Weekly Notebook

Linear

First Semester 2024

KAIZEN TEAM

Chapter 1: System of linear equations & matrices

Linear equation:

An equation is called linear if the powers of all variables in the equation are 1, and it does not involve any product of variables

Examples-

- $\sqrt{2}x + 3y - 2z = 3$
- $x_1 - 2x_2 + 3x_3 = 7$

General form for linear equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

Special part:

$3x + 3y = 0$ is called homogeneous linear equation

Linear system:

A finite set of linear equations

$$\left.\begin{aligned} 2x_1 + 3x_2 - x_3 &= 1 \\ x_1 + 2x_2 + 3x_3 &= 2 \\ x_1 + 2x_3 &= 5 \end{aligned}\right\}$$ system of Linear equations or Linear system

System of linear equation

- One solution
- No solution
- infinitly many solutions

* if there is at least one solution of the system it's called consistent
* if the system has no solution it's called inconsistent

$$\left.\begin{aligned} x - 2y &= 0 \\ 2x + y &= 0 \\ x + y &= 0 \end{aligned}\right\}$$ homogenous linear system (always consistent)

Exersizes -

1- find the solution for the system

$x + 2y = 4$ multiplying it by -2 $\qquad -2x - 4y = -8$

$2x + 4y = 9$ $\qquad + \; 2x + 4y = 9$

$0 = 1$??

So there is no solution

2. find the solution for the system

$3x - y = 3$ $\qquad 3x - \cancel{y} = 3$

$x + y = 5$ $\qquad + \; x + \cancel{y} = 5$

$4x = 8$ $\qquad x = 2 \,,\; y = 3$

$(2,3)$ is a solution for the system

intersection point (one solution)

3- find the solution for the system

$3x - 2y = 9$ multiplying it by -2 $\qquad -6x + 4y = -18$

$6x - 4y = 18$ $\qquad + \; 6x - 4y = 18$

$0 = 0$

So there is infinitely many solution

we can assume : $x = t$

$3t - 2y = 9$

$3t = 9 + 2y$

$\frac{2y}{2} = \frac{3t - 9}{2}$, at $t = 0$ $\quad (0, \frac{-9}{2})$

Basics of Matrices :-

In general we will write matrices as $A = (a_{ij})_{n\times m}$

$n\times m$: size of the matrix

n: num of rows

m: num of columns

a_{ij}: the entery in the i^{th} row and j^{th} column

* $n=1$, A is called a row matrix (row vector)
* $m=1$, A is called a column matrix (column vector)

* Diagonal $a_{nn} = a_{11}, a_{22}, a_{33}$

example: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

* Identity matrix : the diagonal should be 1, & all of the inputs are zero

example: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

* The transport of A (A^T) : we switch the rows & columns

example: $\begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 3 & 4 \end{bmatrix}$

* Matrix multiplication

If $A: a_{ij}(n\times r)$, $B = b_{ij}(r\times m)$ then $A.B = C_{ij}(n\times m)$ rows of the first one & columns of the 2nd one.

Example:-

$A: \begin{bmatrix} 1 & 2 \\ 10 & 1 \end{bmatrix}$ $B: \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$C: \begin{bmatrix} 2+2 \\ 20+1 \end{bmatrix}_{2\times 1} = \begin{bmatrix} 4 \\ 21 \end{bmatrix}_{2\times 1}$

# of columns of the first Matrix should be equal to # of rows of 2nd.

$A_{4\times 5} \times B_{5\times 5} = AB_{4\times 5}$

* Inverse of Matrix

let $A \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Example $A:\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ find A^{-1}

$$\frac{1}{(1\times4)-(2\times3)} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

Notes:-

if the $\det(A) \neq 0$ then A has an inverse

if the $\det(A) = 0$ the A has no inverse

if $A \cdot A^{-1} = I_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

how we use the matrices to solve linear systems.

- Elementary Row Operations (e.r.o)

1- multiply a row by non zero constant

2- Interchange two rows

3- Add a multiple of one equation to another

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$a_{31}x_3 + a_{32}x_2 + \cdots + a_{3n}x_n = b_3$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{bmatrix}$$

Augmented Matrix

we can solve linear system by using matrix if we can convert the Augmented Matrix to Trianglar form by using (e.r.o)

note that all a & b are real constants

$$\begin{bmatrix} 1 & a_{12} & a_{13} & b_1 \\ 0 & 1 & a_{23} & b_2 \\ 0 & 0 & 1 & b_3 \end{bmatrix}$$

Solve the following system of Linear equation

$$2x - y + 3z = 13$$
$$x + 2y = -3$$
$$3x + y - z = -2$$

step 1:
$$\left(\begin{array}{ccc|c} 2 & -1 & 3 & 13 \\ 1 & 2 & 0 & -3 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

step 2: should be zero

$R_2 \rightarrow -2R_2 + R_1$

$R_3 \rightarrow -3R_2 + R_3$

$$\left(\begin{array}{ccc|c} 2 & -1 & 3 & 13 \\ 0 & -5 & 3 & 19 \\ 0 & -5 & -1 & 7 \end{array}\right)$$

step 3: should be zero

$R_3 \rightarrow -R_2 + R_3$

$$\left(\begin{array}{ccc|c} 2 & -1 & 3 & 13 \\ 0 & -5 & 3 & 19 \\ 0 & 0 & -4 & -12 \end{array}\right)$$

step 4: Back sub.

$$2x - y + 3z = 13$$
$$-5 + 3z = 19$$
$$-4z = -12$$

from step 4 I can find $z = 3$ & then going back to the above equations & find y, x.

IMP

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 4 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

If we face this form during → triangular form (Linear system has infinitely many solution) because we have the no. of equations less than the no. of unknowns.

$x + 3y + 2z = 4$

$y - 3z = 2$

letting $z = t \Rightarrow y = 2 + 3t \Rightarrow x = -2 - 11t$

$(-2 - 11t, \; 2 + 3t, \; t)$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 4 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

this linear system has no solution

$0z = 1$???

X wrong

Q (IMP) for which value of "a" will the following system has: no solution? exactly one solution? infinity many solution?

$x + 2y - 3z = 4$

$3x - y + 5z = 2$

$4x + y + (a^2 - 14)z = a + 2$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{array}\right]$$

$R_2 \Rightarrow -3R_1 + R_2$

$R_3 \Rightarrow -4R_1 + R_3$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -8 & a^2+2 & a-14 \end{array}\right]$$

$-R_2 + R_3$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & a^2-16 & a-4 \end{array}\right]$$

test $a = 4$ $\quad 0, 0, 0, 0 \Rightarrow$ infinitly many solution

test $a = -4$ $\quad 0, 0, 0, -8 \Rightarrow$ no solution

test $a \neq \pm 4$ $\quad$ one solution

Gaussian Elimination :

⇒ Reduced row echlon form (r.r.e.f)

we say the matrix is in reduced row echelon form (r.r.ef) if the following conditions holds:

1-

1- The first nonzero number in the row is a 1, & we can call it Leader 1

Example: $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

2- Any Rows with All zero are placed at the bottom of the matrix

Example: $\begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

3- Every Leading 1 is to the right of the one above it

Example: $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

4- Each column that contains a leader 1, has zero everywhere else.

