1-2 Echelon Forms

(4 Properties) r.r.e.f

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

- A matrix is in reduced row-echelon form
  - If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. We call this a leader 1.
  - If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
  - In any two successive rows that do not consist entirely of zeros, the leader 1 in the lower row occurs farther to the right than the leader 1 in the higher row.
  - Each *column* that contains a leader 1 has zeros everywhere else.
- A matrix that has the *first three properties* is said to be in row-echelon form.

1-2 Example 2

- Matrices in **row-echelon form** (any real numbers substituted for the *'s) :

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

- Matrices in **reduced row-echelon form** (any real numbers *substituted for the *'s.*) :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

1-2 Example 3

- Solutions of linear systems

2 leaders → 2 ind equ
→ 2 dep var
→ 1 free var (3-2)

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

none of them (inconsistent)

3 leaders → 3 ind equ
→ 3 dep var
→ 1 free var (4-3)

$$\begin{bmatrix} 1 & 0 & 0 & 4 & -1 \\ 0 & 1 & 0 & 2 & 6 \\ 0 & 0 & 1 & 3 & 2 \end{bmatrix}$$

R.R.E.F

3 leaders →
3 ind equ →
3 dep var →
2 free var (5-3)

$$\begin{bmatrix} 1 & 6 & 0 & 0 & 4 & -2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow$$

1 dep equ

R.R.E.F

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

R.R.E.F
(inconsistant)

3 leaders →
3 ind equ →
3 dep var →
0 free var (3-3)

1-2 Elimination Methods

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

- Step1. Locate the leftmost column that does not consist entirely of zeros.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Leftmost nonzero column

- Step2. Interchange the top row with another row, to bring a nonzero entry to top of the column found in Step1

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

The 1th and 2th rows in the preceding matrix were interchanged.

1-2 Example 5

- From the computations in example 4 , a row-echelon form of the augmented matrix is given.

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- To solve the system of equations:

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\ x_3 + 2x_4 + 3x_6 &= 1 \\ x_6 &= 1/3 \end{aligned}$$

1-2 Homogeneous Linear Systems

- A system of linear equations is said to be homogeneous if the constant terms are all zero.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\ \vdots \quad\quad \vdots \quad\quad & \quad\quad \vdots \quad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0 \end{aligned}$$

- Every homogeneous system of linear equation is **consistent**, since all such system have $x_1 = 0, x_2 = 0, \ldots, x_n = 0$ as a solution.
  - This solution is called the trivial solution. Zero solution
  - If there are another solutions, they are called nontrivial solutions.
- There are *only two possibilities* for its solutions:
  - There is **only** the trivial solution
  - There are **infinitely** many solutions in addition to the trivial solution

1-2 Example 7 (Gauss-Jordan Elimination)

- Two important points:
  - None of the three row operations alters the final column of zeros, so the system of equations corresponding to the reduced row-echelon form of the augmented matrix must also be a homogeneous system.
  - If the given homogeneous system has m equations in n unknowns with $m < n$, and there are r nonzero rows in reduced row-echelon form of the augmented matrix, we will have $r < n$. It will have the form:

$$\begin{aligned} x_{k_1} &+ \sum() = 0 & x_{k_1} &= -\sum() \\ x_{k_2} &+ \sum() = 0 & x_{k_2} &= -\sum() \\ &\vdots & &\vdots \\ x_{k_r} &+ \sum() = 0 & x_{k_r} &= -\sum() \end{aligned}$$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad\quad \vdots \quad\quad & \quad\quad \vdots \quad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

  - (Theorem 1.2.1)

Theorem 1.2.1

- A homogeneous system of linear equations with more unknowns than equations has **infinitely many solutions**.

- Remark
  - This theorem applies only to homogeneous system!
  - A nonhomogeneous system with more unknowns than equations need not be consistent; however, if the system is consistent, it will have infinitely many solutions.
  - e.g., two parallel planes in 3-space

operations ال
لـ تؤثر على ال
system
↓
hom. بقى

* (r Nonzero rows in R.R.E.F) → r→ ※ of independant equations
* max(r) = m
* (m<n) → (※ of equations < ※ of unknown)
↳ (r≤m)
(so that r<n) ※

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1-3 Definition and Notation

- A matrix is a rectangular array of numbers. The numbers in the array are called the entries in the matrix
- A general $m \times n$ matrix A is denoted as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- The entry that occurs in row i and column j of matrix A will be denoted a_{ij} or $\langle A \rangle_{ij}$. If a_{ij} is real number, it is common to be referred as **scalars**
- The preceding matrix can be written as $[a_{ij}]_{m \times n}$ or $[a_{ij}]$

1-3 Matrix Products as Linear Combinations

- Let

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- Then

$$A\mathbf{x}=\begin{bmatrix} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n \\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n \\ \vdots \\ a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n \end{bmatrix}=x_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}+x_2\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}+\cdots+x_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

- *The product $A\mathbf{x}$ of a matrix A with a column matrix $\mathbf{x}$ is a linear combination of the column matrices of A with the coefficients coming from the matrix $\mathbf{x}$*

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Variables

$a_1 / a_2 / \dots a_n$

$x_1 / x_2 / \dots x_n$ scaler (real numbers)

1-3 Example 8

The matrix product

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}=\begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

can be written as the linear combination of column matrices

$$2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}-1\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}+3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}=\begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

The matrix product

$$\begin{bmatrix} 1 & -9 & -3 \end{bmatrix}\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix}=\begin{bmatrix} -16 & -18 & 35 \end{bmatrix}$$

can be written as the linear combination of row matrices

$$1\begin{bmatrix} -1 & 3 & 2 \end{bmatrix}-9\begin{bmatrix} 1 & 2 & -3 \end{bmatrix}-3\begin{bmatrix} 2 & 1 & -2 \end{bmatrix}=\begin{bmatrix} -16 & -18 & 35 \end{bmatrix}$$

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$(1\times -1)+(-9\times 1)+(1\times 2)=-6$

$(1\times 3)+(-9\times 2)+(-3\times 1)=-18$

$(1\times 2)+(-9\times -3)+(-3\times -2)=35$

طريقة الضرب

$[-6 \quad -18 \quad 35]$ (as a row)

1-3 Example 9

We showed in Example 5 that

$$AB=\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}\begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}=\begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

The column matrices of AB can be expressed as linear combinations of the column matrices of A as follows:

$$\begin{bmatrix} 12 \\ 8 \end{bmatrix}=4\begin{bmatrix} 1 \\ 2 \end{bmatrix}+0\begin{bmatrix} 2 \\ 6 \end{bmatrix}+2\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 27 \\ -4 \end{bmatrix}=1\begin{bmatrix} 1 \\ 2 \end{bmatrix}-1\begin{bmatrix} 2 \\ 6 \end{bmatrix}+7\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 30 \\ 26 \end{bmatrix}=4\begin{bmatrix} 1 \\ 2 \end{bmatrix}+3\begin{bmatrix} 2 \\ 6 \end{bmatrix}+5\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 13 \\ 12 \end{bmatrix}=3\begin{bmatrix} 1 \\ 2 \end{bmatrix}+1\begin{bmatrix} 2 \\ 6 \end{bmatrix}+2\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

♦

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$(3\times 1)+(1\times 2)+(2\times 4)=13$

$(3\times 2)+(1\times 6)+2(0)=12$

طريقة الضرب

(as a column)

1-3 Matrix Form of a Linear System

- Consider any system of m linear equations in n unknowns:

$$\begin{aligned} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n&=b_1 \\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n&=b_2 \\ &\vdots \\ a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n&=b_m \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n \\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n \\ \vdots \\ a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n \end{bmatrix}=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad \Rightarrow \quad A\mathbf{x}=\mathbf{b}$$

- The matrix A is called the coefficient matrix of the system
- The augmented matrix of the system is given by

$$[A\,|\,\mathbf{b}]=\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

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طريقة الضرب

(as a column)

$A = \begin{bmatrix} 1 & 4 & 12 \\ 2 & 7 & 0 \\ 3 & 10 & 1 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & 10 \\ 12 & 0 & 1 \end{bmatrix}$ الدليل $\Leftarrow$ Trace of A = Trace of A^T * ملاحظة : (مهم جدًا)

※ trace (A) = 9 (trace A^T) = 9

1-3 Example 10

- A function using matrices
  - Consider the following matrices

$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $x = \begin{bmatrix} a \\ b \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

  - The product y = Ax is $\begin{bmatrix} a \\ -b \end{bmatrix}$
  - The product y = Bx is $\begin{bmatrix} b \\ -a \end{bmatrix}$

1-3 Definitions قلب الاعمدة الى صفوف والصفوف الى اعمدة

- If A is any $m \times n$ matrix, then the transpose of A, denoted by A^T, is defined to be the $n \times m$ matrix that results from interchanging the rows and columns of A
  - That is, the first column of A^T is the first row of A, the second column of A^T is the second row of A, and so forth
- If A is a square matrix, then the trace of A, denoted by $\text{tr}(A)$, is defined to be the sum of the entries on the main diagonal of A. The trace of A is undefined if A is not a square matrix.
  - For an $n \times n$ matrix $A = [a_{ij}]$, $\text{tr}(A) = \sum_{i=1}^{n} a_{ii}$

