system.

we will find from ($P_0 \xrightarrow{\text{to}} P_8$)

$$* \; P_n = \begin{cases} \frac{3^n}{n!} \times P_0 & n = 1,2,3,4,5 \\ \frac{3^n}{5!\,5^{n-5}} \times P_0 & n = 6,7,8 \end{cases}$$

$P_1 = \frac{\lambda_0}{\mu_1} \times P_0 = \frac{6}{2} P_0 = \boxed{3P_0}$

$P_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} \times P_0 = \frac{6}{2} \times \frac{6}{2 \times 2} \times P_0 = \boxed{\frac{9}{2} P_0}$

$P_3 = \frac{9}{2} \times \frac{6}{3 \times 2} \times P_0 = \boxed{\frac{9}{2} P_0}$

$P_4 = \frac{9}{2} \times \frac{6}{4 \times 2} \times P_0 = \boxed{\frac{27}{8} P_0}$

$P_5 = \frac{27}{8} \times \frac{6}{5 \times 2} \times P_0 = \boxed{\frac{81}{40} P_0}$

$P_6 = \frac{81}{40} \times \frac{6}{10} \times P_0 = \boxed{\frac{243}{200} P_0}$

هنا μ اصبحت ثابتة

$P_7 = \frac{243}{200} \times \frac{6}{10} \times P_0 = \boxed{\frac{729}{1000} P_0}$

$P_8 = \frac{729}{1000} \times \frac{6}{10} \times P_0 = \boxed{\frac{2187}{5000} P_0}$

→ Now we want to find $P_0 = 1 - \sum_{n=1}^{n=8} P_n \longrightarrow P_0 = 1 - 19.7814\, P_0 \longrightarrow \boxed{P_0 = 0.4812}$

now we substitute in $P_1 \ldots P_8$

$N=8 \rightarrow$ (Maximum capacity of the system (Que + Service))
$c=8 \rightarrow$ (the Park spaces (parallel servers))

2. The effective arrival rate for cars that actually use the lot.

$$\lambda_{effective} = \lambda - \lambda_{lost} \longrightarrow \lambda_{eff} = 6 - 0.12 = 5.88 \text{ cars/hour}$$

λ: 6 cars/hr من السؤال

$$\lambda_{lost} = \lambda \times P_8 = 6 \times 0.02 = 0.12 \text{ cars/hour}$$

بالساعة كم سيارة بتجي و بتروح

لأنه السيارات ماراح تقدر تدخل لما يكون عندي 8 جوا
(it is the maximum)

* $\lambda_{eff} \longrightarrow$ يعني كم سيارة من يلي بيجو فعلاً بدخلو

Note: $P_0 + P_1 + P_2 + P_3 + P_4 + P_5 + P_6 + P_7 + P_8 = 1$

* So actually the reason behind having λ and λ_{eff} is that I am limited with number of cars that enters, not ∞.

3. The average number of cars in the lot.
Average السيارات يلي قدرو يدخلو

$$L_s = \sum_{n=0}^{n=8} n \cdot P_n = 0P_0 + 1P_1 + 2P_2 + 3P_3 + 4P_4 + 5P_5 + 6P_6 + 7P_7 + 8P_8 = 3.1286 \text{ cars}$$

expected value (on average)

Note: $P_3 = 0.30 \rightarrow$ معناها 30% من الوقت عندي ٣ سيارات

4. The average time a car waits for a parking space inside the lot.

we must use (eff) to calculate who actually entered

$$W_q = W_s - \frac{1}{\mu} \quad \text{OR} \quad W_q = \frac{L_q}{\lambda_{eff}}$$

$$W_s = \frac{L_s}{\lambda_{eff}} = 0.53$$

$$L_q = \sum_{n=c+1}^{n=8} (n-c) \cdot P_n$$

$\sum_{n=0}^{n=8}$ * كونها محصورة بال Park

فهي effective لأنه بس يلي دخلو و لحد ال 8
بدون geo لا تساوي ال effective
مدري

$$W_q = W_s - \frac{1}{\mu}$$

$$W_q = 0.53 - \frac{1}{2} = 0.0326 \text{ hour}$$

5. The average number of *occupied* parking spaces.
busy

$$\bar{c} = L_s - L_q = 2.94$$

$\left(\frac{\lambda_{eff}}{\mu}\right)$
يعني لو فجأة دخلت
الحل on average كم مشغول

