Engineering mathmatics

These notes were prepared to help students better understand the course. However, please note that they are not sufficient on their own. It is strongly recommended to practice the suggested questions provided by the instructor to fully grasp the material and prepare well for the exam.

classifications of DEs

equation that at least has one dirrevative

$2y=x \Rightarrow$ not D.E

$2y'=x \Rightarrow$ D.E

$y''+2xy'=e^x \Rightarrow$ Second order D.E

$(x+1)dx=y\,dy$

$\frac{dy}{dx}=\frac{x+1}{y} \Rightarrow$ First order D.E

partial derivatives can not be included here

ordinary differential equation: functions with one independent variable (ODE) يعني ما في partial

order: The highest dirrevative ⇒ مهم

degree: the power of the highest derrivative ⇒ مش موضوعنا هنا

Q1: what is the order of each one of these DE

① $y''+2xy'=x^2$

order: second order

② $y'''+2(y'')^2=e^x$

order: third order

③ $x\,dx-y\,dy=0$

$x\,dx=y\,dy$

$\frac{dy}{dx}=\frac{y}{x}$

First order D.E

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when the DE is linear?

Linearity: the differential equation is linear if it is linear in y & its derivatives

$y''+2x^2y'+e^xy=\sin x$

linear second order

بنشوف ال y والمشتقات إذا كانوا بال linear form معناها linear

$y'''+2x^2y''+y=e^x$

third order linear D.E

يعني بشوف ال y وما بيهمني ال x ⇐ linear ⇐ مش مهم

\# $y''+2xy''+y^2=1$ لانو هون ال derivative بصيغة ال

non-linear

Note

when we have $(y.y') \Rightarrow$ non linear أي تنين y إذا ضربتهم ببعض non-linear

مهمة

\# $y''+yy'=1 \Rightarrow$ second order non linear D.E

\# $y''+\sin(x)y=1 \Rightarrow$ second order linear D.E

\# $y''+\sin(xy)=1 \Rightarrow$ non-linear second order D.E

\# $y'''-2xy'=\sin x \Rightarrow$ linear third order D.E

\# $y''-(y')^2=x \Rightarrow$ second order non linear D.E

\# $y^{(4)}-\sin y=x \Rightarrow$ Non linear 4^{th}-order

\# $y''+yy'=x \Rightarrow$ second order non linear D.E

Ex: show that $y(x)=e^{2x}-1$ is a solution of

$$y''-4y=4$$

$y'(x)=2e^{2x}$

$y''(x)=4e^{2x}$

$4e^{2x}-4(e^{2x}-1)=4$

$\cancel{4e^{2x}}-\cancel{4e^{2x}}+4=4$

$4=4$

so $y(x)=e^{2x}-1$ is a solution

Ex. $y''+y=0$

is $y(x)=\sin x$ a solution?