Example: $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

⇒ A matrix that has the first three properties is called row-echelon form

Examples:-

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$
R.R.E.F

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$
R.E.F

$$\begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
R.R.E.F

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
R.E.F

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
R.R.E.F

Gaussian Elimination $\Rightarrow$ used to convert Matrix to ROW ECHELON FORM

Gaussian Jordan Elimination $\Rightarrow$ used to convert Matrix to R.R.E.F.

Q. Suppose that the augmented matrix for a linear system in the unknows x, y, z has been reduced by elementary row operations

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 7 \end{bmatrix} \qquad \begin{aligned} x &= 5 \\ y &= 3 \\ z &= 7 \end{aligned}$$

Free variables & Leading variables:-

$\Rightarrow$ Leading variables: The variables whose columns in the R.R.E.F contain leading 1

$\Rightarrow$ Free variables: The variables whose columns in the R.R.E.F does not contain a leading 1

Q Solve the linear system by Gauss Jordan elimination

$$\begin{aligned} 3x - y + z + 7w &= 13 \\ -2x + y - z - 3w &= -9 \\ -2x + y - 7w &= -8 \end{aligned} \Rightarrow \left[\begin{array}{cccc|c} 3 & -1 & 1 & 7 & 13 \\ -2 & 1 & -1 & -3 & -9 \\ -2 & 1 & 0 & -7 & -8 \end{array}\right] \text{multiply by } \tfrac{1}{3}$$

Convert to zero

$$\left[\begin{array}{cccc|c} 1 & -1/3 & 1/3 & 7/3 & 13/3 \\ -2 & 1 & -1 & -3 & -9 \\ -2 & 1 & 0 & -7 & -8 \end{array}\right] \begin{matrix} 2R_1+R_2 \\ 2R_1+R_3 \end{matrix} \Rightarrow \left[\begin{array}{cccc|c} 1 & -1/3 & 1/3 & 7/3 & 13/3 \\ 0 & 1/3 & -1/3 & 5/3 & -1/3 \\ 0 & 1/3 & 2/3 & -7/3 & 2/3 \end{array}\right] \rightarrow 3R_2$$

Convert to zero

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 4 \\ 0 & 1 & -1 & 5 & -1 \\ 0 & 0 & 1 & -4 & 1 \end{array}\right] \begin{matrix} 1/3R_2+R_1 \\ \\ -1/3R_2+R_3 \end{matrix} \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 4 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -4 & 1 \end{array}\right]$$

Convert to zero by R_2+R_3

free var. = w , so w = t

we want tofind A^{-1} ? (using ROW OPERATION)

$$A = \begin{bmatrix} 2 & 4 & 7 \\ 1 & 5 & 0 \\ 4 & 9 & 5 \end{bmatrix}$$

$$E_1(\tfrac{1}{2}) \Big/ \frac{R_1}{2} \quad \begin{bmatrix} 1 & 2 & 3.5 \\ 1 & 5 & 0 \\ 4 & 9 & 5 \end{bmatrix}$$

$$E_{21}(-1) \leq -R_1 + R_2 \quad \begin{bmatrix} 1 & 2 & 3.5 \\ 0 & 3 & -3.5 \\ 4 & 9 & 5 \end{bmatrix} , \quad E_{31}(-4) \begin{bmatrix} 1 & 2 & 3.5 \\ 0 & 3 & -3.5 \\ 0 & 1 & -9 \end{bmatrix}$$

instead of this long method I canuse

$$\left[\begin{array}{ccc|ccc} 2 & 4 & 7 & 1 & 0 & 0 \\ 1 & 5 & 0 & 0 & 1 & 0 \\ 4 & 9 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\frac{R_1}{2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3.5 & 1/2 & 0 & 0 \\ 1 & 5 & 0 & 0 & 1 & 0 \\ 4 & 9 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} \\ -R_1+R_2 \\ -4R_1+R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 3.5 & 1/2 & 0 & 0 \\ 0 & 3 & -3.5 & -1/2 & 1 & 0 \\ 0 & 1 & -9 & -2 & 0 & 1 \end{array}\right]$$

$$\frac{R_2}{3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3.5 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 7/6 & -1/6 & 1/3 & 0 \\ 0 & 1 & -9 & -2 & 0 & 1 \end{array}\right]$$

complete →

$-2R_2+R_1$
$-R_2+R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 5.83 & 0.83 & -0.67 & 0 \\ 0 & 1 & -1.167 & -0.167 & 0.33 & 0 \\ 0 & 0 & -7.83 & -1.83 & -0.33 & 1 \end{array}\right]$$

$\frac{R_3}{-7.83}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 5.83 & 0.83 & -0.67 & 0 \\ 0 & 1 & -1.167 & -1.167 & 0.33 & 0 \\ 0 & 0 & 1 & 0.23 & 0.04 & -0.127 \end{array}\right]$$

$1.167 R_3 + R_2$
$-5.83R_3 + R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.5 & -0.9 & 0.75 \\ 0 & 1 & 0 & 0.1 & 0.38 & -0.15 \\ 0 & 0 & 1 & 0.23 & 0.04 & -0.12 \end{array}\right]$$

R.R.E.F $\qquad A^{-1}$

Theorem 1.6.2

$Ax = b \longrightarrow x = A^{-1}b$

It will give
exactly one sol.
because A is invertible

suppose that we have 3 systems of L.E & they all have the same coef. matrix (A) but, they have different R.H.S (A should be invertible)

Ex8

$x_1 + 2x_2 + 3x_3 = 4$
$2x_1 + 5x_2 + 3x_3 = 5$
$x_1 + \quad + 8x_3 = 9$

$x_1 + 2x_2 + 3x_3 = 1$
$2x_1 + 5x_2 + 3x_3 = 6$
$x_1 + \quad + 8x_3 = -6$

$$\begin{array}{ccc|c|c} & A & & b_1 & b_2 \end{array}$$
$$\left[\begin{array}{ccc|c|c} 1 & 2 & 3 & 4 & 1 \\ 2 & 5 & 3 & 5 & 6 \\ 1 & 0 & 8 & 9 & -6 \end{array}\right]$$

$-2R_1+R_2$, $-R_1+R_3$
$$\left[\begin{array}{ccc|c|c} 1 & 2 & 3 & 4 & 1 \\ 0 & 1 & -3 & -3 & 4 \\ 0 & 2 & 5 & 5 & -7 \end{array}\right]$$

$-2R_2+R_1$, $-2R_2+R_3$
$$\left[\begin{array}{ccc|c|c} 1 & 0 & 9 & 10 & -7 \\ 0 & 1 & -3 & -3 & 4 \\ 0 & 0 & -1 & -1 & 1 \end{array}\right]$$

$-R_3$
$$\left[\begin{array}{ccc|c|c} 1 & 0 & 9 & 10 & -7 \\ 0 & 1 & -3 & -3 & 4 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

$3R_3+R_2$, $-9R_3+R_1$
$$\left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

Theorem 1.6.3

Let $A \to$ square matrix

↳ If B is square matrix satisfying $BA = I$, then $B = A^{-1}$

↳ If B is square matrix satisfying $AB = I$, then $B = A^{-1}$

Q find A^{-1} for $A^2 - 2A + I = 0$

$I = -A^2 + 2A$

$I = A(-A + 2I)$

$\times A^{-1} \quad \times A^{-1}$

$A^{-1} = -A + 2I$

Theorem 1.6.5

If A & B is square matrices of the same size

If AB is invertible then A & B must also be invertible

↳ $AB = B^{-1}A^{-1}$ (invertible)

$A \to$ square matrix $\quad B \to$ square matrix

$AB \to$ invertible (given)

$\det(AB) \neq$ zero

$\det(AB) = \det(A) \cdot \det(B)$

If $\det(AB) \neq 0 \longrightarrow \det(A) \neq 0,\ \det(B) \neq 0$ (otherwise, $\det(AB) = 0$ so the matrix is ininvertible)

So here we will have

$A \to$ square matrix
↳ $\det(A) \neq 0$
So this is invertibe

$B \to$ square matrix
↳ $\det(B) \neq 0$
& this is invertible

- exam Q:

Let A & B be matrices. If ~~A~~ AB is invertible, BA is also invertible ??
Check if this sentence is always correct?