6. The average utilization of the parking lot.

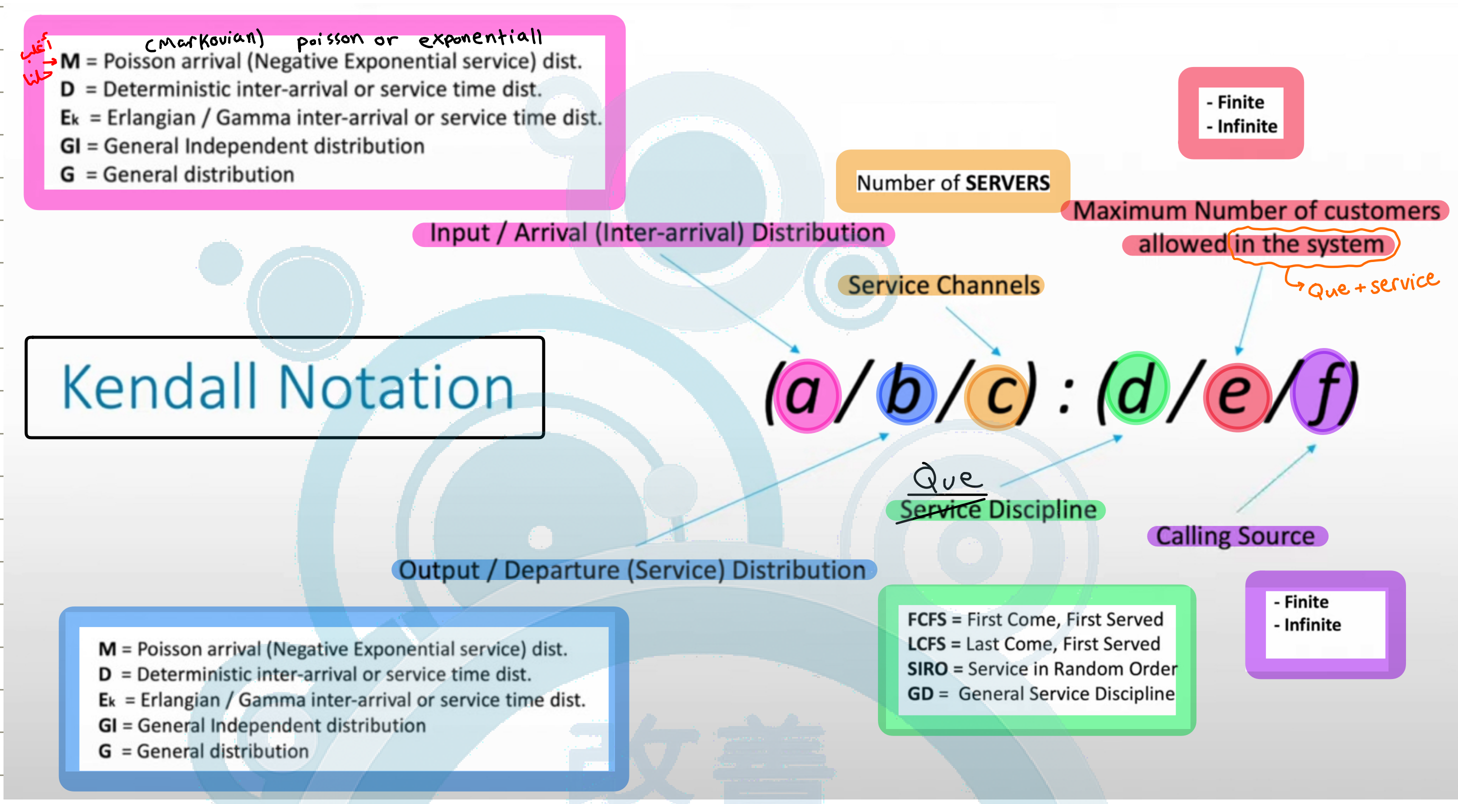
$$\frac{\lambda_{effective}}{\mu} = \frac{\text{Used}}{\mu} = \frac{\bar{c}}{c} = \frac{2.94}{5} = 0.588$$

فرع اضافي *7. Compute the average number of carsthat will not be able to enter the parking lot during an 8 hour period

$\lambda_{lost} \times 8$ hours

(μ_n) is sometimes needed and usually it is a multi servers case

Section (18.6) :- Kendall Lee Notation



Kendall Lee notation

$$(a / b / c): (d / e / f)$$

where

a = Arrivals distribution

b = Departures (service time) distribution

c = Number of parallel servers 1= 1, 2, ..., infinity

d = Queue discipline

e = Maximum number (finite or infinite) allowed in the system

(in-queue plus in-service)

f = Size of the calling source (finite or infinite)

$$(a / b / c): (d / e / f)$$

The standard notation for representing the arrivals and departures distributions (symbols *a* and *b*) is

- *M* = Markovian (or Poisson) arrivals or departures distribution (or equivalently exponential interarrival or service time distribution)
- *D* = Constant (deterministic) time
- *Ek* = Erlang or gamma distribution of time (or, equivalently, the sum of independent exponential distributions)
- *GI* = General (generic) distribution of interarrival time
- *G* = General (generic) distribution of service time

$$(a / b / c): (d / e / f)$$

The queue discipline notation (symbol *d*) includes

- *FIFO* = First In, First-out
- *LIFO* = Last In, First-out
- *SIRO* = Service In random order
- *GD* = General discipline (i.e., any type of discipline)

* The Single Server Models :-

Kendal Notation

$(M/M/1):(GD/\infty/\infty)$

لما يكون عندي هون n
مو ∞ بصير عندي $\lambda_{effective}$

$\lambda_n = \lambda$
$\mu_n = \mu$ $\rightarrow$ $\rho = \frac{\lambda}{\mu}$

$P_n = \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} P_0 = \rho^n P_0$

$P_1 = \frac{\lambda_0}{\mu_1} P_0$

$P_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} = \frac{\lambda^2}{\mu^2}$

$P_n = \left(\frac{\lambda}{\mu}\right)^2 P_0$

$P_n = \rho^n P_0$

$\boxed{P_n = \rho^n (1-\rho)}$

$P_0 + P_1 + P_2 + P_3 + \cdots\cdots = 1$

$P_0 \left(1 + \rho + \rho^2 + \rho^3 + \cdots\cdots\right) = 1$

$P_0 \left(\frac{1}{1-\rho}\right) = 1 \Rightarrow P_0 = (1-\rho)$

$L_s = E(n) = \frac{\rho}{1-\rho}$

$L_s = L_q + \rho$

$(\bar{c} = \rho)$

Thursday 2/11

$P_0 = \rho^n (1-\rho)$, $\rho < 1$

$E(n) = L_s = \sum_{n=0}^{\infty} n P_n = \frac{\rho}{1-\rho}$, $\rho < 1$

لو فرضنا $\rho > 1$ ما بزبط
منطقياً ورياضياً

$\leftarrow$ منطقياً على فرض الـ Que
راح يضل growing وما راح
اقدر اوصل الـ steady state

$L_q = L_s - L_{\text{being served}} = L_s - \bar{c} = L_s - \rho$

$L_s = W_s \lambda$

$L_q = W_q \lambda$

$L_q = L_s - \rho$

$\rho = \frac{\lambda}{\mu} = \frac{10 \text{ cars/hr}}{20 \text{ cars/hr}}$ *

$\boxed{\rho < 1}$

$\rho = \frac{\lambda}{\mu} = \frac{10 \text{ car/hr}}{5 \text{ cars/hr}}$ *

$\boxed{\rho > 1}$

$\therefore$ I will never have a steady state

*Example

$(M/D/10):(GD/20/\infty)$

- M: Poisson Arrivals or Exponential interarrival time
- D: constant service time
- 10: 10 Parallel servers
- GD: Que discipline is GD
- 20: limit of 20 customers on the entire system
- ∞: the size of the source from which customers arrive is infinite

Let's consider $\boxed{\mu = 10 \text{ customers / hour}}$

Calculate the following for each λ [customers / hour] (λ = 1, 5, 9)

		$\lambda=1$	$\lambda=5$	$\lambda=9$
$\rho = c = \frac{\lambda}{\mu}$	1. ρ =	0.1	0.5	0.9
$L_s = \frac{\rho}{1-\rho}$	2. Ls =	0.1111	1	9
$L_q = L_s - \rho$	3. Lq =	0.01111	0.5	8.1
$W_s = \frac{L_s}{\lambda}$	4. Ws =	0.11111 hr	0.2	1 hr
$W_q = \frac{L_q}{\lambda}$	5. Wq =	0.01111 hr	0.1	0.9

Example

Single Server Model

Sports fans arrive at a ticket counter at the Amman Sports City Arena by a Poisson process with rate 105 customers per hour. Customers are served by a single cashier. The service time has an exponential distribution with mean 30 seconds.