$y'=\cos x$

$y''=-\sin x$

$y''+y=0$

$-\sin x+\sin x=0$

$0=0$ so $y(x)=\sin x$ is a solution

seprable D.E

is any first order D.E which can be written as:-

$$f(x)\,dx = g(y)\,dy$$

\# which of the following ODEs is sep

① $x\cos^2 y\,dx + e^x dy = 0$

$x\cos^2 y\,dx = -e^x dy$

$\frac{x}{-e^x}dx = \frac{dy}{\cos^2 y}$

seprable D.E

② $(x+xy)dx = x\,dy$

$x(1+y)dx = x\,dy$

$dx = \frac{dy}{(1+y)}$ seprable D.E

③ $x^2(1+y^2)dx = (x^2+1)\,dy$

$\frac{x^2\,dx}{(x^2+1)} = \frac{dy}{1+y^2}$

seprable

④ $x^2y\,dx = (x^2+x^2y)dy$

$x^2y\,dx = x^2(1+y)\,dy$

$$\boxed{dx = \frac{(1+y)\,dy}{y}}$$

⑤ $(x^2+x^2y)dx + (x+y)dy = 0$

$x^2(1+y)dx = -(x+y)\,dy = 0$

not seprable

⑥ $x\cos^2 y\,dx + e^x dy = 0$

$x\cos^2 y\,dx = -e^x dy$

$\frac{-x\,dx}{e^x} = \frac{dy}{\cos^2 y}$ seprable

$-xe^{-x}dx = \sec^2 y\,dy$

⑦ $2xy\,dx + (x^2+1)dy = 0$

$2xy\,dx = -(x^2+1)dy$

$\frac{2x}{x^2+1} = \frac{-dy}{y}$ seprable

⑧ $(x+y^2)dx + x^2e^y dy = 0$

not seprable

⑨ $(xy+x)\,dx = (x^2+1)\,y\,dy$

$x(y+1)\,dx = (x^2+1)\,y\,dy$

$\frac{x\,dx}{(x^2+1)} = \frac{y\,dy}{y+1}$

seprable

Example: solve the following ODEs

① $y' = y^3\cos^2 x$

$\frac{dy}{dx} = y^3\cos^2 x$

$\int\frac{dy}{y^3} = \int\cos^2 x\,dx$ sep

$\int y^{-3}dy = \frac{1}{2}\int(1+\cos 2x)\,dx$

$\frac{y^{-2}}{-2} = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + C$

$\frac{-1}{2y^2} = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + C$

② $e^{2x-y}dx = e^{x+y}dy$

$e^{2x}\cdot e^{-y}dx = e^x\cdot e^y dy$

$e^x dx = \frac{e^y}{e^{-y}}dy$

$\int e^x dx = \int e^{2y}dy$ sep

$e^x = \frac{e^{2y}}{2} + C$

③ $\frac{dy}{dx} = \frac{xy\sin x}{y+1}$, $y(0)=1$

$$\frac{dy\,(y+1)}{y} = x\sin x\,dx$$

$$\int \frac{y+1}{y}\,dy = \int x\sin x\,dx$$

بنشتق　　بنكامل

$x \quad + \quad \sin x$
$1 \quad - \quad -\cos x$
$0 \qquad\;\; -\sin x$

$$\int 1+\frac{1}{y}\,dy = x\sin x$$

$$y+\ln|y| = -x\cos x+\sin x + C \rightarrow \text{general solution}$$

$$1+0 = 0+0+C$$

$$\boxed{C=1}$$

$$\therefore\ y+\ln|y| = -x\cos x+\sin x+1 \rightarrow \text{particular solution}$$

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② $(x^2y+y)dy + (y+1)dx = 0$

$$y(x^2+1)dy = -(y+1)dx$$

$$\frac{-y\,dy}{y+1} = \frac{dx}{x^2+1}$$

substitution
$u = y+1$
$du = dy$

$$\int \frac{1-u}{u}\,du = \int \frac{dx}{x^2+1}$$

$$\int \frac{1}{u} - 1\,du = \int \frac{dx}{x^2+1}$$

$$\ln|u| - u = \tan^{-1}(x) + C$$

$$\ln|y+1| - (y+1) = \tan^{-1}(x) + C$$

③ $\frac{dy}{dx} = x+1+xy^2+y^2$

$$\frac{dy}{dx} = x+1+y^2(x+1)$$

$$\frac{dy}{dx} = (x+1)(y^2+1)$$

$$\int \frac{dy}{y^2+1} = \int dx\,(x+1) \quad \text{sep}$$

$$\tan^{-1}|y| = \frac{x^2}{2} + x + C$$

First order linear D.E

$p(x)$ & $f(x)$ are continuous functions

$$y' + p(x)y = f(x)$$

so this is first order liner D.E

how to solve this equasion

① معامل ال y' ⇐ واحد ⇐ إذا مش واحد بنقسم عليه

& we will call this equasion (*)