حاصل جمع ال (main diagonal) بشرط ان تكون A square

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الاثبات

** profe that Tr(A) = Tr(A^T)

$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$

سؤال امتحان

Tr(A) = $a_1 + a_5 + a_9$

Tr(A^T) = $a_1 + a_5 + a_9$

so that Tr(A) = Tr(A^T) ※

$A^T = \begin{bmatrix} a_1 & a_4 & a_7 \\ a_2 & a_5 & a_8 \\ a_3 & a_6 & a_9 \end{bmatrix}$

1-3 Example 11 & 12

- Transpose: $(A^T)_{ij} = (A)_{ij}$

$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}$ 3x2

- Trace of matrix:

$B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$

$A^T = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}$ 3x2

tr(B) = -1 + 5 + 7 + 0 = 11

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[1] A (3*4) B (4*5)

AB defined but

BA not defined

[2] A (2*4) B (4*2)

both of them defined but not commutative because they are not same size

[3] Both of them are defined and have the same size but their entries not equal

example: $A=\begin{bmatrix}4 & 10\\ 0 & 1\end{bmatrix}$ $B=\begin{bmatrix}2 & 5\\ 4 & 1\end{bmatrix}$

$AB_{2*2}=\begin{bmatrix}48 & 30\\ 4 & 1\end{bmatrix}$ $BA_{2*2}=\begin{bmatrix}8 & 31\\ 4 & 1\end{bmatrix}$

1-4 Properties of Matrix Operations

- For real numbers a and b, we always have $ab = ba$, which is called the ***commutative law for multiplication***. For matrices, however, AB and BA need not be equal.
- Equality can fail to hold for three reasons:
  - The product AB is defined but BA is undefined.
  - AB and BA are both defined but have different sizes.
  - It is possible to have $AB \neq BA$ even if both AB and BA are defined and have the same size.

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Theorem 1.4.1 (Properties of Matrix Arithmetic)

- Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid:
  - $A + B = B + A$ (commutative law for addition)
  - $A + (B + C) = (A + B) + C$ (associative law for addition)
  - $A(BC) = (AB)C$ (associative law for multiplication)
  - $A(B + C) = AB + AC$ (left distributive law)
  - $(B + C)A = BA + CA$ (right distributive law)
  - $A(B - C) = AB - AC$, $(B - C)A = BA - CA$
  - $a(B + C) = aB + aC$, $a(B - C) = aB - aC$
  - $(a+b)C = aC + bC$, $(a-b)C = aC - bC$
  - $a(bC) = (ab)C$, $a(BC) = (aB)C = B(aC)$
- Note: the cancellation law is not valid for matrix multiplication!

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** يجب الانتباه لشروط الضرب والجمع والطرح والضرب بشابت

$A=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$ $B=\begin{bmatrix}2 & 0\\ 1 & 0\end{bmatrix}$ $C=\begin{bmatrix}4 & 0\\ 1 & 0\end{bmatrix}$

** (مهم)

$AB=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$ $AC=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$

AB = AC doesn't mean B = C

(AC / AB متساويات من حيث ال size + entries)

لكن هذا لا يعني (B = C) ← الا اذا ضربنا بال (invertable)

$A^{-1}(AB) = A^{-1}(AC)$ ⇐ B = C الحالة الوحيدة

* عندنا القسمة على matrix نهائيًا

1-4 Proof of $A(B + C) = AB + AC$

- show the same size: $A_{m\times n}$ $B_{n\times L}$ $C_{n\times L}$ $(B+C)_{n\times L}$
- show the corresponding entries are equal: $A(B+C)_{m\times L}$

$A_{2\times 2}=\begin{bmatrix}a_1 & a_2\\ a_3 & a_4\end{bmatrix}$ $B_{2\times 3}=\begin{bmatrix}b_1 & b_2 & b_3\\ b_4 & b_5 & b_6\end{bmatrix}$

$C_{2\times 3}=\begin{bmatrix}c_1 & c_2 & c_3\\ c_4 & c_5 & c_6\end{bmatrix}$

ما يفرق ال size اللي بناخذه لل matrices (يفضل 3x3)

1-4 Example 2

As an illustration of the associative law for matrix multiplication, consider

$$A=\begin{bmatrix}1 & 2\\ 3 & 4\\ 0 & 1\end{bmatrix},\quad B=\begin{bmatrix}4 & 3\\ 2 & 1\end{bmatrix},\quad C=\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}$$

Then

$$AB=\begin{bmatrix}1 & 2\\ 3 & 4\\ 0 & 1\end{bmatrix}\begin{bmatrix}4 & 3\\ 2 & 1\end{bmatrix}=\begin{bmatrix}8 & 5\\ 20 & 13\\ 2 & 1\end{bmatrix}\quad\text{and}\quad BC=\begin{bmatrix}4 & 3\\ 2 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}=\begin{bmatrix}10 & 9\\ 4 & 3\end{bmatrix}$$

Thus,

$$(AB)C=\begin{bmatrix}8 & 5\\ 20 & 13\\ 2 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}=\begin{bmatrix}18 & 15\\ 46 & 39\\ 4 & 3\end{bmatrix}$$

and

$$A(BC)=\begin{bmatrix}1 & 2\\ 3 & 4\\ 0 & 1\end{bmatrix}\begin{bmatrix}10 & 9\\ 4 & 3\end{bmatrix}=\begin{bmatrix}18 & 15\\ 46 & 39\\ 4 & 3\end{bmatrix}$$

so $(AB)C = A(BC)$, as guaranteed by Theorem 1.4.1c.

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$B+C_{(2\times 3)}=\begin{bmatrix}b_1+c_1 & b_2+c_2 & b_3+c_3\\ b_4+c_4 & b_5+c_5 & b_6+c_6\end{bmatrix}$

$A(B+C)_{2\times 3}=\begin{bmatrix}a_1(b_1+c_1)+a_2(b_4+c_4) & \cdots\\ a_3(b_1+c_1)+a_4(b_4+c_4) & \cdots\end{bmatrix}$

ونكمل لاثبات النظرية

$AB_{(2\times 3)}=\begin{bmatrix}a_1b_1+a_2b_4 & a_1b_2+a_2b_5 & a_1b_3+a_2b_6\\ a_3b_1+a_4b_4 & a_3b_2+a_4b_5 & a_3b_3+a_4b_6\end{bmatrix}$

$AC=\begin{bmatrix}a_1c_1+a_2c_4 & a_1c_2+a_2c_5 & \cdots\\ a_3c_1+a_4c_4 & a_3c_3+a_4c_5 & \cdots\end{bmatrix}$

1-4 Zero Matrices

- A matrix, all of whose entries are zero, is called a **zero matrix**
- A zero matrix will be denoted by 0
- If it is important to emphasize the size, we shall write $0_{m\times n}$ for the $m\times n$ zero matrix.
- In keeping with our convention of using **boldface symbols** for matrices with one column, we will denote a zero matrix with one column by **0**

1-4 Example 3

- The cancellation law does not hold

$$A=\begin{bmatrix}0 & 1\\ 0 & 2\end{bmatrix} \quad B=\begin{bmatrix}1 & 1\\ 3 & 4\end{bmatrix} \quad C=\begin{bmatrix}2 & 5\\ 3 & 4\end{bmatrix} \quad D=\begin{bmatrix}3 & 7\\ 0 & 0\end{bmatrix}$$

- $AB=AC=\begin{bmatrix}3 & 4\\ 6 & 8\end{bmatrix}$
- $AD=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

اذا بضربها باي matrix بتعطيني نفس ال matrix بشرط تكون defined

Theorem 1.4.2 (Properties of Zero Matrices)

- Assuming that the sizes of the matrices are such that the indicated operations can be performed ,the following rules of matrix arithmetic are valid
  - $A+0=0+A=A$
  - $A-A=0$
  - $0-A=-A$
  - $A0=0;\quad 0A=0$

** الحالة الوحيدة اللي يكون فيها $AO=OA$ same size + square ان تكون كلا المعكوسين (AO OA)

1-4 Identity Matrices

- A square matrix with 1's on the main diagonal and 0's off the main diagonal is called an **identity matrix** and is denoted by I, or I_n for the $n\times n$ identity matrix
- If A is an $m\times n$ matrix, then $AI_n=A$ and $I_mA=A$
  - Example 4

* لازم اتأكد اذا defined (سؤال امتحان)

- An identity matrix plays the same role in matrix arithmetic as the number 1 plays in the numerical relationships $a\cdot 1=1\cdot a=a$

بما انها square يكفي اما ان نضع عدد الصفوف او عدد الاعمدة

$AO \neq OA$

2*3 3*2 — 3*2 2*3

2*2 — 3*3

square ✓

same size ✗

A * identity = A (بشرط defined)

but A + identity ≠ A

** سؤال امتحان

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

$$\det(A) = a_1 a_4 - a_3 a_2$$

Theorem 1.4.3

- If R is the reduced row-echelon form of an $n \times n$ matrix A, then either R has a row of zeros or R is the identity matrix I_n

r.r.e.f + square

↳ identity (I) → invertible (unique r.r.e.f)
→ (rows ind.) (det ≠ 0)