$\lambda = 105$ customers/hour

$\frac{1}{\mu} = 30$ seconds

$\mu = \frac{1}{30\text{ s}} \times \frac{60\text{ s}}{1\text{ min}} \times \frac{60\text{ min}}{1\text{ hour}}$

$\mu = 120$ customer served/hour

Calculate:

1. **The service rate** $= \mu = 120$ customer/hour
2. **The utilization** $= \rho = \frac{\lambda}{\mu} = \frac{105}{120} = 0.875$
3. $\boldsymbol{Ls = \frac{\rho}{(1-\rho)}} = 7$ counters
4. $\boldsymbol{Lq = L_s - \rho} = 7 - 0.875 = 6.125$ counters
5. $\boldsymbol{Ws = \frac{L_s}{\lambda}} = \frac{7}{105} = 0.0666$
6. $\boldsymbol{Wq = \frac{L_q}{\lambda}} = \frac{6.125}{105} = 0.0583$

Example :

$\lambda = 10$ cars/hour

* **Keyword :-** $(M/M/1):(\;/\;/\;)$ يعني

Single server model

- An average of 10 cars per hour arrive at a single-server drive-in teller. Assume that the average service time for each customer is 4 minutes, and both interarrival times and service times are exponential. Answer the following questions:

service time $= \frac{1}{\mu}$

$\frac{1}{\mu} = 4$ min

1. What is the probability that the teller is idle? ما عنده حد إذاً (P_0)
2. What is the average number of cars waiting in line for the teller? (A car that is being served is not considered to be waiting in line.) L_q
3. What is the average amount of time a drive-in customer spends in the bank parking lot (including time in service)?
4. * On the average, how many customers per hour will be served by the teller?

$E(x) = \lambda = 10$ cars/hour

$\mu = \frac{1 \text{ customer}}{4 \text{ min}} \times \frac{60 \text{ min}}{1 \text{ hour}} = 15 \frac{\text{customer}}{\text{hour}}$

$\rho = \frac{\lambda}{\mu} = \frac{10}{15} = 0.667$

1) $\boxed{\text{Probability teller is idle} = P_0 = 1 - \rho} = 1 - 0.667 = 0.334$

2) $L_q = L_s - \rho = 2 - 0.667 = 1.33$

$L_s = \frac{\rho}{\rho - 1} = \frac{0.667}{1 - 0.667} = 2$

$(M/M/1):(GD/\infty/\infty)$

GD: General Disipline

لأنه ما بفرق معي كيف بدخلو

3) $W_s = \frac{L_s}{\lambda} = \frac{2}{10} = 0.2$ hour

* 4) $= \mu \times \rho = 15 \times 0.667 = 10$

مش μ لحال لأنها بتكون

Assuming that he is busy all the time (ulitilization is 100%)

but here he is only busy for $\rho = 0.667$ of the time

which means if I take μ → I am serving 15 customers/hour but I only have 10 customers arriving

* هنا تساوي (λ) لكن مو دائماً حترتبط لأنه server واحد و $(\mu > \lambda)$

Example: Single Server Example 18.6-2 (Modified)

Automata car wash is a one-bay facility. Cars arrive according to a Poisson distribution with a mean of 4 cars per hour and may wait in the facility's parking lot or on the street bordering the wash facility if the bay is busy. The time for washing and cleaning a car is exponential, with a mean of 10 minutes. This means that, for all practical purposes, there is no limit on the size of the system.

$\lambda = 4$ cars/hour

$\mu = \frac{1}{10\text{ min}} \times \frac{60\text{ min}}{1\text{ hour}} = 6$ cars/hr

Calculate

1. The utilization $= \rho = \frac{\lambda}{\mu} = \frac{4}{6} = 0.667$
2. $L_s = \frac{\rho}{1-\rho} = \frac{0.667}{1-0.667} = 2$ car
3. $L_q = L_s - \rho = 2 - 0.667 = 1.333$ car
4. $W_s = \frac{L_s}{\lambda} = \frac{2}{4} = 0.5$ hour
5. $W_q = \frac{L_q}{\lambda} = \frac{1.333}{4} = 0.333$ hour
6. If we have 2 parking spaces, what is the probability that an arriving car will enter the facility? $\rightarrow$ P_0✓, P_1✓, P_2✓ بدخلو
7. If we have 4 parking spaces, what is the probability that an arriving car will not enter the facility?
8. The manager of the facility wants to determine the size of the parking lot.

(6) Probability that an arriving car will enter the facility ≥ 0.9

$P_{entering} = $ for 2 spaces $\rightarrow$ $P_0 + P_1 + P_2 = .7 = 70\%$

منطقي جداً
لما أكبره
بدخل أكثر

bay (service)

فرع اضافي $P_{entering} = $ for 3 spaces $\rightarrow$ $P_0 + P_1 + P_2 + P_3 = .8 = 80\%$

Note:-
$P_{entering} = \ldots\ldots$
$P_{not\ entering} = 1 - P_{entering}$

$P_0 = 1 - \rho = 0.3333$
$P_1 = 0.2222$
$P_2 = 0.1481$
$P_3 = 0.0988$
$P_4 = 0.0659$
$P_5 = 0.0439$ — 0.91
$P_6 = 0.029$
$P_7 = 0.019$ — 0.95