② find the integration factor $e^{\int p(x)dx}$

③ Multiply (*) by $e^{\int p(x)dx}$

④ simplify & integrate

Example: Solve the following ODEs

① $x\frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$

$$\frac{xy' + 3y}{x} = \frac{\frac{\sin x}{x^2}}{x}$$

$$y' + \frac{3y}{x} = \frac{\sin x}{x^3}$$

⇒ finding the integrating factor

$$e^{\int \frac{3}{x}} = e^{3\ln x} = x^3$$

$$x^3y' + 3x^2y = \sin x$$

هسا بنحط ال

$$\left(\begin{pmatrix}\text{integrating}\\ \text{factor}\end{pmatrix} \cdot \begin{pmatrix}\text{function} \\ \text{اللي هو } y\end{pmatrix}\right)'$$

$$\int (x^3y)' = \int \sin x$$

$$\boxed{x^3y = -\cos x + C}$$

final answer

② $y' + \frac{4x}{1+x^2}y = \frac{1}{(1+x^2)^3}$

$$\Rightarrow \text{finding } e^{\int p(x)} = e^{\int \frac{4x}{1+x^2}} = e^{2\ln|1+x^2|} = (1+x^2)^2$$

$$\left((1+x^2)^2 y\right)' = \frac{1}{1+x^2}$$

$$(1+x^2)^2 y = \tan^{-1}(x) + C$$

هدام امتحان
ضع دائرة بختصر
الحل

Note :- هرات بكون صعب نكتب المعادلة على شكل

$y'+p(x)y=f(x)$

ممكن يطلع معي بالو هادا ال (form)

$x'+p(y)x=f(y)$

$x'=\frac{dx}{dy}$

③ $\frac{dy}{dx}=\frac{y}{4y^2-2x}$

$\frac{dx}{dy}=\frac{4y^2-2x}{y}$

$\frac{dx}{dy}=4y-2\frac{x}{y}$

$x'-\frac{2}{y}x=4y$

$e^{\int p(y)}=e^{\int\frac{2}{y}}=e^{2\ln|y|}=y^2$

$\int(y^2\cdot x)'=\int 4y^3$

$y^2\cdot x=\frac{4y^4}{4}+c$

$\Rightarrow y^2x=y^4+c$

Bernoulli DE

$y'+p(x)y=f(x)y^n \Rightarrow y^2\ y^3\ y^{-2}\ y^{-1}$

$n\neq 0 \Rightarrow$ لانو بيبطل الو قيمة

$n=1\Rightarrow$ بتصير linear

هلأ Binoulli

لأني بودّيها عالجهة التانية

Q: Find all values of a so that this D.E is linear

$y'+2xy=x^2y^{3a-2}$

$3a-2=0$ | $3a-2=1$
$\frac{3a}{3}=\frac{2}{3}$ | $\frac{3a}{3}=\frac{3}{3}$
$a=\frac{2}{3}$ | $a=1$

so $R-\{\frac{2}{3},1\}$

$u(x)=(y(x))^n\Rightarrow u'(x)=n(y(x))^{n-1}(y'(x))$

$u'=ny^{n-1}y'$

$u(x)=y^2\Rightarrow u'(x)=2yy'$

So

$y'+p(x)y=f(x)y^n,\ n\neq 0,1$

Let $u=y^{1-n}\Rightarrow u'=(1-n)y^{-n}y'$

بنضرب المعادلة ب (الأخضر)