↳ not identity → not invertible (row of zeros) (or column of zeros)
→ (rows dep.) (det = 0)
→ (matrix singular)

1-4 Invertible

- If A is a square matrix, and if a matrix B of the same size can be found such that $AB = BA = I$, then A is said to be *invertible* and B is called an *inverse* of A. If no such matrix B can be found, then A is said to be *singular*.

not invertible

- Remark:
  - The inverse of A is denoted as A^{-1}
  - Not every (square) matrix has an inverse
  - An inverse matrix has exactly one inverse

(لم نجد B)

* non singular → inv.

unique r.r.e.f

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** Zeros matrix not invertible because determinant = 0

1-4 Example 5 & 6

- Verify the inverse requirements

$$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \rightarrow \det = 1 \neq 0$$

inv inv

det = 1 ← ≠ 0

- A matrix with no inverse

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$$ is singular

column of zeros

1-4 Theorems

- Theorem 1.4.4
  - If B and C are both inverses of the matrix A, then $B = C$

- Theorem 1.4.5
  - The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

    is invertible if $ad - bc \neq 0$, in which case the inverse is given by the formula

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

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سؤال امتحان:

$$A_{2*3} = \begin{bmatrix} 3 & 4 & 5 \\ 2 & 3 & 4 \end{bmatrix} \qquad B_{2*3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

AB is a :

a) zero matrix
b) A
c) I
d) none of the above

if AB^T defined (zero matrix)

because It's not defined

Proof Theorem 1.4.4

B and C are both inverses of the matrix A

↳ $B = A^{-1}$ $C = A^{-1}$

$AB = AA^{-1} = I$ $AC = AA^{-1} = I$

A inv.

$A^{-1}(AB) = A^{-1}(AC) \rightarrow B = C$ ※

Proof Theorem 1.4.6

A invertible
- square
- det ≠ 0

B invertible
- square
- det ≠ 0

$\det(AB) = \det(A)\det(B) \neq 0$

+ AB is a square matrix → so AB is inv. #

Theorem 1.4.6

- If A and B are invertible matrices of the same size ,then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

- Example 7

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \quad AB = \begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix}$$

سؤال امتحان :

* if A, B, and C are inv. + same size, prove that, $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

$D = AB$ (defined because A and B are the same size + square)

$D^{-1} = B^{-1}A^{-1}$

according to the Theorem 1.4.6

Then $(DC)^{-1} = C^{-1}D^{-1} = C^{-1}B^{-1}A^{-1}$ #

1-4 Powers of a Matrix

- If A is a square matrix, then we define the nonnegative integer powers of A to be

$$A^0 = I \quad A^n = \underbrace{AA\cdots A}_{n \text{ factors}} \quad (n > 0)$$

- If A is invertible, then we define the negative integer powers to be

$$A^{-n} = (A^{-1})^n = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{n \text{ factors}} \quad (n > 0)$$

- Theorem 1.4.7 (Laws of Exponents)
  - If A is a square matrix and r and s are integers, then $A^rA^s = A^{r+s}$, $(A^r)^s = A^{rs}$

* ما باخذ ال matrix under the root

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** سؤال مقترح : If A, B, C, D, are inv + same size, prove that $(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$ (بنفس الطريقة)

D = AB
E = CD
→ (defined because, A, B, C, D are the same size + square, according to the Theorem 1.4.6)

$D^{-1} = B^{-1}A^{-1}$

$E^{-1} = D^{-1}C^{-1}$

Then $(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$ #

* if A is inv. Then $A^2, A^3, \ldots A^n$ must be inv. according to the Theorem 1.4.6 (صح)

-if A is inv. Prove that A^{-3} inv. too ; سؤال امتحان

$(A^3)^{-1} = A^{-1}.A^{-1}.A^{-1}$ According to the Theorem 1.4.6 (A^{-3}) is inv. too (because A is inv.)

Theorem 1.4.8 (Laws of Exponents)

- If A is an invertible matrix, then:
  - A^{-1} is invertible and $(A^{-1})^{-1} = A$
  - A^n is invertible and $(A^n)^{-1} = (A^{-1})^n$ for $n = 0, 1, 2, \ldots$
  - For any nonzero scalar k, the matrix kA is invertible and $(kA)^{-1} = (1/k)A^{-1}$

انباتهما

انباتهما

1-4 Example 8

- Powers of matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

* خطأ فادح اكعب واربع كل عنصر بال matrix

- $A^3 = ?$ $A.A.A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}.\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}.\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$
- $A^{-3} = ?$

→ according to the Theorem 1.4.6 $(KA)^{-1} = K^{-1}A^{-1}$ #

اثبات آخر نقطة من Theorem 1.4.8

* if A is inv. profe that A^{-1} is inv too.

A inv. → square → so that A^{-1} is square too
→ det ≠ 0

$\det(A) \neq 0 \rightarrow \det(A^{-1}) = \frac{1}{\det(A)} \rightarrow$ so $\det(A^{-1}) \neq 0$

Then A^{-1} is inv too ←

* If A is inv., profe that KA is inv. and $(KA)^{-1} = (1/K)A^{-1}$. (K is nonzero scaler)

$(KA) * (1/K)A^{-1} = (K * \frac{1}{K})AA^{-1} = I$

so that KA is inv. and it's inverse $(1/K)A^{-1}$ #

* Note:

$\det(AB) = \det(A) * \det(B)$

$\det(A+B) \neq \det(A) + \det(B)$ (غلط جزما)

$\det(2A) = 2\det(A)$

$\det(A*A) = (\det(A))^2$

If AB is invartible matrix, A, and B is invartible matrices too.

If the statment is always correct?! No

AB غير صحيحة لو اذا كانت (are square

A_{2*3} B_{3*2} AB_{2*2} (inv.)

1-4 Polynomial Expressions Involving Matrices

- If A is a square matrix, say $m \times m$, and if
$$p(x) = a_0 + a_1x + \ldots + a_nx^n$$
is any polynomial, then we define
$$p(A) = a_0I + a_1A + \ldots + a_nA^n$$
where I is the $m \times m$ identity matrix.
- That is, $p(A)$ is the $m \times m$ matrix that results when A is substituted for x in the above equation and a_0 is replaced by a_0I

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1-4 Example 9 (Matrix Polynomial)

If
$$p(x) = 2x^2 - 3x + 4 \text{ and } A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$$
then
$$p(A) = 2A^2 - 3A + 4I = 2\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}^2 - 3\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 8 \\ 0 & 18 \end{bmatrix} - \begin{bmatrix} -3 & 6 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 0 & 13 \end{bmatrix}$$

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Theorems 1.4.9 (Properties of the Transpose)

- If the sizes of the matrices are such that the stated operations can be performed, then
  - $((A^T)^T = A$
  - $(A + B)^T = A^T + B^T$ and $(A - B)^T = A^T - B^T$
  - $(kA)^T = kA^T$, where k is any scalar
  - $(AB)^T = B^TA^T$

* سؤال مقترح: prove that $(ABC)^T = C^TB^TA^T$

$AB = D \rightarrow D^T = B^TA^T$

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$(DC)^T = C^TD^T = C^TB^TA^T$

according to the Theorem 1.4.9

**

$A_{m\times n} / B_{m\times n} \rightarrow (A+B)_{m\times n}$

$(A+B)^T_{n\times m}$

$A^T_{n\times m} / B^T_{n\times m} \rightarrow A^T_{n\times m} + B^T_{n\times m}$

If orginal matrix is inv., show that → after row operations, the matrix will result inv. too

* after row operations will reduce equavilent matrix

** سؤال بوظب

$\det(A) = \det(A^T) \neq 0$

A is a square matrix

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \qquad A^T = \begin{bmatrix} a_1 & a_4 & a_7 \\ a_2 & a_5 & a_8 \\ a_3 & a_6 & a_9 \end{bmatrix}$$

(A inv.) → so that (A^T is square too)

A^T → square + $\det(A^T) \neq 0$ → is inv too

✻

Theorem 1.4.10 (Invertibility of a Transpose)

- If A is an invertible matrix, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$

- Example 10

$$A = \begin{bmatrix} -5 & -3 \\ 2 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} -5 & 2 \\ -3 & 1 \end{bmatrix}$$

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always inv. because it produced by Identity matrix

1-5 Elementary Row Operation

- An elementary row operation (sometimes called just a row operation) on a matrix A is any one of the following three types of operations:
  - Interchange of two rows of A
  - Replacement of a row $\mathbf{r}$ of A by $c\mathbf{r}$ for some number $c \neq 0$
  - Replacement of a row $\mathbf{r}_1$ of A by the sum $\mathbf{r}_1 + c\mathbf{r}_2$ of that row and a multiple of another row $\mathbf{r}_2$ of A

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1-5 Elementary Matrix

- An $n \times n$ elementary matrix is a matrix produced by applying exactly one elementary row operation to I_n
  - E_{ij} is the elementary matrix obtained by interchanging the i-th and j-th rows of I_n
  - $E_i(c)$ is the elementary matrix obtained by multiplying the i-th row of I_n by $c \neq 0$
  - $E_{ij}(c)$ is the elementary matrix obtained by adding c times the j-th row to the i-th row of I_n, where $i \neq j$

✻ $E_3(1)$ not elemetary ✻

E_{13} → يعني تبديل بين ال (row1 /row3)

$E_2(4)$ → ضرب ال (row2) بـ (4) ← $4R_2$

$E_{12}(3)$ → $3R_2 + R_1$ ضرب ال (row2) بـ (3) وجمعه مع (Row1) وتخزين القيمة بـ (row1)

1-5 Example 1

- Elementary Matrices and Row Operations

Listed below are four elementary matrices and the operations that produce them.