NO, because I can assume $A_{3\times2}$, $B_{2\times3}$
So in such a case A is not invertible as like B (not invertible)
they are not square matrices
but AB $\rightarrow$ 3x3 (square matrix)

invertible / NOT invertible

$$x_1 + x_2 + 2x_3 = b_1$$
$$x_1 + \quad + x_3 = b_2$$
$$2x_1 + x_2 + 3x_3 = b_3$$

Q: find b_1, b_2, & b_3 such that the system of equations is consistent

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

If I looked here, If i sum $R_1 + R_2$
It will gives me R_3
so R_3 is dependent $\longrightarrow$ matrix A is not invertible
means there is no way this system gives me a single sol. (trivial solution)

assuming that b_1, b_2 & b_3 = zero's
(homogenous system is always consistent)

$$x_1 + 2x_2 + 3x_3 = b_1$$
$$2x_1 + 5x_2 + 3x_3 = b_2$$
$$x_1 + \quad + 8x_3 = b_3$$

A is invertible, Q: find b_1, b_2 & b_3 such the system of equations is consistent

A $\longrightarrow$ invertible $\longrightarrow$ the system will be consistent with a unique solution
so b_1, b_2, & b_3 can take any values

Special cases of Matrices

- Diagonal
- triangular
- Symmetric

Diagonal Matrix

- main diagonal elements $\longrightarrow$ values
- off diagonal elements $\longrightarrow$ 0

(It can get zeros but the important thing that, off diagonal element's don't get any value)

Zero matrix, Identity matrix $\longrightarrow$ Diagonal Matrices

in diagonal Matrix usually these two conditions appears $\longrightarrow$ (the main diagonal element $\neq 0$)

- square
- $\det(\text{Matrix}) \neq 0$

why?!

Simply because $\det(\text{matrix}) =$ حاصل ضرب main diagonals elements

So If one of the main diagonal element is zero

So $\det(\text{matrix}) = \text{zero}$

So the diagonal matrix is not invertible

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

Q: find the inverse?

$$\left[\begin{array}{ccc|ccc} d_1 & 0 & 0 & 1 & 0 & 0 \\ 0 & d_2 & 0 & 0 & 1 & 0 \\ 0 & 0 & d_3 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/d_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/d_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/d_3 \end{array}\right]$$

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

Q: find the powers of diagonal matrix

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \times \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} = \begin{bmatrix} d_1^2 & 0 & 0 \\ 0 & d_2^2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

Triangular Matrices

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} \square & \circ & \circ & \circ \\ \triangle & \square & \circ & \circ \\ \triangle & \triangle & \square & \circ \\ \triangle & \triangle & \triangle & \square \end{bmatrix} \end{array}$$

$\square$ element's have equal rows & columns

If $i<j$ $\circ$, $a_{ij}=0 \rightarrow$ L.T.M

If $j<i$ $\triangle$, $a_{ij}=0 \rightarrow$ U.T.M

Symmetric Matrix

A square matrix in which $A^T = A$

$$\begin{bmatrix} 7 & -3 \\ -3 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 7 & -3 \\ -3 & 7 \end{bmatrix}$$

- zero matrix $\rightarrow$ square $\rightarrow$ symmetrix

CHAPTER 2: ELEMENTARY LINEAR ALGEBRA

- WHAT IS THE MINOR OF MATRIX?!

The minor denoted by M_{ij} is the determinant of the $(n-1)\times(n-1)$ formed by deleting the ith row & the jth column from A.

Example:-

$$A=\begin{bmatrix} 3 & 6 & 2 \\ 2 & 0 & 9 \\ 2 & 7 & 1 \end{bmatrix} \qquad M_{23}=\begin{vmatrix} 3 & 6 \\ 2 & 7 \end{vmatrix}$$

$3\times7 - 2\times6$

$M_{23} = 9$

- What IS THE COFACTOR?

C_{ij} is $(-1)^{i+j} M_{ij}$

For the previous example

$C_{ij} = (-1)^{2+3}(9) = -9$

$$\begin{bmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \end{bmatrix}$$

CHECKERBOARD

Find the determinant for

$$A=\begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

we want to chose the column that has the most 0's. OR ROW

$$-0 + 1\begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix} - 0 + 0$$

$$1\left(-2\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}\right) = -6$$

IF A is $n\times n$ triangular matrix (upper triangular, lower trangular, diagonal) the the $\det(A)$ is the product of the entries on the main diagonal.

find the determinant :-

$$\begin{bmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta-\cos\theta & \sin\theta+\cos\theta & 1 \end{bmatrix} \begin{matrix} + \\ - \\ + \end{matrix}$$

$$1 \begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix} = 1(\sin^2\theta + \cos^2\theta) = 1$$

Show that the Matrices

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, B = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$$

commute if & only if $\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix} = 0$

$$AB = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} ad & ae+bf \\ 0 & cf \end{bmatrix}$$

$$BA = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} da & db+ec \\ 0 & fc \end{bmatrix}$$

For $AB = BA$, we need the corresponding entries to be equal.

$ad = da$

$ae + bf = db + ec \longrightarrow ae + db = ec - bf$

$cf = fc$

This can expressed as a Matrix

$$\begin{bmatrix} b & a-c \\ e & d-f \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

TO CLARIFY IT

$$X = \begin{bmatrix} a \\ b \end{bmatrix}, Y = \begin{bmatrix} c \\ d \end{bmatrix}, Z \begin{bmatrix} e \\ f \end{bmatrix} \rightarrow Y^T(X-Z) = 0$$

$$X - Z = \begin{bmatrix} a-e \\ b-f \end{bmatrix}, Y^T = \begin{bmatrix} c & d \end{bmatrix} \rightarrow \begin{bmatrix} c & d \end{bmatrix}\begin{bmatrix} a-e \\ b-f \end{bmatrix} = c(a-e) + d(b-f)$$

prove that $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are colinear points if & only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

DETERMINANT ZERO

Assuming the points are colinear, This is means that the slope between Any two points are the same.

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2}$$

$$\frac{y_j - y_i}{x_j - x_i} = \frac{\det\begin{bmatrix} x_i & y_i \\ x_j & y_j \end{bmatrix}}{\det\begin{bmatrix} 1 & x_i \\ 1 & y_i \end{bmatrix}}$$

$$\det\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \cdot \det\begin{bmatrix} 1 & x_2 \\ 1 & x_3 \end{bmatrix} = \det\begin{bmatrix} x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} \cdot \det\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}$$

$$\det\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \cdot \det\begin{bmatrix} 1 & x_2 \\ 1 & x_3 \end{bmatrix} - \det\begin{bmatrix} x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} \cdot \det\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} = 0$$

THIS IMPLIES THAT THE DET OF THE GIVEN MATRIX IS ZERO

* Back ward Dir.