$(1-n)y^{-n}y'+(1-n)p(x)y^{1-n}=(1-n)f(x)$

$u'+(1-n)p(x)u=(1-n)f(x)$

$u'+p(x)u=f(x)$
↳ this is linear in (u) D.E

$y'+p(x)y=f(x)y^n$ الحل باختصار

Let $u=y^{1-n}$ Then

$u'+(1-n)p(x)u=(1-n)f(x)$

$u'+(1-n)up x=(1-n)f(x)$

Example: Solve the following ODEs

① $x^2y'+2xy=y^3$

$y'+\frac{2}{x}y=\frac{y^3}{x^2}$

$u=y^{1-n}=y^{1-3}=y^{-2}\Rightarrow u'=-2y^{-3}y'$

$u'-2\cdot\frac{2}{x}u=\frac{-2}{x^2}$

$u'-\frac{4}{x}u=\frac{-2}{x^2}$

linear in u

$e^{\int\frac{-4}{x}}=e^{-4\ln x}=x^{-4}$

$x^{-4}u'-4x^{-5}u=-2x^{-6}$

$\int(u\cdot x^{-4})'=\int -2x^{-6}$

$u\cdot x^{-4}=\frac{-2x^{-5}}{-5}+c$

$ux^{-4}=\frac{2}{5}x^{-5}+c \Longrightarrow$

آخر شي لازم نرجع (u) الى صورتها الأساسية

$y^{-2}x^{-4}=\frac{2}{5}x^{-5}+c$

② $y' + x^2 y = \dfrac{e^{-x^3}\sinh x}{3y^2}$

$y' + x^2 y = \frac{1}{3} e^{-x^3} \sinh x \, y^{-2}$

$u = y^{1--2} = y^3 \Rightarrow u' = 3y^2 y'$

$u' + 3u\, x^2 = \frac{1}{3} e^{-x^3} \sinh x \cdot 3$

Linear in u

$e^{\int 3x^2} = e^{3\int x^2} = e^{\frac{3x^3}{3}} = e^{x^3}$

$e^{x^3} u' + 3u e^{x^3} x^2 = \sinh x$

$\int (e^{x^3} \cdot u)' = \int \sinh x$

$e^{x^3} \cdot u = \cosh x + C$

$e^{x^3}(y^3) = \cosh x + C$

يرجعوا مرات صعب نكتبها عالصيغة العامة فبنضطر نكتبها

$x' + p(y)\, x = f(y)\, x^n \quad , n \neq 0,1$

Let $u = x^{1-n}$

then the differintial equastion can be reduced to

$u' + (1-n)\, u\, p(y) = (1-n) f(y)$

Q: Solve the following DE

$\dfrac{dy}{dx} = \dfrac{y}{x^2 y^3 - x}$

$\dfrac{dx}{dy} = \dfrac{x^2 y^3 - x}{y}$

$\dfrac{dx}{dy} = x^2 y^2 - \dfrac{x}{y}$

$x' + \dfrac{x}{y} = x^2 y^2$

$u = x^{1-2} = x^{-1} \Rightarrow u' = -1\, x^{-2}$

$u' + -1\, u\, \frac{1}{y} = -y^2$

$u' - \dfrac{u}{y} = -y^2$

$u' - \dfrac{u}{y} = -y^2$

$e^{\int \frac{-1}{y}} = e^{-\ln(y)} = y^{-1} = \frac{1}{y}$

$= -y^2 \cdot \frac{1}{y}$

$= -y$

$\int (u \cdot \frac{1}{y})' = \int -y$

$u \cdot \frac{1}{y} = \dfrac{-y^2}{2} + C$

$u \cdot \frac{1}{y} = \dfrac{-y^2}{2} + C$

$x^{-1} \cdot \frac{1}{y} = -\dfrac{y^2}{2} + C$

ممكن يكون ضارب المعادلة

بال Multible choices بـ xy

$1 = \frac{-1}{2} x y^3 + C$

② $\dfrac{dy}{dx} = \dfrac{x}{y - yx^2}$

$\dfrac{dy}{dx} = \dfrac{x}{y(1-x^2)}$

$y(1-x^2)\, dy = x\, dx$

$\int y\, dy = \frac{-1}{2}\int \dfrac{-2x}{1-x^2}\, dx$

$y^2 = -\frac{1}{2} \ln|1-x^2| + C$

can be solved as
Bernoullie in x or seprable

chapter one is all about ordinary D.E

Homogenous D.E

$y' = f\left(\frac{y}{x}\right) \Rightarrow$ homogeneous

$$f\left(\frac{y}{x}\right) = \left(\frac{y}{x}\right)^2 + 2\left(\frac{y}{x}\right) + 1$$

$$f = \sin\left(\frac{y}{x}\right) + 1$$

$f = \dfrac{x+y}{x-y} \Rightarrow \frac{1}{x}$ لضرب البسط والمقام ب

$$f = \frac{1 + \left(\frac{y}{x}\right)}{1 - \frac{y}{x}}$$

so it is a function of $\left(\frac{y}{x}\right)$

if we have rational function where البسط/المقام both are polynomial

$f = \dfrac{x^2 + 2xy}{y^2 - x^2} \Rightarrow \dfrac{1 + 2\left(\frac{y}{x}\right)}{\left(\frac{y}{x}\right)^2 - 1}$

so this is function of $\left(\frac{y}{x}\right)$

بنشوف الـ degree لكل term

$x^2 \Rightarrow 2$

$2xy \Rightarrow 2$

$y^2 \Rightarrow 2$

$x^2 \Rightarrow 2$

so this is function $f\left(\frac{y}{x}\right)$

لأن كلهم نفس الدرجة إذا بدنا نقدر بنقسم كلهم على x^2

the degree for this is (0)