$R_2(-3)$ $\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$ — Multiply the second row of I_2 by -3.

$R_2(4)$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ — Interchange the second and fourth rows of I_4.

$R_{L3}(3)$ $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ — Add 3 times the third row of I_3 to the first row.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ — Multiply the first row of I_3 by 1.

1-5 Elementary Matrices and Row Operations

- Theorem 1.5.1
  - Suppose that E is an $m \times m$ elementary matrix produced by applying a particular elementary row operation to I_m, and that A is an $m \times n$ matrix. Then EA is the matrix that results from applying that same elementary row operation to A

EX : $A = \begin{bmatrix} 4 & 1 \\ 5 & 4 \\ 2 & 0 \end{bmatrix}$ (3*2)

$(E_{13}) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (3*3)

$E_{13} * A = \begin{bmatrix} 2 & 0 \\ 5 & 4 \\ 4 & 1 \end{bmatrix}$

* نستنتج ان تم تبديل السطر الثالث بالاول في المصفوفة الناتجة

** (العملية التي تتم على E تتم على A :)

1-5 Example 2 (Using Elementary Matrices)

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$$

and consider the elementary matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

which results from adding 3 times the first row of I_3 to the third row. The product EA is

$$EA = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix}$$

which is precisely the same matrix that results when we add 3 times the first row of A to the third row. ♦

1-5 Inverse Operations

- If an elementary row operation is applied to an identity matrix I to produce an elementary matrix E, then there is a second row operation that, when applied to E, produces I back again

Row operation on I That produces E	Row operation on E That produces I
Multiply row i by c≠0	Multiply row i by 1/c
Interchange row i and j	Interchange row i and j
Add c times row i to row j	Add -c times row i to row j

$E_{13} * E_{31} = I$

$E_2(4) * E_2(\frac{1}{4}) = I$

$E_{12}(3) * E_{12}(-3) = I$

1-5 Inverse Operations

- Examples

$$\begin{bmatrix}1&0\\0&1\end{bmatrix} \longrightarrow \begin{bmatrix}1&0\\0&7\end{bmatrix} \longrightarrow \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

↑ Multiply the second row by 7 ↑ Multiply the second row by $\frac{1}{7}$

$$\begin{bmatrix}1&0\\0&1\end{bmatrix} \longrightarrow \begin{bmatrix}0&1\\1&0\end{bmatrix} \longrightarrow \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

↑ Interchange the first and second rows ↑ Interchange the first and second rows

$$\begin{bmatrix}1&0\\0&1\end{bmatrix} \longrightarrow \begin{bmatrix}1&5\\0&1\end{bmatrix} \longrightarrow \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

↑ Add 5 times the second row to the first ↑ Add −5 times the second row to the first

◆

Theorem 1.5.2

invertible

- Elementary Matrices and Nonsingularity
  - Each elementary matrix is nonsingular, and its inverse is itself an elementary matrix. More precisely,
  - $E_{ij}^{-1} = E_{ji}\ (= E_{ij})$
  - $E_i(c)^{-1} = E_i(1/c)$ with $c \neq 0$
  - $E_{ij}(c)^{-1} = E_{ij}(-c)$ with $i \neq j$

Theorem 1.5.3(Equivalent Statements)

- If A is an $n \times n$ matrix, then the following statements are equivalent, that is, all true or all false
  - A is invertible
  - $Ax = 0$ has only the trivial solution ⟶ $A^{-1}AX = A^{-1}0$
  - The reduced row-echelon form of A is I_n
  - A is expressible as a product of elementary matrices

$X = 0$

A inv. اذا ال

I = لها r.r.ef دائماً

عبارة عن elementary حضروبين ببعض ال (A)
matrices

$$(E_4E_3E_2E_1A)^{-1} = A^{-1}E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} = I$$

$$A^{-1} = IE_4E_3E_2E_1 \ *$$

* لو لخدت ال inverse لـ وضربتها row operations بال (I) بترد تعطيني ال (matrix A)

1-5 A Method for Inverting Matrices

- To find the inverse of an invertible matrix A, we must find a sequence of elementary row operations that reduces A to the identity and then perform this same sequence of operations on I_n to obtain A^{-1}

- Remark
  - Suppose we can find elementary matrices $E_1, E_2, \ldots, E_k$ such that
    $$E_k \ldots E_2 E_1 A = I_n$$
    then
    $$A^{-1} = E_k \ldots E_2 E_1 I_n$$

* برهان

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$$A\bar{A}^{-1} = I \rightarrow \det(A\bar{A}^{-1}) = \det(I)$$

$$\det(A).\det\bar{A}^{-1}) = \det(I) = 1$$

then $\det(\bar{A}^{-1}) = \frac{1}{\det(A)}$

(طريقة لايجاد A^{-1})

1-5 Example 4
(Using Row Operations to Find A^{-1})

- Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

- Solution:
  - To accomplish this we shall adjoin the identity matrix to the right side of A, thereby producing a matrix of the form $[A \mid I]$
  - We shall apply row operations to this matrix until the left side is reduced to I; these operations will convert the right side to A^{-1}, so that the final matrix will have the form $[I \mid A^{-1}]$

1-5 Example 4

The computations are as follows:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] \leftarrow \text{We added } -2 \text{ times the first row to the second and } -1 \text{ times the first row to the third.}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \leftarrow \text{We added 2 times the second row to the third.}$$

1-5 Example 4 (continue)

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \leftarrow \text{We multiplied the third row by } -1.$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \leftarrow \text{We added 3 times the third row to the second and } -3 \text{ times the third row to the first.}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \leftarrow \text{We added } -2 \text{ times the second row to the first.}$$

Thus,

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

1-5 Example 5

- Consider the matrix

$$A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$$

Apply the procedure of example 4 to find A^{-1}

A not inv.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{11}{4} & -\frac{1}{2} & \frac{3}{4} & 0 \\ 0 & 1 & \frac{9}{8} & \frac{2}{8} & -\frac{1}{8} & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

↓

اذا لم ينتج الطرف الايسر I

A not inv. ∴

dependant → not inv.

(single solution)

~~(Back sub.) واستطيع ايجاد الحلول باستخدام~~

1-5 Example 6

- According to example 4, A is an invertible matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

- $\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_1 + 5x_2 + 3x_3 &= 0 \\ x_1 + \quad + 8x_3 &= 0 \end{aligned}$ has only trivial solution

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* حتى لو غيرت الـ (right hand side) بضل الـ (system has single sol) وان كانت (A inv) بشو ما احط قيم بضل " single "

Theorems 1.6.1

- Every system of linear equations has either no solutions, exactly one solution, or in finitely many solutions.

Theorem 1.6.2

Single Solu.

- If A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix $\mathbf{b}$, the system of equations $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely, $\mathbf{x} = A^{-1}\mathbf{b}$.

* اذا كانت A inv. ← (has only single Sol.)

اما $AX = b \leftarrow X = A^{-1}b$ ← (non trivial)

او $AX = 0 \leftarrow X = A^{-1}0 \leftarrow X = 0$

(trivial solu.)

Remark :

ممنوع اضرب بهذه الطريقة

$AXA^{-1} = bA^{-1}$ (not defined) X

Example

$$A=\begin{bmatrix}3&1&0\\2&4&5\\0&1&2\end{bmatrix}\quad b_1=\begin{bmatrix}4\\10\\2\end{bmatrix}\quad b_2=\begin{bmatrix}5\\2\\0\end{bmatrix}\rightarrow\left[\begin{array}{ccc|c|c}3&1&0&4&5\\2&4&5&10&2\\0&1&2&2&0\end{array}\right]$$

$$\Rightarrow\left[\begin{array}{ccc|c|c}1&\frac{1}{3}&0&\frac{4}{3}&\frac{5}{3}\\0&\frac{-8}{3}&5&\frac{22}{3}&\frac{-4}{3}\\0&1&2&2&0\end{array}\right]\Rightarrow\left[\begin{array}{ccc|c|c}1&\frac{1}{3}&0&\frac{4}{3}&\frac{5}{3}\\0&1&\frac{-15}{8}&\frac{-11}{4}&\frac{1}{2}\\0&1&2&2&0\end{array}\right]\Rightarrow\left[\begin{array}{ccc|c|c}1&0&\frac{5}{8}&\frac{9}{4}&\frac{3}{2}\\0&1&\frac{-15}{8}&\frac{-11}{4}&\frac{1}{2}\\0&0&\frac{31}{8}&\frac{19}{4}&\frac{-1}{2}\end{array}\right]$$

$$\Rightarrow\left[\begin{array}{ccc|c|c}1&0&\frac{5}{8}&\frac{9}{4}&\frac{3}{2}\\0&1&\frac{-15}{8}&\frac{-11}{4}&\frac{1}{2}\\0&0&1&\frac{38}{31}&\frac{-4}{31}\end{array}\right]\Rightarrow\left[\begin{array}{ccc|c|c}1&0&0&4.21&1.29\\0&1&0&-3.4&0.5\\0&0&1&1.23&0.129\end{array}\right]$$

x_1 x_2

1-6 Example 1

Consider the system of linear equations

$$\begin{aligned}x_1+2x_2+3x_3&=5\\2x_1+5x_2+3x_3&=3\\x_1\qquad+8x_3&=17\end{aligned}$$