If $\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0$

this means that $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are colinear

What is the adjoint Matrix?

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

COFACTOR MATRIX

THE TRANSPOSE OF THIS MATRIX IS CALLED $\rightarrow$ the adjoint of A (adj(A))

Let's say that the cofactor Matrix is $\begin{bmatrix} 63 & 16 & 14 \\ 1 & 1 & -9 \\ 54 & -25 & -12 \end{bmatrix}$

$$C^T = \begin{bmatrix} -63 & 1 & 54 \\ 16 & 1 & -25 \\ 14 & -9 & -12 \end{bmatrix} = adj(A)$$

TO FIND $A^{-1} = \frac{1}{\det(A)} \cdot adj(A) = \frac{1}{-79} \begin{bmatrix} -63 & 1 & 54 \\ 16 & 1 & -25 \\ 14 & -9 & -12 \end{bmatrix}$

THEOREM 2.2.2

- Let A be a square Matrix, Then $\det(A) = \det(A^T)$

consider A is a square Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

If we want to find it's determinant using the cofactor expansion along the first row:

$\det(A) = a_{11} \cdot \text{Cofactor}(A_{11}) + a_{12}\,\text{Cofactor}(A_{12}) + \cdots + a_{1n}\,\text{Cofactor}(A_{1n})$

where $A_{ij} \rightarrow$ sub Matrix obtained by removing the ith row & jth column of A

consider $A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$

$\det(A^T) = a_{11}\,\text{cofactor}(A^T_{11}) + a_{21}\,\text{Cofactor}(A^T_{21}) + a_{n1}\,\text{Cofactor}(A^T_{n1})$

A_{ij}^T is the same as A_{ji} because transposing swaps rows & columns. Therefore the cofactor expansion of A^T along the first column is equivalent to the cofactor expansion of A along the first row. $\det(A) = \det(A^T)$

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\det(B) = k\det(A)$$

$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{12} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\det(B) = -\det(A)$$

$$\begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\det(B) = \det(A)$$

THEOREM 2.2.4

let E be an $n \times n$ matrix (Elementary)

a) If E results from multiplying a row of I_n by a non zero number k $\det(E) = k$

b) If E results from interchanging two rows of I_n, the $\det(E) = -1$

c) If E results from adding a multiple of one row of I_n to another, $\det(E) = 1$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \qquad \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} \qquad \begin{vmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$\det = 3$ $\qquad$ $\det(E) = -1$ $\qquad$ $\det = 1$

Compute the determinant

$$A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & 7 & 0 & 6 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -5 \end{bmatrix}$$

adding -3 times the first column then added to the forth column

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 7 & 0 & 0 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -26 \end{bmatrix}$$

This is lower triangular Matrix

$\det = 1 \times 7 \times 3 \times -26 = -546$

Evalute det(A) where

$$A = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}$$

I want to simplify it to determine the det(A) easily

$-3R_2 + R_1 \longrightarrow [\,0 \quad -1 \quad 1 \quad 3\,]$

$-2R_2 + R_3 \longrightarrow [\,0 \quad 0 \quad 3 \quad 3\,]$

$-3R_2 + R_4 \longrightarrow [\,0 \quad 1 \quad 8 \quad 0\,]$

$$\begin{bmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{bmatrix}$$

↑

$$-1 \begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix} = -1\Big(-1\big(3\times 0 - 38\big) - 1\big(-3\big) + 3\big(3\big)\Big)$$

$$= -36$$

use row reduction to show that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$

$R_2 = R_2 - aR_1$

$R_3 = R_3 - a^2R_1 \quad \rightarrow A' = \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & (c-a)(c-b) \end{vmatrix}$

$R_3 = R_3 - bR_2$

since A' is upper triangular Matrix so $\det = 1 \times (b-a)(c-a)(c-b)$

Find $\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g+3a & h+3b & i+3c \end{vmatrix}$, given that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$

- Multiply the ~~first~~ 2nd row of $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ by 2 $= [\,2d \quad 2e \quad 2f\,]$

- Add 3 times the first row of ↑ to the third $= [\,g+3a \quad h+3b \quad i+3c\,]$

now given that the determinant of $A = -6$

then $\det(B) = 2\det(A) = 2 \times -6 = -12$

Find $\begin{vmatrix} a+d & b+e & c+f \\ -d & -e & -f \\ g & h & i \end{vmatrix}$ $\quad \det(B) = (a+d)\,\text{Cofactor}(B_{11}) + (b+e)\,\text{Cofactor}(B_{12}) + (c+f)\,\text{Cofactor}(B_{13})$

$\det(B) = -\det(A) = 6$

CRAMER'S RULE

I CAN USE IT WHEN, NO. OF INDEBENDENT EQ = NO. OF UNKNOWNS

$$A = \begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 0 \\ 0 & 2 & 4 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

NOTE That :

I CAN find the unique Solution

BY EITHER $X = A^{-1} b$

OR $x_1, x_2, x_3 \leftarrow \frac{\det(A_i)}{\det(A)}$

1- First of All I have to Find the $\det(A)$ Using the cofactor expansion method

$$3(8) - 4(4) + 5(2) = 18$$

2- Find A_1

$$A_1 = \begin{bmatrix} 3 & 4 & 5 \\ 2 & 2 & 0 \\ 1 & 2 & 4 \end{bmatrix}$$

$\det(A_1) = 3(8) - 4(8) + 5(4-2) = 2$, $x_1 = \frac{\det(A_1)}{\det(A)} = \frac{2}{18}$

3- Find A_2

$$A_2 = \begin{bmatrix} 3 & 3 & 5 \\ 1 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix}$$

$\det(A_2) = 3(8) - 3(4) + 5(1) = 17$, $x_2 = \frac{\det(A_2)}{\det(A)} = \frac{17}{18}$

4- Find A_3

$$A_3 = \begin{bmatrix} 3 & 4 & 3 \\ 1 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

$\det(A_3) = 3(2-4) - 1(4-6) = -4$, $x_3 = \frac{\det(A_3)}{\det(A)} = \frac{-4}{18}$

Q: for a homogenous sys. of 3 linear eq. & 3 unknowns, which of the following statement is ALWAYS CORRECT?

1- The system is inconsistent

2- The system has a unique solution → we can't tell

3- The system has an infinite number of equations

4- The system can be solved using Cramer's Rule → we can't tell

5- None of the following presented choises ✓

IF we changed the Q → (3 independent L.EQ & 3 unknowns) here i can choose 2 & 4 are Both correct.

Let A be 3x3 matrix. Show if the following expression is ALWAYS CORRECT

$$\det(AA^T) \geq 0$$

$\det(AB) = \det(A) \cdot \det(B)$ → A & B are Both square

$\det(AA^T) = \det(A) \cdot \det(A^T)$

$\det(A) = \det(A^T)$ → Theorem 1.4.10

$\det(AA^T) = (\det(A))^2$

$(\text{any value})^2 \geq 0$ will always be greater than or equal to zero

+ve -ve

$$\det(AA^TA) \geq 0$$

$\det(AA^TA) = \det(AA^T) \cdot \det(A)$

$\det(AA^T) = (\det(A))^2$

$\det(AA^TA) = (\det(A))^3$ It can have a positive or negative value according to the determinant of A.