$P_n = \left(\frac{\lambda}{\mu}\right)^n \times P_0$ — if μ, λ are constant

(7) for 4 spaces $\rightarrow$ $P_{entering} = P_0 + P_1 + P_2 + P_3 + P_4 = $

$\therefore P_{not\ entering} = 1 - P_{entering} = $ ✓

(8) it depends on my desire, and here we chose to cover 90%

احنا فرضناها
عشان يكون نسبة أفضل

$P_0 + P_1 + P_2 + P_3 + P_4 + P_5 \geq 0.9$

$(1-\rho) + (1-\rho)\rho + (1-\rho)\rho^2 + \ldots\ldots + (1-\rho)\rho^s \geq 0.9$ — s not ∞

$(1-\rho)\left[1 + \rho + \rho^2 + \rho^3 + \ldots\ldots\ldots + \rho^s\right] \geq 0.9$

$\cancel{(1-\rho)}\left[\frac{1-\rho^{s+1}}{\cancel{1-\rho}}\right] \geq 0.9$ — From here Solve on Calculator

بقدر اضل اجمع الـ P
لحد ما اوصل للنسبة
يلي بدي اياها
(لكن مرات بكون الاتي
طويل و صعب)

$(1 - \rho^{s+1}) \geq 0.9$

$\underset{\text{less than one}}{\ln \rho^{s+1}} \geq \underset{\text{less than one}}{\ln 0.9}$

الان سوف اضع ln
و ln أي اشي أقل من
واحد بكون سالب
فبنقلب الاشارة

يمكن x $\rightarrow$ $(-\rho^{s+1} \geq -0.1)\ x -1$ الاشارة تنقلب

$s+1(\ln \rho) \geq \ln 0.1$

$s+1 \geq \frac{\ln 0.1}{\ln 0.6667}$

$s+1 \geq 5.679$

$\rightarrow s \geq 4.679 \approx 5$

أو بطلع الـ P و بضل أجمع
لحد ما اوصل للي بدي اياه
لكن مرات بكون صعب لأنه؟
ما بنعرف لوين نوقف

$(M/M/1_{server}):(GD/\infty/\infty)$ — GD: general discipline; ∞: in the system; ∞: calling population

$(M/M/1):(GD/N/\infty)$

Single Server Finite Size Poisson Queuing Situation

يعني ما الي دخل ب يلي برا

$$\lambda_n = \begin{cases} \lambda & n = 0, 1, 2, \ldots, N-1 \\ 0 & n = N, N+1, \ldots \end{cases}$$

$$\mu_n = \mu \qquad n = 0, 1, \ldots$$

$$\rho = \frac{\lambda}{\mu}$$

* ρ could be >1, <1, $=1$

لأن في N → we are limited

$$P_n = \begin{cases} \rho^n P_0 & n \leq N \\ 0 & n > N \end{cases}$$

$$P_0 = \begin{cases} \dfrac{1-\rho}{1-\rho^{N+1}}, & \rho \neq 1 \\ \dfrac{1}{N+1} & \rho = 1 \end{cases}$$

$$\lambda_{lost} = \lambda P_N$$

we have effective since we are limited →

$$\lambda_{eff} = \lambda - \lambda_{lost}$$

$$L_s = \begin{cases} \dfrac{\rho\left[1-(N+1)\rho^N + N\rho^{N+1}\right]}{(1-\rho)(1-\rho^{N+1})}, & \rho \neq 1 \\ \dfrac{N}{2}, & \rho = 1 \end{cases}$$

← مو حفظ لكن ضروري نعرف نطبق عليهم

$$L_s = W_s \cdot \lambda_{effective}$$

$$L_q = W_q \cdot \lambda_{effective}$$

$$W_s = W_q + \frac{1}{\mu}$$

Example: Example 18.6-4 + Problem 18-59

Consider the car wash facility of Example 18.6-2. Suppose that the facility has a total of four parking spaces. If the parking lot is full, newly arriving cars balk to other facilities.

Key word (λ_{lost})

The owner wishes to determine the impact of the limited parking space on losing customers to the competition.

a) Probability that an arriving car will go into the wash bay immediately on arrival.

b) Expected waiting time until a service starts.

c) Expected number of empty parking spaces.

d) Probability that all parking spaces are occupied.

e) Percent reduction in average service time that will limit the average time in the system to about 10 minutes.

4 Parking Spaces

a) P_0

b) W_q

c) $4 - L_q$

d) $P_5 = \rho^n P_0$

$$P_0 = \begin{cases} \dfrac{1-\rho}{1-\rho^{N+1}} & \rho \neq 1 \\ \dfrac{1}{N+1} & \rho = 1 \end{cases}$$

$\lambda = 4$ cars/hour

e) $W_s = 10$ min

$W_s = W_q + \frac{1}{\mu}$

$\mu = 6$

$\mu = 7$

$\mu = 8$

$\boxed{\mu = 9}$ car/hour

$L_s = W_s \lambda_{effective}$

$L_s = 0.752$

x60

$\frac{1}{\mu}$	λ	μ	ρ	λ_{eff}	L_s	W_s (hr)	W_s (min)
10 min	4	6	0.667	3.807		0.37	
	4	9	0.44			.19	11
6 min		10				0.16	9.69

* The owner wishes to determine loosing

Key word

[illegible]

$\rightarrow \lambda_{lost} =$

reduction $= 40\%$

$(10 - 6 = 4)$

Example

$C=1$ $N=10$

A one-man barber shop has a total of 10 seats. Interarrival times are exponentially distributed, and an average of 20 prospective customers arrive each hour at the shop.

Those customers who find the shop full do not enter. The barber takes an average of 12 minutes to cut each customer's hair. Haircut times are exponentially distributed.