In matrix form this system can be written as $A\mathbf{x}=\mathbf{b}$, where

$$A=\begin{bmatrix}1&2&3\\2&5&3\\1&0&8\end{bmatrix},\quad \mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix},\quad \mathbf{b}=\begin{bmatrix}5\\3\\17\end{bmatrix}$$

In Example 4 of the preceding section we showed that A is invertible and

$$A^{-1}=\begin{bmatrix}-40&16&9\\13&-5&-3\\5&-2&-1\end{bmatrix}$$

By Theorem 1.6.2 the solution of the system is

$$\mathbf{x}=A^{-1}\mathbf{b}=\begin{bmatrix}-40&16&9\\13&-5&-3\\5&-2&-1\end{bmatrix}\begin{bmatrix}5\\3\\17\end{bmatrix}=\begin{bmatrix}1\\-1\\2\end{bmatrix}$$

or $x_1=1, x_2=-1, x_3=2$.

1-6 Linear Systems with a Common Coefficient Matrix

- To solve a sequence of linear systems, $A\mathbf{x}=\mathbf{b}_1$, $A\mathbf{x}=\mathbf{b}_2$, …, $A\mathbf{x}=\mathbf{b}_k$, with common coefficient matrix A
- If A is invertible, then the solutions $\mathbf{x}_1=A^{-1}\mathbf{b}_1$, $\mathbf{x}_2=A^{-1}\mathbf{b}_2$, …, $\mathbf{x}_k=A^{-1}\mathbf{b}_k$
- A more efficient method is to form the matrix $[A|\mathbf{b}_1|\mathbf{b}_2|\ldots|\mathbf{b}_k]$
- By reducing it to reduced row-echelon form we can **solve all k systems at once** by Gauss-Jordan elimination.

(Back.Sub باستخدام inv. واذا لم تكن)

لحل نحله $[A|b_1]$ لايجاد الحل الاول ايضاً $[A|b_2]$ لايجاد الحل الثاني , او نستخدم الصيغة $[A|b_1|b_2]$ لايجاد الحلول معًا باستخدام (G.J.E.M)

(method 1)

(method 2) (اذا A^{-1} موجودة يفضل هذه الطريقة اسرع) $x_1=A^{-1}b_1$ $x_2=A^{-1}b_2$ او

** مع ملاحظة ان الطريقة الاولى بدون اي شروط بينما الطريقة الثانية يجب ان تكون (A inv.)

1-6 Example 2

- Solve the system

$$\begin{aligned}x_1+2x_2+3x_3&=4\\2x_1+5x_2+3x_3&=5\\x_1\qquad+8x_3&=9\end{aligned}\qquad\begin{aligned}x_1+2x_2+3x_3&=1\\2x_1+5x_2+3x_3&=6\\x_1\qquad+8x_3&=-6\end{aligned}$$

Theorems 1.6.3

- Let A be a square matrix
  - If B is a square matrix satisfying $BA=I$, then $B=A^{-1}$
  - If B is a square matrix satisfying $AB=I$, then $B=A^{-1}$

سؤال امتحان :

If $(A^2-2A+I=0)$, prove that A is inv. and find the inverse of A ?!

$$I=-A^2+2A=A(2-A)$$ *

* since the multiplication of (A) by (A^{-1}) gives I, Then A is inv. and It's inverse equal (2-A)

Theorem 1.6.4 (Equivalent Statements)

- If A is an $n \times n$ matrix, then the following statements are equivalent
  - A is invertible
  - $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
  - The reduced row-echelon form of A is I_n
  - A is expressible as a product of elementary matrices
  - $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$
  - $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$

مهم جدًا

{ هواي بجرد (right hand side) ← A inv لم تكن اذا * ← (always) has single sol. ← A inv اذا ال *

Theorem 1.6.5

- Let A and B be square matrices of the same size. If AB is invertible, then A and B must also be invertible.

- Let A be a fixed $m \times n$ matrix. Find all $m \times 1$ matrices b such that the system of equations $Ax=b$ is consistent.

AB is inv. → AB square
→ $\det(AB) \neq 0$

Summary

(مهم جدًا جدًا)

A inv → has single solu. (بغض النظر عن ال R.H.S)

A not inv → According to the right side

so that A, and B are squre

and $\det(AB) = \det(A) \cdot \det(B) \neq 0$

Then A and B are inv. *

→ inconsistent (L.H.S same, R.H.S different)

→ consistent with ∞ * of solution (one of rows or columns dependant on another)

1-6 Example 3

- Find b_1, b_2, and b_3 such that the system of equations is consistent.

$$x_1 + x_2 + 2x_3 = b_1$$
$$x_1 + \quad + x_3 = b_2$$
$$2x_1 + x_2 + 3x_3 = b_3$$

* نجرّ فيجا اذا ال system
نجد ال det ← (inv. or not)

R_3 dependant on ← not inv. ← system ال *
R_1 and R_2 ($R_3 = R_2 + R_1$)

* اذن مستحيل اخليك ال consistent with ← مستحيل يكون
∞ * of solu.

single

1-6 Example 4

- Find b_1, b_2, and b_3 such that the system of equations is consistent.

$$x_1 + 2x_2 + 3x_3 = b_1$$
$$2x_1 + 5x_2 + 3x_3 = b_2$$
$$x_1 \quad + 8x_3 = b_3$$

لانّ $\det = 0$ (not inv.)

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \rightarrow 1(-1) - 1(1) + 2(1) = 0$$

$\det = 0$ (sys. not inv.)

* اذن لو جعلنا $b_3 = b_2 + b_1$ ايضاً بستنتج (consistent with ∞)

* لو طلبو inconsistent ← بخلي ال R.H.S اي اشي ماعدا الشرط السابق
($b_3 \neq b_2 + b_1$)

inconsistant → بخلي معادلتين بالطرف اليسار نفس المعادلات والطرف اليمين معادلات مختلفة
(اشي مش منطقي) او $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 4 \end{array}\right]$

consistant → اما الصفوف او الاعمدة صفار
او 7 صف يعتمد على صف آخر او عمود يعتمد على عمود آخر (نفس المعادلات بالطرفين) او من مضاعفاتهم او كما في المثال $d_3 = d_1 + d_2$

one matrix + r.r.e → not diagonal (حتى لو square)

r.r.e.f → diagonal (if it is square)

zero matrix → diagonal (if it is square)

I → always diagonal

↑

* $\det(D) = d_1 d_2 d_3$

* اذا طلب مني (D^2) بقدر اربع الـ entries دغري

** (diagonal بال وسط)

1-7 Diagonal Matrix

- A square matrix A is $m \times n$ with $m = n$; the (i,j)-entries for $1 \le i \le m$ form the main diagonal of A
- A diagonal matrix is a square matrix all of whose entries *not* on the main diagonal equal zero. By $\text{diag}(d_1, \ldots, d_m)$ is meant the $m \times m$ diagonal matrix whose (i,i)-entry equals d_i for $1 \le i \le m$

* كل العناصر الي مش بال diagonal لازم صفر

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1-7 Properties of Diagonal Matrices

- A general $n \times n$ diagonal matrix D can be written as $D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$
- A diagonal matrix is invertible if and only if all of its diagonal entries are nonzero $D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix}$
- Powers of diagonal matrices are easy to compute $D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$

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$D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$ ، diagonal matrix be inv. if and only if (entries on the main diagonal not equal zero)

* ملاحظة مهمة :

$\det \neq 0$ $d_1 \neq 0$ $d_2 \neq 0$ $d_3 \neq 0$ (cause it already square)

I → (always both)

zero matrix → both (diagonal) (if it is square)

r.e.f → upper (if it's square)

r.r.e.f → both (if it is square)

ones matrix → none

1-7 Properties of Diagonal Matrices

- Matrix products that involve diagonal factors are especially easy to compute

1. $\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_1a_{12} & d_1a_{13} & d_1a_{14} \\ d_2a_{21} & d_2a_{22} & d_2a_{23} & d_2a_{24} \\ d_3a_{31} & d_3a_{32} & d_3a_{33} & d_3a_{34} \end{bmatrix}$

2. $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_2a_{12} & d_3a_{13} \\ d_1a_{21} & d_2a_{22} & d_3a_{23} \\ d_1a_{31} & d_2a_{32} & d_3a_{33} \\ d_1a_{41} & d_2a_{42} & d_3a_{43} \end{bmatrix}$