Chapter 3: Euclidean vectors

HOW TO FIND ANY COMP. OF A VECTOR

terminal point − initial point

DEFINITION 1: If n is positive integer, then an ordered n-turple is a sequence of n real numbers $(v_1, v_2, v_3, \ldots v_n)$
The set of all ordered n-tuples is called n-space & is denoted R^n.

What we mean by (two vectors are defined in the same space)

If we have $v = (v_1, v_2, v_3 \ldots)$ and $w = (w_1, w_2, w_3 \ldots)$
If $v_1 = w_1$, $v_2 = w_2$ and so on. (I mean that they have the same entries)
so they're equal to each other & have the same size.

NOTE

vectors are added (or subtracted) by adding (or subtracting) their corresponding components, & vectors is multiplied by scalar by multipling each component by that ↑.

LINEAR COMBINATION

If I can express the vectors $v_1, v_2, v_3 \ldots$ in this form

$$w = k_1 v_1 + k_2 v_2 + \cdots + k_n v_n$$

where $k_1, k_2, \ldots k_n$ these are scalars (coefficients of the L.C)
There are two cases of Q's

Case 1
(Given k, find w)

case 2
(Given w, find k)

Q1

$V_1 = (3,5,0)$ $K_1 = 2$

$V_2 = (0,2,4)$ $K_2 = 3$

$V_3 = (1,2,3)$ $K_3 = 1$

$W = 2(3,5,0) + 3(0,2,4) + 1(1,2,3)$

Q2

$W = (2, 4.5, 8)$, can this be expressed as a L.C of V_1, V_2, V_3

$(2, 4.5, 8) = K_1(3,5,9) + K_2(0,2,4) + K_3(1,2,3)$

$$\left.\begin{aligned} 2 &= 3K_1 + 0K_2 + 1K_3 \\ 4.5 &= 5K_1 + 2K_2 + 2K_3 \\ 8 &= 9K_1 + 4K_2 + 3K_3 \end{aligned}\right\}$$ This is system of L.EQ

If this system $\longrightarrow$ consistent

That's mean that the w can be expressed as L.C of V_1, V_2, V_3

If this system is $\longrightarrow$ not consistent

That's means that the w can't be expressed as L.C

$$\begin{bmatrix} 3 & 0 & 1 \\ 5 & 2 & 2 \\ 9 & 4 & 3 \end{bmatrix}$$

If the matrix is inv $\rightarrow \det(A) \neq 0 \rightarrow$ consistent

using cofactor expansion method

$3(6-8) + (2) = -4$

$A \rightarrow$ inv $\rightarrow$ unique solution

$$\left[\begin{array}{ccc|c} 3 & 0 & 1 & 2 \\ 5 & 2 & 2 & 4.5 \\ 9 & 4 & 3 & 8 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0.5 \\ 0 & 1 & 0 & 0.5 \\ 0 & 0 & 1 & 0.5 \end{array}\right]$$

So $K_1 = K_2 = K_3 = 0.5$.

$v_1 = (1, 2, 3)$
$v_2 = (3, 2, 5)$
$w = (0, 2, 4)$

$(0, 2, 4) = k_1(1, 2, 3) + k_2(3, 2, 5)$

$0 = 1k_1 + 3k_2$
$2 = 2k_1 + 2k_2$ here we have (eq > unknowns)
$4 = 3k_1 + 5k_2$

we will take the first two equations & Find the values of k_1 & k_2

$k_1 = -3k_2 \longrightarrow k_1 = 3$
$1 = -3k_2 + 2k_2 \longrightarrow k_2 = -1$

substitute in the third equation

$3(3) + -1(5) = 4$

4 = third element in w so w can be expressed as a L.C of v_1, v_2 ...

IF I have instead of $(0, 2, 4) \longrightarrow (0, 2, 6)$
$4 \neq 6$ can't be expressed as a L.C

$(3, 2, 1) = k_1(3, 4, 15) + k_2(4, 0, 8)$

$3 = 3k_1 + 4k_2$
$2 = 4k_1 + 0k_2$ equations > unknowns (overdetermined sys)
$1 = 15k_1 + 8k_2$

- two vectors are parallel if one is scalar multiple of the other.

In other word, two vectors u and v if there exists a scalar c

$v = c \cdot u$

proof $\longrightarrow$ $u = (u_1, u_2)$ and $v = (v_1, v_2)$

they are parallel if $\frac{v_1}{u_1} = \frac{v_2}{u_2}$

Example: for what value of t is the given vector parallel to $u = (4, -1)$

a) $(8t, -2)$.

$v = c \cdot u$

$8t = c \cdot 4$ (x component)

$-2 = c(-1)$ (y component)

For the last one I can tell $c = 2$

$8t = 8 \longrightarrow t = 1$

Let $u = (2, 1, 0, 1, -1)$, $v = (-2, 3, 1, 0, 2)$ Find a & B so that

$au + bv = (-8, 8, 3, -1, 7)$

$a(2, 1, 0, 1, -1) + b(-2, 3, 1, 0, 2) = (-8, 8, 3, -1, 7)$

$2a - 2b = -8$

$a + 3b = 8$

$0a + b = 3 \quad \rightarrow b = 3$

$a + 2b = -1$

$-a + 2b = 7$

$2a - 2(3) = -8 \Rightarrow a = -1$

$a + 3(3) = 8 \Rightarrow a = -1$

$a + 2(3) = -1 \Rightarrow a = -7$

the sys is not consistent

ExamQ

$v = (3,4,5)$

$u = (4,0,8)$

$w = (5,7,10)$

1- Find all possible value of L.C of these vectors?

$L.C = k_1(3,4,5) + k_2(4,0,8) + k_3(5,7,10)$

I can take $k_1 = 1$, $k_2 = 1$, $k_3 = (-\infty, \infty)$.

So there is ∞ combinations

2- Give me one example for one linear combination

I can take any no. like $k_1 = 1$, $k_2 = 0$, $k_3 = 0$

$= (3,4,5)$

3- $y = (3,2,1)$, can this vector be written as a L.C of vuw?

يعني المقصود ان نستبدل y بـ L.C

$(3,2,1) = k_1(3,4,5) + k_2(4,0,8) + k_3(5,7,10)$

$3 = 3k_1 + 4k_2 + 5k_3$

$2 = 4k_1 + 0k_2 + 7k_3$

$1 = 5k_1 + 8k_2 + 10k_3$

consistent → can be written as L.C

inconsistent → can NOT be written as L.C

$$\begin{bmatrix} 3 & 4 & 5 \\ 4 & 0 & 7 \\ 5 & 8 & 10 \end{bmatrix}$$

det( ) ≠ 0 → inv → single sol.

det( ) = 0 → we can't know if it is consistent OR not

So then we will DO GJEM

$det = 3(-56) - 4(40-35) + 5(32) = -28$

4- $y = (3,2,1)$, can this vector be written as a L.C of vu?

$(3,2,1) = k_1(3,4,5) + k_2(4,0,8)$

$3 = 3k_1 + 4k_2$

$2 = 4k_1 + 0k_2$

$1 = 5k_1 + 8k_2$

3 equations with 2 unknowns (overdetermined sys.)

IN THIS SYS. If no one of the equations dependent to other equation there is no solution.