$f = \dfrac{x^2 - xy + 5}{y^2 + x^2} \Rightarrow$ this is not function of $\left(\frac{y}{x}\right)$

$y' = f\left(\frac{y}{x}\right) \Rightarrow$ we call this equation Homogenuous

بنعوض مكانها الي بالأخضر

لـ u

$u = \frac{y}{x} \Rightarrow y = xu$

$\hookrightarrow y' = x.u' + u$

then this DE should turn to seprable

Q: which of the following ODEs is Homogeneous?

① $\dfrac{dy}{dx} = \dfrac{y-x}{x}$ $\quad f\left(\frac{y}{x}\right) = \left(\frac{y}{x}\right) - 1$

Homogeneous

② $\dfrac{dy}{dx} = \dfrac{2x^2 - y^2}{xy + x^2}$

$$\frac{dy}{dx} = \frac{2 - \left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right) + 1}$$

$\therefore$ Homogeneous

③ $\dfrac{dy}{dx} = \dfrac{x - 2y + 1}{y - x}$

the degree of the constant term is zero so that

$\therefore$ Not homogeneous

Ex: Solve the Following ODEs

① $\dfrac{dy}{dx} = \dfrac{xy + y^2 + x^2}{x^2}$

$\boxed{y' = \left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 + 1}$ ##

$u = \frac{y}{x} \Rightarrow y = u \cdot x$

$y' = u \cdot (1) + x u'$

$\cancel{u} + xu' = \cancel{u} + u^2 + 1$

$xu' = u^2 + 1$

$x \dfrac{du}{dx} = u^2 + 1$

$$\int \frac{du}{u^2+1} = \int \frac{dx}{x}$$

$$\tan^{-1}(u) = \ln|x| + c$$

$$u = \tan(\ln|x| + c)$$

$$\frac{y}{x} = \tan(\ln|x| + c)$$

ممكن نبسط أكثر يكون ضارب الطرفين في (x)

② $(x+y)\,dy = (x-y)\,dx$

$$\frac{dy}{dx} = \frac{x-y}{x+y}$$

$$\frac{dy}{dx} = \frac{1-\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)}$$

$$y' = \frac{1-\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)}$$

$$u = \frac{y}{x} \Rightarrow y = ux$$
$$y' = u.(1) + x.u'$$

$$u + xu' = \frac{1-u}{1+u} - u$$
$$-u$$

$$xu' = \frac{1-u}{1+u} - \frac{u(1+u)}{1+u}$$

$$xu' = \frac{1-u-u-u^2}{1+u}$$

$$xu' = \frac{1-2u-u^2}{1+u}$$

$$x\frac{du}{dx} = \frac{1-2u-u^2}{1+u}$$

$$\int \frac{du\,(1+u)}{1-2u-u^2} = \int \frac{dx}{x}$$

$$\frac{-1}{2}\ln|1-2u-u^2| = \ln|x| + c$$

$$\frac{-1}{2}\ln\left|1-2\left(\frac{y}{x}\right)-\left(\frac{y}{x}\right)^2\right| = \ln|x| + c$$