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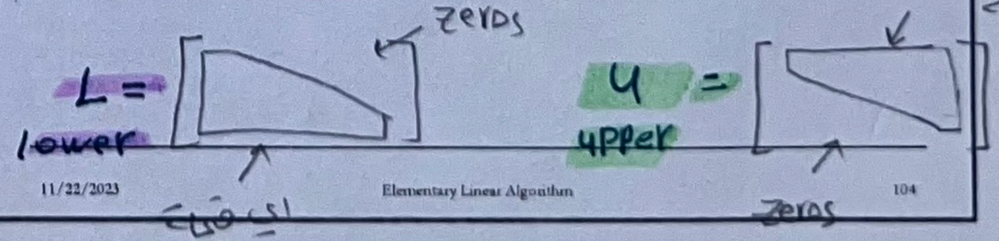
1-7 Triangular Matrices

- A $m \times n$ lower-triangular matrix L satisfies $(L)_{ij} = 0$ if $i < j$, for $1 \le i \le m$ and $1 \le j \le n$
- A $m \times n$ upper-triangular matrix U satisfies $(U)_{ij} = 0$ if $i > j$, for $1 \le i \le m$ and $1 \le j \le n$
- A unit-lower (or –upper)-triangular matrix T is a lower (or upper)-triangular matrix satisfying $(T)_{ii} = 1$ for $1 \le i \le \min(m,n)$



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1 لحتى اضرب (D * matrix) الصف الاول بـ (d_1) ، الثاني بـ (d_2) ، الثالث بـ (d_3) .....

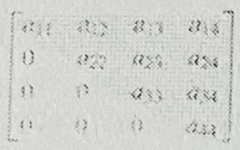
2 لحتى اضرب الـ (matrix * D) العمود الاول بـ (d_1) ، الثاني بـ (d_2) ، الثالث بـ (d_3) ... .

- upper * Lower = matrix (عادية)
- upper * upper = upper
- Lower * Lower = Lower

$$A = \begin{bmatrix} a_1 & 0 & 0 \\ a_4 & a_2 & 0 \\ a_5 & a_6 & a_3 \end{bmatrix} \rightarrow \text{Lower}$$

$$A^T = \begin{bmatrix} a_1 & a_4 & a_5 \\ 0 & a_2 & a_6 \\ 0 & 0 & a_3 \end{bmatrix} \rightarrow \text{upper}$$

1-7 Example 2 (Triangular Matrices)



- The diagonal matrix
  - both upper triangular and lower triangular
- A square matrix in row-echelon form is upper triangular

Theorem 1.7.1

- The transpose of a lower triangular matrix is upper triangular, and the transpose of an upper triangular matrix is lower triangular
- The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular
- A triangular matrix is invertible if and only if its diagonal entries are all nonzero
- The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular

$$A = \begin{bmatrix} a_1 & 0 & 0 \\ a_4 & a_2 & 0 \\ a_5 & a_6 & a_3 \end{bmatrix}$$

(triangular matrix) تفرض

$$\det(A) = a_1 \begin{vmatrix} a_2 & 0 \\ a_6 & a_3 \end{vmatrix} - 0 \begin{vmatrix} a_4 & 0 \\ a_5 & a_3 \end{vmatrix} + 0 \begin{vmatrix} a_4 & a_2 \\ a_5 & a_6 \end{vmatrix}$$

$$= a_1 a_2 a_3$$

lower ← lower ∩ inverse

upper ← upper ∩ inverse

* triangular matrix always square and it's inv. if and only if $(a_1 / a_2 / a_3) \neq 0$ nonzeros element

1-7 Example 3

- Consider the upper triangular matrices

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

A^{-1} (upper) B not inv. det = 0

1-7 Symmetric Matrices

- A (square) matrix A for which $A^T = A$, so that $\langle A \rangle_{ij} = \langle A \rangle_{ji}$ for all i and j, is said to be symmetric.
- Example 4

$$\begin{bmatrix} 7 & -3 \\ -3 & 7 \end{bmatrix} \quad \begin{bmatrix} 1 & 4 & 5 \\ 4 & -3 & 0 \\ 5 & 0 & 7 \end{bmatrix} \quad \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$$

* اذا كانت A ← symmetric ← (A^{-1} always symmetric) ← and $A = A^T$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} a_1 & a_4 & a_5 \\ a_4 & a_2 & a_6 \\ a_5 & a_6 & a_3 \end{bmatrix} \qquad A^T = \begin{bmatrix} a_1 & a_4 & a_5 \\ a_4 & a_2 & a_6 \\ a_5 & a_6 & a_3 \end{bmatrix}$$

* اذا كانت A ← symmetric ← $(A^{-1})^T \neq A^{-1}$

** لو طلبو اثباتها (بالرموز)

Theorem 1.7.2

- If A and B are symmetric matrices with the same size, and if k is any scalar, then
  - A^T is symmetric
  - $A + B$ and $A - B$ are symmetric
  - kA is symmetric
- Remark
  - The product of two symmetric matrices is symmetric if and only if the matrices commute, i.e., $AB = BA$
- Example 5

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -4 & 3 \\ 3 & -1 \end{bmatrix}$$

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اثباتها

$$A = \begin{bmatrix} a_1 & a_3 \\ a_3 & a_2 \end{bmatrix} \qquad B = \begin{bmatrix} b_1 & b_3 \\ b_4 & b_2 \end{bmatrix}$$

$$AB = \begin{bmatrix} a_1b_1 + a_3b_3 & a_1b_3 + a_3b_2 \\ a_3b_1 + a_2b_3 & a_3b_3 + a_2b_2 \end{bmatrix}$$

$$BA = \begin{bmatrix} b_1a_1 + b_3a_3 & b_1a_3 + b_3a_2 \\ b_3a_1 + b_2a_3 & b_3a_3 + b_2a_2 \end{bmatrix}$$

$\langle AB \rangle_{12} = \langle BA \rangle_{21}$

$\langle AB \rangle_{12} = \langle AB \rangle_{12}$

Theorem 1.7.3

- If A is an invertible symmetric matrix, then A^{-1} is symmetric.
- Remark:
  - In general, a symmetric matrix needs not be invertible. →
  - The products AA^T and A^TA are always symmetric

* بغض النظر عن الـ size

اثباتها

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معنى شرط كل sym. matrix تكون inv

$$A_{2*3} = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \qquad A^T_{3*2} = \begin{bmatrix} a_1 & a_4 \\ a_2 & a_5 \\ a_3 & a_6 \end{bmatrix}$$

$$A * A^T \ (2\times2) = \begin{bmatrix} a_1^2 + a_2^2 + a_3^2 & a_1a_4 + a_2a_5 + a_3a_6 \\ a_1a_4 + a_2a_5 + a_6a_3 & a_4^2 + a_5^2 + a_6^3 \end{bmatrix}$$

$$A^TA \ (3\times3) = \begin{bmatrix} a_1^2 + a_4^2 & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

1-7 Example 6

Let A be the 2×3 matrix

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}$$

Then

$$A^TA = \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -2 & -11 \\ -2 & 4 & -8 \\ -11 & -8 & 41 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 21 & -17 \\ -17 & 34 \end{bmatrix}$$

Observe that A^TA and AA^T are symmetric as expected.

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Theorem 1.7.4 (symmetric) A^TA ونكمل الاول والتابع

- If A is an invertible matrix, then AA^T and A^TA are also invertible

ملاحظة مهمة: $AA^T \neq A^TA$

* سؤال امتحان:

If the following statment is always correct ?! $AA^T = A^TA$

not correct →

الا اذا كانت A ← symmetric الاساس (مهم جدًا)

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* diagonal matrices → always symmetric

* zero + ones matrices → symmetric if they are square

* triangular aren't symmetric unless they are diagonal

اذا كانت الـ (matrix A) ← Symmetric

$AA^T = A^TA = A^2$

$(A = A^T)$

اثباتها بأول الصفحة

Q1 : prove that, if A is upper matrix, then A^{-1} is upper matrix too.

(النقطة الأخيرة من Theorem 1.7.1)

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{bmatrix}$$

$$\det(A) = a_1 a_4 a_6$$

$$C = \begin{bmatrix} a_4a_6 & 0 & 0 \\ -a_2a_6 & a_1a_6 & 0 \\ a_2a_5 - a_3a_4 & -a_1a_5 & a_1a_4 \end{bmatrix}$$

$$adj = \begin{bmatrix} a_4a_6 & -a_2a_6 & a_2a_5 - a_3a_4 \\ 0 & a_1a_6 & -a_1a_5 \\ 0 & 0 & a_1a_4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} * adj \longrightarrow \begin{bmatrix} \frac{1}{a_1} & \frac{-a_2}{a_1a_4} & \frac{a_2a_5 - a_3a_4}{a_1a_4a_6} \\ 0 & \frac{1}{a_4} & \frac{-a_5}{a_4a_6} \\ 0 & 0 & \frac{1}{a_6} \end{bmatrix}$$

A^{-1} upper ✻

Q2 : prove that, AB will result ordinary matrix . if A (upper) , B (Lower)

(سؤال مقترح)

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{bmatrix} \qquad B = \begin{bmatrix} b_1 & 0 & 0 \\ b_4 & b_2 & 0 \\ b_5 & b_6 & b_3 \end{bmatrix}$$

$$AB_{3*3} = \begin{bmatrix} a_1b_1 + a_2b_4 + a_3b_5 & a_2b_2 + a_3b_6 & a_3b_3 \\ a_4b_1 + a_5b_4 & a_4b_2 + a_2b_6 & a_5b_3 \\ a_6b_5 & a_6b_6 & a_6b_3 \end{bmatrix}$$ ✻

Q3 : prove that, if A is inv. Then A^T is inv too.