1. On the average, how many haircuts per hour will the barber complete? λ_{eff} → (يلي فعلاً دخلو المحل وراح ينعطى الهم ال service)
2. On the average, how much time will be spent in the shop by a customer who enters? W_s و منها بلاقي λ_{eff}

$\lambda = 20$ customer/hour

$\frac{1}{\mu} = 12$ mins $\longrightarrow \mu = \frac{1}{12} \times 60 \rightarrow \mu = 5$ customers/hour

$(M/M/1):(GD/N/\infty)$

$c=1$ $N=10$

$\rho = \frac{\lambda}{\mu} = \frac{20}{5} = 4$

1. On the average, how many haircuts per hour will the barber complete?

$\lambda_{eff} = \lambda - \lambda_{lost} = 20 - 15 = 5$ customer/hour

$\lambda_{lost} = \lambda P_N = 20 P_{10} = 20 (4)^{10} (7.15 \times 10^{-7}) \rightarrow \lambda_{lost} = 14.99 = 15 \frac{\text{customer}}{\text{hour}}$

$P_0 = \frac{1-4}{1-4^{11}} = 7.15 \times 10^{-7}$

2. On the average, how much time will be spent in the shop by a customer who enters?

$L_s = 9.66$ customer

$W_s = \frac{L_s}{\lambda_{eff}} = \frac{9.66}{5} = 1.932$ hour

12/11/2023

*The Multiple Server Model :-

Section : (18.6.3)

$$(M/M/C):(GD/\infty/\infty)$$

Markovian

Multi Server

ممكن الـ Que يتفجر
إذاً سوف احط شرط لمنع ذلك

$$\left(\frac{\rho}{c} < 1\right)$$

Multi Server
infinite Size
Poisson Queuing
Situation

$$P_n = \begin{cases} \frac{\rho^n}{n!} P_0 &, n < c \\ \frac{\rho^n}{c!\, c^{n-c}} P_0 &, n \geq c \end{cases}$$

$$\lambda_n = \{ \lambda \quad n \geq 0$$

$$\mu_n = \begin{cases} n\mu & n < c \\ c\mu & n \geq c \end{cases}$$

Service Rate per server

$$\rho = \frac{\lambda}{\mu}$$

$$P_0 = \frac{1}{\left[\sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c}{c!}\left(\frac{1}{1-\frac{\rho}{c}}\right)\right]}$$

$$L_q = \frac{\rho^{c+1}}{(c-1)!\,(c-\rho)^2} \cdot P_0$$

$$L_s = L_q + \rho$$

$$W_s = \frac{L_s}{\lambda} \quad \text{or} \rightarrow \left(W_s = W_q + \frac{1}{\mu}\right)$$

$$W_q = \frac{L_q}{\lambda}$$

قاعدي نفسها
λ effective

*Performance Measures :-

- L_s number of customers in the system (system Length)
- L_q number of customers in the Que (Que Length)
- W_s expected waiting time in the system
- W_q expected waiting time in the Que

*جاي زيه بالإمتحان

Example

- [Sports fans arrive at a ticket counter at the Amman City Arena by a Poisson process with rate 105 customers per hour. Customers are served by a single cashier. The service time has an exponential distribution with mean 30 seconds.]
- Now, Amman City Arena would like to evaluate the benefits of adding a second server.

$\lambda = 105$ customers per hour

$\mu = \frac{1}{30 \text{ second}} \times \frac{60 \text{ second}}{1 \text{ min}} \times \frac{60 \text{ min}}{1 \text{ hour}} = 120 \frac{\text{customer}}{\text{hour}}$

$\rho = \bar{c} = \frac{\lambda}{\mu} = 0.875$

$C = 2$ servers

For Multiple Servers

W_s for single server $= 4$ min

(c) servers ثم يتفرق إلى

single Que

$W_q = \frac{L_q}{\lambda}$

$L_q = \frac{\rho^{c+1}}{(c-1)!\,(c-\rho)^2} \cdot P_o$

$=$

$= 0.20696$ customer

$P_o = \frac{1}{\left[\sum_{n=0}^{c-1}\left(\frac{\rho^n}{n!}\right) + \frac{\rho^c}{c!}\left(\frac{1}{1-\frac{\rho}{c}}\right)\right]} = 0.391$

$L_s = L_q + \rho = 0.20696 + 0.875 = 1.08196$ customer

$W_q = \frac{L_q}{\lambda} = \frac{0.20696}{105} \times 60 = 0.11826$ min

للتحويل إلى min

$W_s = \frac{L_s}{\lambda} = \frac{1.08196}{105} \times 60 = 0.618$ min

للتحويل إلى min

بالامتحان حسب شو طالب السؤال

Example :

Example 18.6-5

2 companies

2 cabs 2 cabs

A community is served by two cab companies. Each company owns two cabs, and both share the market equally, with calls arriving at each company's dispatching office at the average rate of eight per hour. The average time per ride is 12 minutes. Calls arrive according to a Poisson distribution, and the ride time is exponential. The two companies have been bought by an investor and will be consolidated into a single dispatching office.

Analyze the new owner's proposal.

$\lambda = 8$ calls/hour

$\mu = \frac{1 \text{ calls}}{12 \text{ min}} \times \frac{60 \text{ min}}{\text{hour}} = 5$ calls/hour

One system , two servers

multiserver الثنين

بس عم يفكر يجمعهم ويسويهم اربعة

So for the first system is:-

$\boxed{(M/M/2):(GD/\infty/\infty)}$ c=2

$\lambda = 8$ calls/hour

$\frac{1}{\mu} = 12$ min $\longrightarrow \mu = 5$ Rides/hour

سوف يكون نفس الناتج لل (2 systems) لأنة تكسي واحد

$C = 2$

$\rho = \frac{\lambda}{\mu} = \frac{8}{5} = 1.6$

$P_0 = \frac{1}{\sum_{n=0}^{c-1}\left(\frac{\rho^n}{n!}\right) + \frac{\rho^c}{c!}\left(\frac{1}{1-\frac{\rho}{c}}\right)} = 0.1111$

$L_q = \frac{\rho^{c+1}}{(c-1)!\,(c-\rho)^2} \cdot P_0$

$= \frac{1.6^{2+1}}{(2-1)!\,(2-1.6)^2} \times 0.1111 = 2.84$

$L_s = L_q + \rho = 2.84 + 1.6 = 4.44$

$W_q = \frac{L_q}{\lambda} = \frac{2.84}{8} \times 60_{\text{min}} = 21.33$ min (للتحويل)

$W_s = \frac{L_s}{\lambda} = \frac{4.44}{8 \frac{\text{call}}{\text{hour}}} \times 60_{\text{min}} = 33.33$ min (للتحويل)

After the merge of systems:-

$\boxed{(M/M/4):(GD/\infty/\infty)}$ c=4

$\lambda = 16$ Calls/hour

$\frac{1}{\mu} = 12$ min $\longrightarrow \mu = 5$ Rides/hour → لأنة تكسي واحد ما بتغير