What is the NORM & UNIT VECTOR

$V = (5, 3, 4)$

$\text{norm}(V) = \sqrt{5^2 + 3^2 + 4^2}$ It's always +ve

$\text{unit vector} = \frac{1}{\text{norm}(V)} V$ (in other word it's called NORMALIZING)

Theorem 3.21

If v is a vector in R^n

a) $\|V\| \geq 0$

b) $\|V\| = 0 \longrightarrow$ if ~~and~~ only If $V = 0$

c) $\|kV\| = |k| \, \|V\|$

* any vector defined in 3D, can be written as L.C of these vectors

3D space $\hat{i}$ $(1,0,0)$ $\hat{j}$ $(0,1,0)$ $\hat{k}$ $(0,0,1)$ (standard unit vector)

Example:-

$(5, 3, 4) \longrightarrow 5\hat{i} + 3\hat{j} + 4\hat{k}$

DISTANCE BETWEEN TWO VECTORS

$d(u,v) = \|u - v\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \dots (u_n - v_n)^2}$

Example → $u = (5, 2, 9, 4)$, $V = (0, 1, 3, 2)$

$u - v = (5, 1, 6, 2)$

$\|u - v\| = \sqrt{5^2 + 1^2 + 6^2 + 2^2} = \sqrt{66} = 8.12$

for the same example find the dot product & find θ.

$u.v = 0 + (2 \times 1) + (9 \times 3) + (4 \times 2) = 37$

$\cos\theta = \frac{u.v}{\|u\| \|v\|} \rightarrow \theta = \cos^{-1}\left(\frac{37}{11.22 \times 3.74}\right) = 28.15°$

Find the angle between a diagonal of a cube & one of its edge

diagonal for the cube $\rightarrow (a, a, a)$

$e = (a, 0, 0)$

$d.e = (a \times a) + (a \times 0) + (0 \times a) = a^2$

$norm(d) = \sqrt{a^2 + a^2 + a^2} = \sqrt{3}\, a$

$norm(e) = \sqrt{a^2 + 0^2 + 0^2} = a$

$\cos\theta = \frac{a^2}{\sqrt{3a} \cdot a} = \frac{a^2}{\sqrt{3}\, a^2} = \frac{1}{\sqrt{3}}$

$\theta = 54.7°$

Find the angle between the main diagonal of the cube and the main diagonal of its faces

diagonal $\rightarrow (a, a, a)$

$F \rightarrow (a, a, 0)$

$d.F = (a \times a) + (a \times a) + (a \times 0) = 2a^2$

$norm(d) = \sqrt{3}\, a$

$norm(f) = \sqrt{a^2 + a^2 + 0^2} = \sqrt{2}\, a$

$\cos\theta = \frac{2a^2}{\sqrt{3}\sqrt{2}\, a^2} = \frac{2}{\sqrt{3}\sqrt{2}}$

$\theta = 35.26°$

NOTE

θ is acute if $u.v > 0$

θ is obtuse if $u.v < 0$

$\theta = \frac{\pi}{2}$ if $u.v = 0$

DOT PRODUCTS AND MATRICES

$\underset{3\times1}{u} \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$, $\underset{3\times1}{v} \begin{bmatrix} 7 \\ 8 \\ 2 \end{bmatrix}$

$u^T \times v \quad \begin{bmatrix} 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 2 \end{bmatrix} = v^T u \quad \begin{bmatrix} 7 & 8 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$

Theorem 3.2.7

If u & v are vectors defined in R^n with Euclidean inner product

$$u.v = \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2$$

$$\|u+v\|^2 = (u+v)(u+v) = \|u\|^2 + 2(u.v) + \|v\|^2$$
$$\|u-v\|^2 = (u-v)(u-v) = \|u\|^2 - 2(u.v) + \|v\|^2$$

$$\frac{1}{4}\left(\|u\|^2 + 2(u.v) + \|v\|^2\right) - \frac{1}{4}\left(\|u\|^2 - 2(u.v) + \|v\|^2\right)$$

$$\cancel{\frac{1}{4}\|u\|^2} + \frac{1}{2}(u.v) + \cancel{\frac{1}{4}\|v\|^2} - \cancel{\frac{1}{4}\|u\|^2} + \frac{1}{2}(u.v) - \cancel{\frac{1}{4}\|v\|^2}$$

$$\frac{1}{2}(u.v) + \frac{1}{2}(u.v) = u.v$$

$Au.v = u.A^Tv$

Suppose that $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 1 \\ -1 & 0 & 1 \end{bmatrix}$, $u = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$ $v = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$

$$Au = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ 5 \end{bmatrix}$$

$$A^Tv = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 4 & 0 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \\ -1 \end{bmatrix}$$

$$\left.\begin{aligned} Au.v &= 7(-2)+10(0)+5(5) = 11 \\ u.A^Tv &= -1(-7)+2(4)+4(-1) = 11 \end{aligned}\right\} \text{The same answer}$$

Theorem 3.2.4 (Chauchy – Schwarz inequality)

If $u = (u_1, u_2, \dots, u_n)$ and $v = (v_1, v_2, \dots, v_n)$

$$\|u.v\| \leq \|u\|.\|v\|$$

proof $\|u.v\| \leq \|u\|.\|v\|$

$(u.v)^2 \leqslant \|u\|^2.\|v\|^2$

applying the arithemetic Mean-geometric mean inequality

$$\frac{a+b}{2} \geqslant \sqrt{ab}$$

Let $a = \|u\|^2$ & $b = \|v\|^2$

$$\frac{\|u\|^2+\|v\|^2}{2} \geqslant \sqrt{\|u\|^2.\|v\|^2}$$

$$\|u\|^2+\|v\|^2 \geqslant 2\|u\|\|v\|$$

using the result into the squared Cauchy-Schwarz inequality

$(u.v)^2 \leqslant \|u\|^2.\|v\|^2$ (taking the square root)

$|u.v| \leqslant \|u\|\|v\|$

Theorem 3.2K

a) $\|u+v\| \leqslant \|u\|+\|v\|$ (triangular inequality)

proof why?!

$$\|u+v\|^2 \leqslant (\|u\|+\|v\|)^2$$

$$\|u+v\|^2 \leqslant \|u\|^2+2\|u\|\|v\|+\|v\|^2$$

$$\leqslant \|u\|^2+2(u.v)+\|v\|^2$$

Applying the cauchy-schawrz inequality which says that: $(u.v)^2 \leqslant \|u\|^2.\|v\|^2$

$$\|u+v\|^2 \leqslant \|u\|^2+2\|u\|\|v\|+\|v\|^2 \leqslant \|u\|^2+2\|u\|\|v\|+\|v\|^2$$

$$\|u+v\|^2 \leqslant (\|u\|+\|v\|)^2$$

$$u+v \leqslant \|u\|+\|v\|$$

b) $d(u,v) \leqslant d(u,w)+d(w,v)$

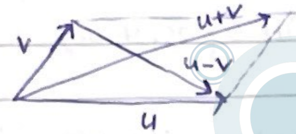
لانو اقصر بين 2 vectors الخط المستقيم الواصل بينهم

$$d(u,v) = \|u-w\| = \|(u-w)+(w-v)\|$$

$$\leqslant \|u-w\|+\|w-v\| = d(u,w)+d(w,v)$$

Theorem 3.2.6 (Parallelogram Equation for vectors)

$$\|u+v\|^2+\|u-v\|^2 = 2\left(\|u\|^2+\|v\|^2\right)$$
$$= 2(u\cdot u) + 2(v\cdot v)$$
$$= 2\left(\|u\|^2+\|v\|^2\right)$$