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \qquad A^T = \begin{bmatrix} a_1 & a_4 & a_7 \\ a_2 & a_5 & a_8 \\ a_3 & a_6 & a_9 \end{bmatrix}$$

$$a_1(a_5a_9 - a_6a_8) - a_2(a_4a_9 - a_6a_7) + a_3(a_4a_8 - a_5a_7) \neq 0 \quad (A \text{ inv})$$

$$a_1(a_5a_9 - a_6a_8) - a_4(a_2a_9 - a_3a_8) + a_7(a_2a_6 - a_3a_5)$$

$$\det(A) = \det(A^T) \neq 0$$ So That A^T is inv too ✻✻

Q4: prove that, if A and B dre symmetric, A+B is symmetric too.

$$A = \begin{bmatrix} a_1 & a_4 & a_5 \\ a_4 & a_2 & a_6 \\ a_5 & a_6 & a_3 \end{bmatrix} \qquad B = \begin{bmatrix} b_1 & b_4 & b_5 \\ b_4 & b_2 & b_6 \\ b_5 & b_6 & b_3 \end{bmatrix}$$

$$A+B = \begin{bmatrix} a_1+b_1 & a_4+b_4 & a_5+b_5 \\ a_4+b_4 & a_2+b_2 & a_6+b_6 \\ a_5+b_5 & a_6+b_6 & b_3+b_3 \end{bmatrix}$$ ✻

Q5: prove that, the inverse of the diagonal matrix will be like:

$$A^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & 0 \\ 0 & \frac{1}{d_2} & 0 \\ 0 & 0 & \frac{1}{d_3} \end{bmatrix} \rightarrow A = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \quad \det(A) = d_1 d_2 d_3$$

$$M = \begin{bmatrix} d_2d_3 & 0 & 0 \\ 0 & d_1d_3 & 0 \\ 0 & 0 & d_1d_2 \end{bmatrix} \qquad C = \begin{bmatrix} d_2d_3 & 0 & 0 \\ 0 & d_1d_3 & 0 \\ 0 & 0 & d_1d_2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} * adj \qquad \text{(Theorem 2.1.2)}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & 0 \\ 0 & \frac{1}{d_2} & 0 \\ 0 & 0 & \frac{1}{d_3} \end{bmatrix}$$ ✻

Q6: Prove that, $\det(KA) = K^n . \det(A)$

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$$

⋮

+ بجيب ال det(A) وبعدين بضربها بـ K

وبرد احسب ال det(KA)

* Cofactor نفس الـ Minor لكن اشارته تعتمد على قيمة (i and j)

$(-1)^{i+j} * M_{ij}$

2-1 Minor and Cofactor (square)

- Definition
  - Let A be $m \times n$
    - The (i,j)-minor of A, denoted M_{ij} is the determinant of the $(n-1) \times (n-1)$ matrix formed by deleting the ith row and jth column from A
    - The (i,j)-cofactor of A, denoted C_{ij}, is $(-1)^{i+j} M_{ij}$
- Remark
  - Note that $C_{ij} = \pm M_{ij}$ and the signs $(-1)^{i+j}$ in the definition of cofactor form a checkerboard pattern:

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

← او هذه الطريقة اسهل

2-1 Example 1

- Let $A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$
- The minor of entry a_{11} is $M_{11} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 16$
- The cofactor of a_{11} is $C_{11} = (-1)^{1+1} M_{11} = M_{11} = 16$
- Similarly, the minor of entry a_{32} is $M_{32} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 26$
- The cofactor of a_{32} is $C_{32} = (-1)^{3+2} M_{32} = -M_{32} = -26$

2-1 Cofactor Expansion

- The definition of a 3×3 determinant in terms of minors and cofactors
  - $\det(A) = a_{11}M_{11} + a_{12}(-M_{12}) + a_{13}M_{13}$
    $= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$
  - this method is called cofactor expansion along the first row of A
- Example 2

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix} = 3(-4) - (1)(-11) + 0 = -1$$

** اذا ضربت اي صف من (A) بالصف المقابل لهما من ال C
det (A) = 0

اما اذا ضربت اي صف من ال (A) بما لا يقابلهما من ال C ← det=0

*(صف او عمود)

سؤال امتحان :
$a_{11}C_{31} + a_{12}C_{32} + a_{13}a_{33}$?!
$= 0$

2-1 Cofactor Expansion

- $\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$
  $= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$
  $= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33}$

- Theorem 2.1.1 (Expansions by Cofactors)
  - The determinant of an $n \times n$ matrix A can be computed by multiplying the entries in any row (or column) by their cofactors and adding the resulting products; that is, for each $1 \le i, j \le n$

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$$

(cofactor expansion along the jth column)

and

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$

(cofactor expansion along the ith row)

2-1 Example 3 & 4

- Example 3
  - cofactor expansion along the first column of A

$$\det(A)=\begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5\begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix} = 3(-4) - (-2)(-2) + 5(3) = -1$$

- Example 4
  - smart choice of row or column

$$A=\begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

  - det(A) = ?

2-1 Adjoint of a Matrix

- If A is any $n \times n$ matrix and C_{ij} is the cofactor of a_{ij}, then the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

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والصفوف اعمدة

is called the *matrix of cofactors from A*. The transpose of this matrix is called the *adjoint of A* and is denoted by *adj(A)*

- Remarks
  - If one multiplies the entries in any row by the corresponding cofactors from a different row, the sum of these products is always zero.

same row → det(A)

** to find det(adj) ⟹ $A^{-1} = \frac{1}{\det(A)} * \det(adj)$ according to the

$A^{-1}.\det(A) = adj \rightarrow [\det(A)]^{n}.\det(A^{-1}) = \det(adj)$ Theorem 2.1.2

↳ $\det(A)^{n-1} = \det(adj)$

(n → # of rows or columns (A square))

Theorem 2.1.2
(Inverse of a Matrix using its Adjoint)

- If A is an invertible matrix, then $A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$
  - Show first that

$$A\,\text{adj}(A) = \det(A)\,I$$

$$A\,\text{adj}(A) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{j1} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{j2} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{jn} & \cdots & C_{nn} \end{bmatrix}$$

$$a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \det(A)$$

$$a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn} = 0 (i \neq j)$$

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A upper → A^{-1} upper (لهم الامبات) → Adj upper → transpose of adjoint lower

Theorem 2.1.3

- If A is an n × n triangular matrix (upper triangular, lower triangular, or diagonal), then det(A) is the product of the entries on the main diagonal of the matrix;
  - $\det(A) = a_{11}a_{22}\ldots a_{nn}$
  - E.g.

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$\begin{vmatrix} 2 & 7 & -3 & 8 & 3 \\ 0 & -3 & 7 & 5 & 1 \\ 0 & 0 & 6 & 7 & 6 \\ 0 & 0 & 0 & 9 & 8 \\ 0 & 0 & 0 & 0 & 4 \end{vmatrix} = ?$$

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Theorem 2.1.4 (Cramer's Rule)

- If $A\mathbf{x} = \mathbf{b}$ is a system of n linear equations in n unknowns such that $\det(\lambda I - A) \neq 0$, then the system has a unique solution. This solution is

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \cdots, x_n = \frac{\det(A_n)}{\det(A)}$$

where A_j is the matrix obtained by replacing the entries in the column of A by the entries in the matrix $\mathbf{b} = [b_1 \;\; b_2 \;\; \cdots \;\; b_n]^T$

شرطين : (لاستخدام Cramer's Rule)

(square) # of equation = # of var. ← m=n ①

Single solu. ← inv. ← det ≠ 0 ← all equations ind. ②

2-1 Example 9

- Use Cramer's rule to solve

$$\begin{aligned} x_1 \quad\quad + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

- Since

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}, A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}$$

- Thus,

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \frac{-10}{11}, x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11}, x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11}$$

chapter 1

1. linear equation must be in this form → $a_1x_1 + a_2x_2 + \dots = b$ → (كل الـ variables مرفوعين للقوة 1)