$c = 4$

$\rho = \frac{\lambda}{\mu} = 3.2$

$P_0 = \frac{1}{\sum_{n=0}^{3}\frac{3.2^n}{n!} + \frac{3.2^3}{3!}\left(\frac{1}{1-\frac{3.2}{3}}\right)} = 0.0273$

$L_q = 2.385$

$L_s = 5.586$

$W_q = 8.94$ min

$W_s = 20.94$ min

(استنتاجنا أن الحالة الثانية أفضل)

دائماً لما ندمج بكون احسن *

So second system is better since (W_s) is less

Tuesday
14/11/2023

$(M/M/c):(GD/N/\infty)$

Multiserver

Multi Server
finite Size
Poisson Queuing
Situation

$$\lambda_n = \begin{cases} \lambda & n = 0, 1, \cdots, N-1 \\ 0 & n \geq N \end{cases}$$

$$\mu = \begin{cases} n\mu & n = 0, 1, \cdots, c-1 \\ c\mu & c \leq n < N \end{cases}$$

زمان (μ): Service Rate per server

الان (μ): Server Rate for the whole system

* لما يكون ∞ ← $\rho < 1$

لما يكون c ← $\frac{\rho}{c} < 1$

لما يكون N و c ← حالتي أقل أو = $\frac{\rho}{c} \leq 1$

$$P_n = \begin{cases} \dfrac{\rho^n}{n!} \cdot P_0 & n = 0, 1, \cdots, c-1 \\[2ex] \dfrac{\rho^n}{c!\, c^{n-c}} \cdot P_0 & n = c, \cdots, N \end{cases}$$

$$P_0 = \begin{cases} \dfrac{1}{\left[\left(\sum_{n=0}^{c-1} \dfrac{\rho^n}{n!} \right) + \dfrac{\rho^c}{c!} \cdot \dfrac{1 - \left(\frac{\rho}{c}\right)^{N-c+1}}{1 - \frac{\rho}{c}} \right]} , & \frac{\rho}{c} \neq 1 \\[4ex] \dfrac{1}{\left[\sum_{n=0}^{c-1} \dfrac{\rho^n}{n!} + \dfrac{\rho^c}{c!} \cdot (N-c+1) \right]} , & \frac{\rho}{c} = 1 \end{cases}$$

$$L_q = \begin{cases} \dfrac{\rho^{c+1}}{(c-1)!\,(c-\rho)^2} \times \left[1 - \left(\frac{\rho}{c}\right)^{N-c+1} - (N-c+1)\left(1 - \frac{\rho}{c}\right)\left(\frac{\rho}{c}\right)^{N-c} \right] \times P_0 & \text{when } \frac{\rho}{c} \neq 1 \\[4ex] \dfrac{\rho^c (N-c)(N-c+1)}{2\, c!} \times P_0 & \text{when } \frac{\rho}{c} = 1 \end{cases}$$

$$L_s = L_q + \frac{\lambda_{eff}}{\mu}$$

$$W_s = \frac{L_s}{\lambda_{eff}}$$

$$W_q = \frac{L_q}{\lambda_{eff}}$$

*
$$\lambda_{eff} = \lambda - \lambda_{lost} = \lambda - \lambda P_N = \lambda(1 - P_N)$$

Example : Example 18.6-6

In the consolidated cab company problem of Example 18.6-5, suppose that new funds cannot be secured to purchase additional cabs. The owner was advised that one way to reduce the waiting time is for the dispatching office to inform new customers of potential excessive delay once the waiting list reaches six customers. The expectation is that these customers will seek service elsewhere, which in turn will reduce the average waiting time for those on the waiting list.

I care about (waiting time) for the customer and the Que
→ I care about all including (L_q) and (L_s)

Assess the situation.

$(M/M/4):(GD/10/\infty)$

$\lambda = 16$

(4+6) $N = 10$

$\lambda_n = 16$

$\mu_n = 5$

$c = 4$

$$\rho = \frac{\lambda_n}{\mu_n} = \frac{16}{5} = 3.2$$

$$\frac{\rho}{c} = \frac{3.2}{4} = 0.8$$

So $\left(\frac{\rho}{c} \neq 1\right)$

to Assess the Situation we want to check W_s, W_q, L_q, L_s

when $\frac{\rho}{c} \neq 1$

$$P_0 = \frac{1}{\sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c}{c!} \cdot \frac{1-\left(\frac{\rho}{c}\right)^{N-c+1}}{1-\frac{\rho}{c}}} = \frac{1}{\sum_{n=0}^{3} \frac{3.2^n}{n!} + \frac{3.2^4}{4!} \cdot \frac{1-\left(\frac{3.2}{4}\right)^{10-4+1}}{1-\frac{3.2}{4}}} = 0.03121$$

$$L_q = \frac{\rho^{c+1}}{(c-1)!\,(c-\rho)^2} \cdot \left[1-\left(\frac{\rho}{c}\right)^{N-c+1} - (N-c+1)\left(1-\frac{\rho}{c}\right)\left(\frac{\rho}{c}\right)^{N-c}\right] \cdot P_0$$

$$= \frac{3.2^{4+1}}{3!\,(4-3.2)^2}\left[1-\left(\frac{3.2}{4}\right)^{10-4+1} - (10-4+1)\left(1-\frac{3.2}{4}\right)\left(\frac{3.2}{4}\right)^{10-4}\right] \times 0.03121 = 1.154$$

$$L_s = L_q + \frac{\lambda_{eff}}{\mu} = 1.154 + \frac{15.428}{5} = 4.2398$$

$$\lambda_{eff} = \lambda(1-P_N) = \lambda(1-P_{10}) = 16(1-0.03574) \longrightarrow \lambda_{eff} = 15.428$$

$$P_{10} = \frac{\rho^n}{c!\,c^{n-c}} \cdot P_0 = \frac{3.2^{10}}{4!\;4^{10-4}} \times 0.03121 = 0.03574$$

when $(n=N)$

$$W_s = \frac{L_s}{\lambda_{eff}} = \frac{4.2398}{15.428} = 0.2748$$

$$W_q = \frac{L_q}{\lambda_{eff}} = \frac{1.154}{15.428} = 0.0747$$

$(M/M/\infty):(GD/\infty/\infty)$

infinity numbers of servers means there is no Que
(NO Que , No W_q)