ORTHOGONALITY

$u\,(2,3,-4)$

$v\,(2,0,1)$

$u\cdot v = 4-4 = 0 \longrightarrow$ That's mean that they are Orthogonal (perpendicular to each other)

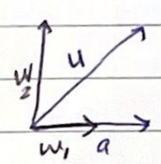
$$\left.\begin{matrix} u \\ v \\ w \\ c \end{matrix}\right\} \quad u\cdot v = u\cdot w = u\cdot c = v\cdot w = v\cdot c = w\cdot c = 0$$ (Orthogonal set)

(ALL POSSIPLE COMPINATION BETWEEN THESE VECTORS = zero)

NOTE: All The Standard unit vectors are orthogonal $\rightarrow i\cdot j = i\cdot k = j\cdot k = 0$

PROJECTION Theorem (3.3.2)

If u & a are vectors in R^n, and if $a \neq 0$, then u can be expressed in exactly one way in the form $u = w_1+w_2$, where w_1 is scalar multiple of a and w_2 is orthogonal to a



$w_1+w_2=u$

$u=(2,-1,3)$

$a=(4,-1,2)$

$W_1 =$ 1- $u\cdot a = 15$

2- $\text{norm}(a)^2 = \left(\sqrt{4^2+1^2+2^2}\right)^2 = 21$

$$W_1 = \frac{u\cdot a}{(\text{norm}\,a)^2} = \frac{15}{21}\cdot a = \left(2.86,\ \frac{-5}{7},\ \frac{10}{7}\right)$$

$$W_2 = u - w_1 = (2,-1,3) - \left(\frac{30}{7}, \frac{-5}{7}, \frac{10}{7}\right) = \left(\frac{-6}{7}, \frac{-2}{7}, \frac{11}{7}\right)$$

Theorem 3.3.3 (Pythagoras in $\mathbb{R}^n$)

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

$$\|u+v\|^2 = (u+v)\cdot(u+v)$$
$$= u\cdot u + u\cdot v + v\cdot u + v\cdot v$$
$$= u\cdot u + 2u\cdot v + v\cdot v$$
$$= \|u\|^2 + 2(u\cdot v) + \|v\|^2$$

If u and v are orthogonal then $(u\cdot v) = 0$

& $\|u+v\|^2 = \|u\|^2 + \|v\|^2$

Theorem 3.4.3

- If A is $m\times n$ vector then the solution set of the homogenous L.S $Ax=0$ Consists of all vectors in R^n that are orthogonal to every row vector of A

$$A = \begin{bmatrix} a_{11} & a_{2} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ → the solution of the homgenous sys is perpendicular on the Rows of Matrix A

- If we have two systems

$Ax = b$ $Ax = 0$

w_1 is a solution for $Ax=b$, proof that w_1+w_2 is a solution for $Ax=b$?

w_2 is a solution for $Ax=0$

$$A(w_1+w_2)$$
$$Aw_1 + Aw_2$$
$$b + 0 = b$$

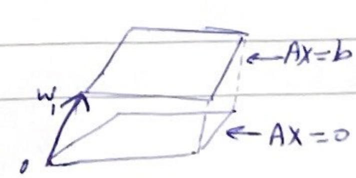
* If A is invertible, then $Ax=b$ has only unique solution , $Ax=0$ (has only the trivial solution)

So If we sum up the trivial & the unique solutions together they will gives us the same unique SOL

- Conversely let x be a solution of $Ax=b$. To show that x is the set w_1+w_2

we must show $x = w_1 + w_2$, $w_2 = x - w_1$

$$Aw_2 = Ax - Aw_1$$
$$b - b = 0$$



- In R^2 the distance D between the point $P_0(x_0, y_0)$ and the line $ax+by+c=0$

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2+b^2}}$$

Example

point $(3, 5)$

line $y = 4x + 7$

$4x - y + 7$

$$D = \frac{|4(3) - 1(5) + 7|}{\sqrt{4^2 + (-1)^2}} = \frac{14}{\sqrt{17}} = 3.4$$

- In R^3 the distance D between the point $P_0(x_0, y_0, z_0)$ and the plane $ax+by+cz+d=0$

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}}$$

Example

point $(3, 2, 1)$

plane $0 = 4x + 5y + 7z - 4$

$$D = \frac{|4\times3 + 5\times2 + 7(1) - 4|}{\sqrt{4^2+5^2+7^2}}$$

(IM) ExamQ: find the distance between two planes

plane1: $4x + 6y + 2z = 5$

Plane2: $2x + 3y + z = 4$

1- first of all I had to If these two planes are parallel to each other

they are parallel because the coeff of x, y, z in plane 2 is $\frac{1}{2}$ coeff of x, y, z in plane 1 (Regardless to the constants $(5,4)$)

2- point on the plane1 (assume $x=4$, $y=0$)

$4(4) + 6(0) + 2z = 5 \longrightarrow z = -5.5$

distance between the point and the plane 2

$2x + 3y + z - 4 = 0$

$$D = \frac{|2(4) + 3(0) - 5.5(1) - 4|}{\sqrt{2^2+3^2+1^2}} =$$

we should notice that 4 & 5
ما يكونو من مضاعفات بعض
لانو هيك بكونه نحكي عن نفس ال plane
والمسافة = zero

CROSS PRODUCT

$u = (u_1, u_2, u_3) \qquad V = (v_1, v_2, v_3)$

$$\begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2)\hat{i} - (u_1v_3 - u_3v_1)\hat{j} + (u_1v_2 - u_2v_1)\hat{k}$$

الناتج بكون orthogonal على u و v مع بعض

RELATIONSHIPS between DOT PRODUCT AND CROSS PRODUCT

- $u \cdot (u \times v) = 0$
  because $(u \times v)$ is orthognal to u
- $v \cdot (u \times v) = 0$
- $\|u \times v\|^2 = \|u\|^2 \|v\|^2 - (u \cdot v)^2$
- $u \times v = -(v \times u)$
- $u \times (v + w) = (u \times v) + (u \times w)$
- $(u + v) \times w = (u \times w) + (v \times w)$
- $k(u \times v) = (ku) \times v = u \times (kv)$
- $u \times 0 = 0 \times u = 0 \rightarrow$ vetor
- $u \times u = 0$

↳ Exam Q (proof this?)

$$\begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = \underbrace{(u_2u_3) - (u_3u_2)}_{0}\hat{i} - \underbrace{((u_1u_3) - (u_3u_1))}_{0}\hat{j} + \underbrace{u_1u_2 - u_2u_1}_{0}\hat{k}$$

$\|u \times v\| = \|u\| \|v\| \sin\theta$

$\|u \times u\| = \|u\| \|u\| \underbrace{\sin\theta}_{= zero}$

norm for any vector is zero if & only if the vector is zero

$\therefore$ ~~norm~~ $\|u \times u\| = 0$ in this case $u \times u = 0$

I can say $u \times u = w$

~~norm~~ $\|w\| = 0$

then $w = 0$

$u \times u = 0$

Chapter 4: General Vector Space

V → objects
- → real numbers
- → matrices
- → functions
- → polynomials
- → sinosidal

there are two operations i can do (Addition, Multiplication by scalar)