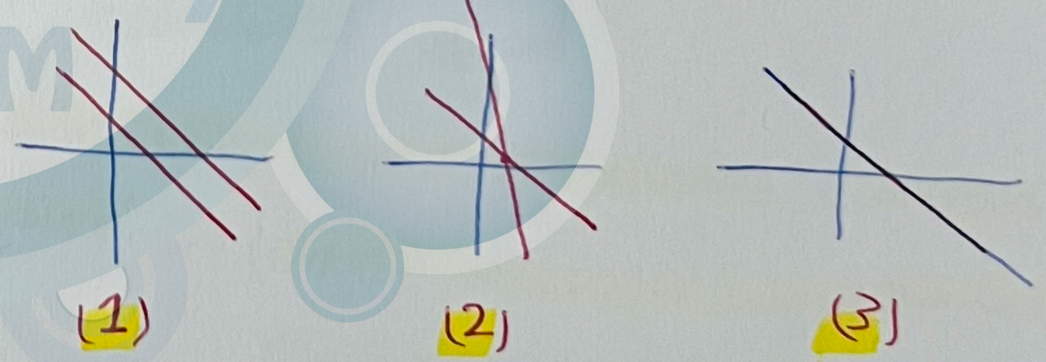
$a_1 / a_2 / a_n / b$ → real numbers / real constant / parameters → (كل الـ linear equ. معرفة بالـ (real space)

$x_1 / x_2 / x_n$ → variables / unknown

2. A linear equations doesn't involve any
   - products
   - roots
   - trigonometric
   - logarithmic
   - exponential

3. SLE
   - consistant → at least one solution
     - singal solution
       - squar (# equ. = # variables) — عدد الصفوف = عدد الأعمدة
       - det ≠ 0 (Equ. are ind) → invertable
     - ∞ # of solutions (general solu.)
   - inconsistant → if it has no solutions

4. A general System of two linear equations
   1. parallel - no solution
   2. only one point is common - one solution
   3. coinside - ∞ # of solutions



5. reduce row-echelon form (r.r.e.f) → (4) ← بجيلك

(row-echelon form)

اذا تحققت اول (٣) شروط تكون r.e.f

١. أول عدد nonzero ولازم يكون واحد (leader)

٢. اذا كان احد الـ rows كله اصفار نضعه بنهاية الـ matrix (dep row)

٣. كل leader يجب ان يكون على يمين الـ leader للصف الذي يسبقه

٤. كل column فيه leader يجب ان تكون باقي العناصر = zeros

6. ملاحظة مهمة

R.R.E.F دائماً (identity + zero matrix) *

لانها تحقق الشروط الاربعة (وبالتالي R.E.F)

not R.R.E.F / not R.E.F ← (ones matrix) *

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

سؤال امتحان ⇐ [0] !!! ⇐ R.R.E.F وبالتالي R.E.F

7 # of free variables = # of variables - # of dependent variables
(independent variables)

# of leaders = # of ind. equ. = # of dep. variables **(مهم)

8 Elimination Methods
- Gaussian elimination → تعطيني R.E.F ونحل (Back-Substitution)بـ
- Gaussian-Jordan elimination → تعطيني R.R.E.F
  ← نفس طريقة الـ G.E لكن بالإضافة الى خطوة اخيرة وهي تصفير باقي العناصر
  لتنتج R.R.E.F ← (leader) المتواجد فيه (column) بال
  ← كيف !! دائماً نضرب الـ (leader) بعكس اشارة العنصر المراد تصفيره ونجمع المعادلتين

9 A homogeneous system of linear equations with more unknown than equations (Theorem 1.2.1)
↳ has ∞ # of solutions (general solution)

* اذا كان الـ homogeneous sys. ← m<n ← ∞ # of solu.

اذا لم يكن homogeneous
- ∞ # of solu
- No solu.

Q1: homogeneous linear system of 3 equations (rows) and 3 unknows (columns)
which of these statements always correct ؟ (single او ∞ اذا كانت هنا لا نستطيع ان نحدد)

a- The system is inconsistent → consistant دائماً homo.sys. خطأ لانه
b- The system has a unique solution (single solu.) → ∞ # of solu. خطأ لانه ممكن ان يعطينا
c- The system has ∞ # of solutions → single solution يعطيني ممكن X اذا كانت احد المعادلات ind.
d- The system can be solved using cramer's Rule → single solu. يكون عندما القاعدة هذه نستخدم

e- none of the presented choices

** اذا كان هناك احد الخيارات (∞ # of solutions or single) يكون الخيار الصحيح
** اذا كان السؤال (4 equations with 3 unknowns) ← دائماً الاجابة (∞ # of solu.) بناءً على (Theorem 1.2.1) الـ

Q2: For a homogeneous system of (4) linear equations and (6) unknowns, which of following statements always correct:

a- The system is inconsistent

b- The system has a unique solution

c- The system has a unique solution or the system is inconsistent

b+ The system has a unique slolution or the system has ∞ # of solutions
(X)

e- None

10 (1-3) Definition

* متى تكون ال Two matrices are equal ← their corresponding entries are equal ← same size

* حاصل ضرب (scaler) بالمصفوفات لا يوجد اي شروط

* يشترط لجمع وطرح ال (matrices) ان تكون (same size)

* يشترط لضرب ال (matrices) $A_{(mxr)} \times B_{(rxn)}$ ← ان تكون عدد الاعمدة بالمصفوفة الاولى = عدد الصفوف بالمصفوفة الثانية (مهم جدًا) **

* Example 6: Determine whether a product is defined Matrices:

A: 3X4 B: 4X7 C: 7X3

AB defined (3X7)

BA (not defined)

AC (not defined)

CA defined (7X4)

BC defined (4X3)

CB (not defined)

(A X BC) defined (7X3)
(7X4) (4X3)

ABC defined (3X3)

(AB) C or A (BC)
(3X7) (7X3) or (3X4) (4X3)

** Final Question: if ABCD = W, and size of A (3X4), C (5X7), W (3X10), Find size of B and C.

A B C D = W ≠
(3X4)(4X6)(5X7)(7X10) (3X10)

* Example 7 :

$$A_{(2\times3)} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \qquad B_{(3\times4)} = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

* ملاحظات مهمة : (وبشرط التأكد من شرط الضرب) **

١- لإيجاد قيمة عنصر محدد بالمصفوفة الناتجة (AB) ⟸ نضرب الصف (from matrix A) من العمود (from matrix B) ونجمع القيم الناتجة

(العنصر الاول بالصف × العنصر الاول بالعمود) + (العنصر الثاني بالصف × العنصر الثاني بالعمود) + ...

$$AB_{(2\times4)} = \begin{bmatrix} a_1b_1 & a_2b_2 & a_3b_3 & a_4b_4 \\ a_5b_5 & a_6b_6 & a_7b_7 & a_8b_8 \end{bmatrix}$$

* مثال : اوجد العنصر (a_7b_7)

$$\begin{bmatrix} 2 & 6 & 0 \end{bmatrix} \times \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

$$= (2\times4) + (6\times3) + (0\times5) = 26$$

٢- لإيجاد قيمة صف كامل بالمصفوفة الناتجة (AB) ⟸ نضرب الصف المطلوب (from matrix A) بكل عمود (from matrix B) كاملة

$$[a_5b_5 \quad a_6b_6 \quad a_7b_7 \quad a_8b_8]$$

* مثال : اوجد قيمة الصف

$$[2 \quad 6 \quad 0] * \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 7 \end{bmatrix} = [8 \quad -4 \quad 26 \quad 12]$$

٣- لإيجاد قيمة عمود كامل بالمصفوفة الناتجة (AB) ⟸ نضرب العمود المطلوب (from matrix B) بكل صف (from matrix A) كاملة

* مثال : اوجد قيمة العمود $\begin{bmatrix} a_2b_2 \\ a_6b_6 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} * \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix} = \begin{bmatrix} 27 \\ -4 \end{bmatrix}$$

٤- لو معك (2 matrices) بجانب بعضها (يجب ان يكون عدد الصفوف نفسه)

فوق بعضها (يجب ان يكون عدد الاعمدة نفسه)

12 Matrix product as linear combinations

1. $A_{(m\times n)} * X_{(n\times 1)}$ — بدلالة الـ (column)
2. $X_{(1\times m)} * A_{(m\times n)}$ — بدلالة الـ (row)

Examples

1. $A_{3\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $\quad X_{3\times 1} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$AX_{(3\times 1)} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

2. $A_{2\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ $\quad X_{1\times 2} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$

$$XA_{(1\times 3)} = x_1 \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Q1: $A_{2\times 2} = \begin{bmatrix} 4 & 5 \\ 6 & x \end{bmatrix}$ $\quad B_{2\times 1} = \begin{bmatrix} x \\ 10 \end{bmatrix}$ $\quad$ Find x? if $AB = \begin{bmatrix} 20 \\ 10 \end{bmatrix}$

$$AB_{2\times 1} = \begin{bmatrix} 4x + 50 \\ 6x + 10x \end{bmatrix} = \begin{bmatrix} 20 \\ 10 \end{bmatrix} \Rightarrow \begin{aligned} 4x + 50 &= 20 \\ 6x + 10x &= 10 \end{aligned}$$

* نجد (x) من المعادلة الاولى و نعوضه بالمعادلة الثانية
  - اذا كانت صحيحة ← المعادلة الثانية dependant
  - اذا لم تحقق المعادلة ← system inconsistant

Q2: Find X? if $AB = \begin{bmatrix} y \\ 10 \end{bmatrix}$

حل معادلتين بمجهولين