Thursday
16/11/2023

Self service

Not like ATM or self check in because they are limited to number of servers

$\lambda_n = \lambda$, $n = 0, 1, \dots$
$\mu_n = n\mu$, $n = 0, 1, \dots$

سوف نستخدمهم بالاشتقاق و بنطلع منهم المعادلات النهائية

في المطار أو اشي self check in ما عم احكي عن هاي الحالات لأنه the numbers of servers is ∞

$$P_n = \frac{\rho^n \times e^{-\rho}}{n!} \quad , n = 0, 1, 2, \dots$$

this is a Poisson Distribution

$E(n) = \rho$ (expected number of customers in the system)

$L_s = \rho$

$$W_s = \frac{L_s}{\lambda} = \frac{\rho}{\lambda} = \frac{1}{\mu}$$

also

$$W_s = \underbrace{W_q}_{zero} + \frac{1}{\mu} = \frac{\rho}{\lambda} = \frac{1}{\mu}$$

$L_q = 0$
$W_q = 0$
because there is no Que

Problem 18-92.

New drivers are required to pass written tests before they are given road driving test. These tests are usually administered in the city hall. Records at the City of Springdale show that the average number of written tests is 100 per 8-hr day. The average time needed to complete the test is about 30 minutes. However, the actual arrival of test takers and the time each spends on the test are totally random.

$\lambda = \frac{100}{8} = 12.5 \ \frac{tests}{hour}$

Determine the following:

$\mu = \frac{1}{30 \min} \times \frac{60 \min}{1 \text{ hour}} \rightarrow \mu = 2 \ \frac{tests}{hour}$

a) The average number of seats the test hall should provide.

b) The probability that the number of test takers will exceed the average number of seats provided in the test hall.

c) The probability that no tests will be administered in any one day.

(a) $L_s = \rho = \frac{\lambda}{\mu} = \frac{12.5}{2} = 6.25$ seats

(b) $P_{n \geq 7} = 1 - P_{n<7} = 1 - [P_0 + P_1 + P_2 + P_3 + P_4 + P_5 + P_6] = 0.436$

عشان نضمن نكون فوق الـ 6.25
(it will exceed from 7 and above)

6.25 7
exceed

(c) $P_0 = \frac{\rho^n \times e^{-\rho}}{n!} = \frac{(6.25)^0 \times e^{-6.25}}{0!} = 0.00193$

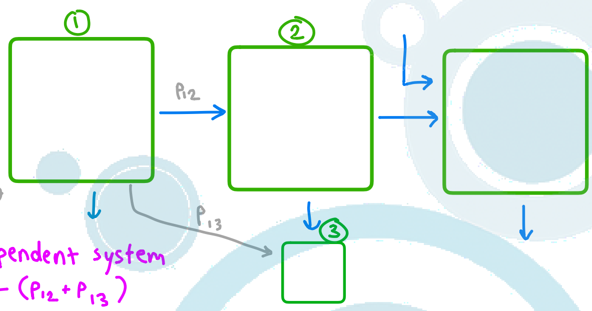
Now we finished the chapter

Sunday
19/11

Jackson Network 8-

(open Network)

Station = Node



بقدر اتعامل معهم مرات زي ال indivisuals models واحد بالـ الماضيين

independent system
$1-(P_{12}+P_{13})$

* Arrivals → Poisson
* servicetime → exponential → (M/M)

- All queues must have unlimited capacity
- can work on each station independently

$(M/M/\text{أي } c):(GD/\infty/\infty)$

SPT → Shortest Processing time but here we are random

EDD → Eearliest Due date

P_{12} بناءً على

Purely Random ← الانتقال من station 1 إلى station 2 → in Jackson

* closed Network → # of customers in the system is constant

(ممنوع حد يطلع أو يدخل)
(أو ممنوع حد يدخل إلا إذا حد طلع)

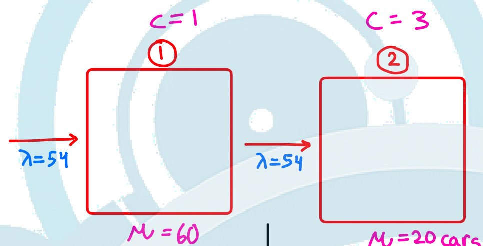
we care about min the objective function

Example

The last two things that are done to a car before its manufacture is complete are installing the engine and putting on the tires. An average of 54 cars per hour arrive requiring these two tasks. One worker is available to install the engine and can service an average of 60 cars per hour. After the engine is installed, the car goes to the tire station and waits for its tires to be attached. Three workers serve at the tire station. Each works on one car at a time and can put tires on a car in an average of 3 minutes. Both interarrival times and service times are exponential.

$\lambda = 54$ cars/hour (same in two stations)

1. Determine the mean queue length at each work station.
2. Determine the total expected time that a car spends waiting for service.



(*to be in Steady State)

1^{st} $\rho < 1$

2^{nd} $\frac{\rho}{c} < 1$

1st Station

$(M/M/1):(GD/\infty/\infty)$ $c=1$

$\lambda = 54$ cars/hour

$\mu = 60$ cars/hour

$\rho = \frac{\lambda}{\mu} = 0.9$

$L_s = \frac{\rho}{1-\rho} = \frac{0.9}{0.1} = 9$

$L_q = L_s - \rho = 9 - 0.9 = 8.1$

$W_q = \frac{L_q}{\lambda} = \frac{8.1}{54} = 0.15 \text{ hour} = 9 \text{ mins}$

2nd Station

$(M/M/3):(GD/\infty/\infty)$ $c=3$

$\lambda = 54$ cars/hour

$\mu = 20$ cars/hour

$\rho = \frac{\lambda}{\mu} = \frac{54}{20} = 2.7$

$$P_0 = \frac{1}{\left[\sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c}{c!}\left(\frac{1}{1-\frac{\rho}{c}}\right)\right]} = 0.02491$$

$$L_q = \frac{\rho^{c+1}}{(c-1)!\,(c-\rho)^2} \cdot P_0 = \frac{2.7^4}{2!\,(3-2.7)^2} \times 0.02491 = 7.35 \text{ cars}$$

$$W_q = \frac{L_q}{\lambda} = \frac{7.35}{54} \times 60 = 8.17 \text{ mins}$$

2. Determine the total expected time that a car spends waiting for service.

$W_q(\text{total}) = W_{q_1} + W_{q_2} = 9 + 8.17 = 17.17 \text{ min}$

Example

All batch jobs submitted to a computer center must first pass through an input processor before moving on to the central processor station where the bulk of the work is performed.