Q: Let V be the real numbers defined from $(-\infty, \infty)$, can V be considered as a real vector space

u, v & w are real numbers, k and L are any scalar

1- u+v is a real number (first axiom)

2- $(u+v)+w = u+(v+w)$

3- $u+v = v+u$

4- zero vector → zero number $0+u = u+0$

5- u → (−u → is defined) → $u + -u = -u + u =$

6- If k is scalar ku = real number

7- $k(u+v) = ku + kv$

8- $(k+L)u = ku + Lu$

9- $k(Lu) = (kL)u$

10- $1u = u$

So V is A VECTOR SPACE

ALL SET OF OBJECTS
- CLOSED UNDER ADDITION & MULTIPLICATION → V.S
- NOT CLOSED UNDER ADDITION & MULTIPLICATION → NOT V.S

Q: Let V defined from $(0, \infty)$

axiom 5 will not satisfy, so It's not a vector space

Q: Let V defined from ~~int~~ & integer $(-\infty, \infty)$ can V be considered as a vector space

axiom 6 will not satisfy, because the scalar k can be fraction $\frac{1}{4}, \frac{1}{2}, \frac{1}{3}$

So It will not give me a integer

Q: is R^n vector space?

$u = (u_1, u_2, \ldots u_n)$

$v = (v_1, v_2, \ldots v_n)$

$w = (w_1, w_2 \ldots w_n)$

Yes, all the axioms are satisfied

Q: $M_{2\times3}$

It is a vector space, all the axioms are satisfied

notes:

axiom 6 → the multiplication is not (matrix X matrix)

It is multiplication by scalar.

ALL FUNCTIONS THAT HAVE A REAL VALUE $(-\infty, \infty)$

THEY ARE CONSIDERED AS A V.S

Q: set of polynomials from $(-\infty, \infty)$

yes

- all real numbers $[0, \infty)$ → NOT V.S (axiom 5)
- all integer numbers $(-\infty, \infty)$ → NOT V.S (axiom 6)
- all polynomials (order 2) → V.S
- all polynomials with any order → V.S
- all polynomials with odd coeff → NOT V.S (axiom 1)
- all polynomials with even coeff → NOT V.S (axiom 6)
- all polynomials with integer coeff → NOT V.S (axiom 6)
- all polynomials with +ve coeff → NOT V.S (axiom 5)
- all polynomials with -ve coeff → NOT V.S
- all matrices that have size 2x3 & all the entries are integer → NOT V.S
- all matrices that have entries are odd numbers → NOT V.S (axiom 1)

Example 5 : Let $V = R^2$ (all vectors defined in 2D)

If $u = (u_1, u_2)$ & $v = (v_1, v_2)$

$\rightarrow u + v = (u_1 + v_1, u_2 + v_2)$

$\rightarrow ku = (ku_1, 0)$

It's not a vector space (axiom 10)

$1u = (u_1, u_2) \neq (u_1, 0)$

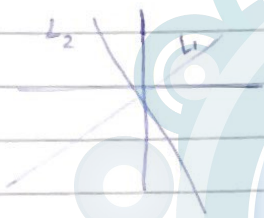
~~Every plane through~~ the origin is a V.S

because if we sum up two planes that pass through the origin point

It will give us a plane that pass through the same point

↳ because the constant part for the planes is zero

We can concider the two Lines that pass through the origin is a V.S



because the intercept zero

(zero + zero = zero)

Example for unusual vector space

Let V be the set of positive real numbers

$u + v = uv$

$ku = u^k$

the set V with these operations satisfies the 10 vector space axioms

- $u + v = uv$
- $(u+v)+w = (uv)w$ & $u+(v+w) = u(vw)$ $\rightarrow (uv)w = u(vw)$
- Since k is any positive real number $k \in R$

  $ku = u^k$
- $u + 1 = u \cdot 1 = u$
- $u \cdot \frac{1}{u} = 1$ which is the multiplicative identity
- $c \cdot (d \cdot u) = c \cdot u^d = cu^d = cd \cdot u$ which satisfy the compatibility property
- Let $c, d \in R$ & $u \in V$ Then $(c+d) \cdot u = u^{c+d} = u^c \cdot u^d = c \cdot u + d$

4.2: SUBSPACES

Ex: If u & v are vectors in the first Quartile & W is vector pass through the origin point

NO, If the scalar is negative → they are not defined in the first quartile (It's closed under addition only) So W is not subspace

Quiz

Give me two subspaces for R^3, we want a subspace for $u=(u_1,u_2,u_3)$

1- R^3

2- $(0,0,0)$, $(u_1,u_2,0)$

3- lines through the origin

4- planes through the origin

NO WAY TO SAY that $M_{2\times2}$ SUBSPACE from $M_{2\times3}$

SUBSPACES OF M_{nn}

1- the set of nxn upper triangular matrices

2- the set of nxn Lower triangular matrices

3- the set of nxn diagonal matrices

Q IS the identity matrix is subspace from the square matrices

NO, Because If we sum up two identities ≠ identity (axiom 1)

Q can we say that inv. matrices subspace from the matrices

NO, because If we sum up two inv. matrices ≠ inv. matrix (axiom 1)

Qxam Q:-

Polynomials with complex cofficient. can we consider this as a real vector space

No, because complex numbers is not a real numbers

So It can be considered as a real vector space

Can we consider the solution of the system of L.E $Ax = b$ is a subspace?

No, if we have x_1, x_2 (solutions of L.E $Ax=b$)

$$A(x_1 + x_2) = Ax_1 + Ax_2 = b + b = 2b$$

can we consider the solution of a homogenous system $Ax = 0$ is a subspace?

Yes, $A(x_1 + x_2) = Ax_1 + Ax_2 = 0 + 0 = 0 \rightarrow$ Closed under addition

$\underline{kx_1}$

$A(kx_1) = kA(0) = 0$ ✓ $\rightarrow$ closed under the scalar multp.

Consider the vectors $u = (1,2,-1)$ and $v = (6,4,2)$ in R^3

Show that $w = (9,2,7)$ is a linear combination of u & v and

$w' = (4,-1,8)$ is NOT a linear comb. of u & v.

$w = k_1 u + k_2 v$

$(9,2,7) = k_1(1,2,-1) + k_2(6,4,2)$

$$\left.\begin{aligned} 9 &= k_1 + 6k_2 \\ 2 &= 2k_1 + 4k_2 \end{aligned}\right\} \begin{aligned} &k_1 = 9 - 6k_2 \\ &2 = 2(9 - 6k_2) + 4k_2 \longrightarrow k_2 = 2 \end{aligned}$$

$7 = -k_1 + 2k_2$ $\qquad k_1 = 9 - 6(2) = -3$

$-(-3) + 2(2)$

$7 = 7$ It's a L.C of u & v

$w' = k_1 u + k_2 v$

$(4,-1,8) = k_1(1,2,-1) + k_2(6,4,2)$

$$\left.\begin{aligned} 4 &= k_1 + 6k_2 \\ -1 &= 2k_1 + 4k_2 \end{aligned}\right\} \begin{aligned} &k_1 = 4 - 6k_2 \\ &-1 = 2(4 - 6k_2) + 4k_2 \longrightarrow k_2 = 9/8 \end{aligned}$$

$k_1 = 4 - 6(9/8) = -2.75$

$8 = -k_1 + 2k_2$

$-(-2.75) + 2(\frac{9}{8}) = 5$

$8 \neq 5$ w' is not a L.C of u & v