Because of errors, only 80% of the jobs go through the central processor; the remaining 20% are rejected. Of the jobs that pass through the central processor successfully, 40% are routed to a printer station where a hard copy is produced.

Jobs arrive randomly at the computer center at an average rate of 10 per minute. To handle the load, each station may have several processors operating in parallel. The times for the three steps have exponential distributions with means as follows: 10 seconds for an input processor, 3 seconds for a central processor, and 70 seconds for a printer. When all the processors at a station are in use, an arriving job must wait in a queue. All queues are assumed to have unlimited capacity.

- Our goal is to find the minimum number of processors of each type
- and compute the average time required for a job to pass through the system.
- Where is the bottleneck?

أبطأ واحد والذي يتحكم بالsystem التي أعلى W_s

L_q, L_s, W_q, W_s * بدي الطلع

$\lambda_1 = 10$ Job per min

إذا نحول كلشي لـ (min)

$\mu_1 = 10$ sec

$\lambda_2 = 8$

$\lambda_3 = 3.2$ (0.4×8)

Station 1: $c = 2$

$\mu = \frac{1}{10} \times 60 = 6$ customer/min

$\lambda = 600$

$\frac{\rho_1}{c} < 1$

$\rho = \frac{\lambda}{\mu} = 1.667$

$\rightarrow c > 1.667$

$\therefore (c = 2)$

* find c from the condition $\frac{\rho}{c} < 1$

(even if $\frac{\rho}{c}$ is integer, we take the next integer)

$$P_0 = \frac{1}{\left[\sum_{n=0}^{n=c-1} \frac{(1.66)^0}{0!} + \frac{1.66^1}{1!} + \frac{1.66^2}{2!}\left(\frac{1}{1-\frac{1.66}{2}}\right)\right]} = 0.0928$$

$L_q = \frac{(1.66)^3}{1!(2-1.66)^2} = 3.672$ (3.7878) ← اجوبة الدكتور

$L_s = 3.672 + 1.66 = 5.33$ (5.4545)

$W_s = \frac{5.33}{10} = 0.5333$ (0.5454)

$W_q =$ (0.37878)

Station 2: $c =$

$\mu = \frac{1}{3} \times 60 = 20$ customer/min

$\lambda = 480$

$\frac{\rho_2}{c} < 1$

$\rho = \frac{\lambda}{\mu} = 0.4$

$\rightarrow c > 0.4$

$\therefore (c = 1)$

$c = 1$

$L_s = 0.667$

$W_s = 0.0833$

$L_q = 0.2667$

$W_q = 0.0333$

Station 3: $c =$

$\mu = \frac{1}{70} \times 60 = 0.857$ customer/min

$\lambda = 192$

$\frac{\rho_3}{c} < 1$

$\rho = \frac{\lambda}{\mu} = 3.733$

$\rightarrow c > 3.733$

$\therefore (c = 4)$

$c = 4$

$L_s = 15.7565$

$W_s = 4.9239$

$L_q = 12.02256$

$W_q = 3.75706$

(2) Sum $W_s = W_{s_1} + W_{s_2} + W_{s_3} = 0.5454 + 0.0833 + 4.9239 = 5.5526$ min

(3) Station (3) is the bottle neck

سوف نقترح انه

→ it is Recomended to add a printer to have 5 printers instead of 4

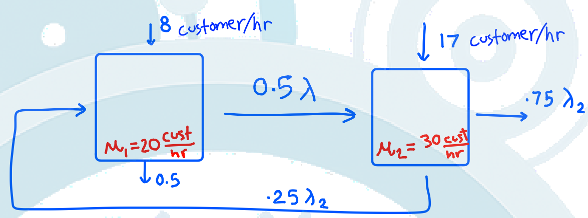
Example

Consider two servers. An average of 8 customers per hour arrive from outside at server 1, and an average of 17 customers per hour arrive from outside at server 2. Interarrival times are exponential.
Server 1 can serve at an exponential rate of 20 customers per hour, and server 2 can serve at an exponential rate of 30 customers per hour. After completing service at server 1, half of the customers leave the system, and half go to server 2.
After completing service at server 2, ¾ of the customers complete service, and ¼ return to server 1.

1. What fraction of the time is server 1 idle?
2. Find the expected number of customers at each server.
3. Find the average time a customer spends in the system.
4. How would the answers to parts (1)-(3) change if server 2 could serve

$(M/M/2):(GD/\infty/\infty)$

$c = 2$
$\mu_1 = 20$
$\mu_2 = 30$



$\frac{3}{4}$ طلعوا
$\frac{1}{4}$ دخلو

*Reminder: we are dealing with Steady State, so we are independent on time, and we don't care about the very first/last moments we care about average when the system has stabilized.

$\lambda_1 = 8 + 0.25\lambda_2$
$\lambda_2 = 17 + 0.5\lambda_1$

by solving for equation 1 and 2 we get
$\lambda_1 = 14$
$\lambda_2 = 24$

Note:-
لو كان (3 servers)
بكونو ثلاث معادلات
بثلاث متغيرات

نحن نحل under Steady State
→ that means (λ) is not a function of time

① Server 1 is idle $(M/M/1):(GD/\infty/\infty)$
$P_0 = 1 - \rho = 1 - 0.7 = 0.3 \longrightarrow$ so 30% of the time

② Ls for each server $(M/M/1):(GD/\infty/\infty)$
$E(n) = L_s = \frac{\rho}{1-\rho} = \frac{0.7}{1-0.7} = \frac{7}{3}$ for the 1st server
$L_s = 4$ for the 2nd server

③ W_s for the system

④ the system will not be able to stabilize since $\mu < \lambda$ and it is not limited so the que will continue increasing (exploding)
$\frac{\lambda}{\mu} > 1$