ممكن يكون ضرب المعادلة
بالسالب 2 باختيارك أو يوخذ
e لكل المعادلة

$$1-2u-u^2 = x^{-2}e^{c}$$

$$\left(1-2\frac{y}{x}-\left(\frac{y}{x}\right)^2 = x^{-2}\,c\right)x^2$$

$$x^2 - 2yx - y^2 = c$$

$$\frac{dy}{dx} = \frac{y}{x}\left(\ln y - \ln x + 1\right)$$

$$y' = \frac{y}{x}\left(\ln\left(\frac{y}{x}\right) + 1\right)$$

$$u = \frac{y}{x} \Rightarrow y = ux$$
$$y' = u + x(u')$$

$$u + xu' = u(\ln u + 1)$$

$$\cancel{u} + xu' = u\ln u + \cancel{u}$$

$$xu' = u\ln u$$

$$x\frac{du}{dx} = u\ln u$$

$$\int \frac{du}{u\ln u} = \int \frac{dx}{x}$$

$$\int \frac{\frac{1}{u}\,du}{\ln u} = \int \frac{1}{x}$$

$$\ln(\ln(u)) = \ln|x| + c$$
$$e \qquad\qquad e$$

$$\ln\left(\ln\left(\frac{y}{x}\right)\right) = e^{c}e^{\ln x}$$
$$e$$

$$\ln\left(\frac{y}{x}\right) = cx$$

$$u = e^{cx}$$

Solve the following ODEs

① $(xy+y^2)\,dx - x^2\,dy = 0$

$$(xy+y^2)\,dx = x^2\,dy$$

$$\frac{dy}{dx} = \frac{xy+y^2}{x^2}$$

$$\frac{dy}{dx} = \frac{y}{x} + \left(\frac{y}{x}\right)^2$$

$$y' = \frac{y}{x} + \left(\frac{y}{x}\right)^2$$

$$u = \frac{y}{x} \Rightarrow y = ux$$
$$y' = u(1) + xu'$$

$$\cancel{u} + xu' = \cancel{u} + u^2$$

$$xu' = u^2$$

$$x\frac{du}{dx} = u^2$$

$$\frac{du}{u^2} = \frac{dx}{x}$$

$$\frac{du}{u^2} = \frac{dx}{x}$$

$$\int u^{-2}\,du = \int \frac{1}{x}\,dx$$

$$\frac{u^{-1}}{-1} = \ln|x| + c$$

$$= \frac{-1}{u} = \ln x + c$$

$$\frac{-1}{\frac{y}{x}} = \ln|x| + c$$

$$\frac{-x}{y} = \ln|x| + c$$

② $$\frac{dy}{dx} = \frac{x\sec\left(\frac{y}{x}\right) + y}{x}$$

$$y' = \sec\frac{y}{x} + \frac{y}{x}$$

$$u = \frac{y}{x} \Rightarrow y = ux$$

$$y' = u + xu'$$

$$\cancel{u} + xu' = \sec u + \cancel{u}$$

$$xu' = \sec u$$

$$x\frac{du}{dx} = \sec u$$

$$\int \frac{du}{\sec u} = \int \frac{dx}{x}$$

$$\int \cos u \, du = \ln|x| + c$$

$$\sin u = \ln x + c$$

$$\sin\frac{y}{x} = \ln x + c$$

③ $$\frac{dy}{dx} = \frac{\sqrt{xy} + y}{x}, \quad x > 0$$

$$y' = \sqrt{\frac{y}{x}} + \frac{y}{x}$$

$$u = \frac{y}{x} \Rightarrow y' = u'x + u$$

$$u'x + \cancel{u} = \sqrt{u} + \cancel{u}$$

$$\frac{du}{dx}x = \sqrt{u}$$

$$\int \frac{du}{\sqrt{u}} = \int \frac{dx}{x}$$

$$\int u^{-\frac{1}{2}} du = \int \frac{dx}{x}$$

$$\frac{u^{\frac{1}{2}}}{\frac{1}{2}} = \ln|x| + c$$

$$\hookrightarrow 2\sqrt{u} = \ln|x| + c$$

$$\underbrace{\frac{dy}{dx} = f(\underbrace{ax+by+c}_{u})}_{} \Rightarrow$$ this function of the whole this variable

ex $(2x-y+3)^3 - 5(2x-y+3)$

↳ semiler to Homogeneous DE

زي ال $\left(\frac{y}{x}\right)$

$$u = ax + by + c$$

$$u' = a + by'$$

$$\boxed{y' = \frac{u' - a}{b}}$$

وبنعوضها بالمعادلة

اللي فوق

then this DE will return to seprable DE

$$f = \frac{2x-y}{2x-y+5} \Rightarrow f(2x-y)$$

$$f = \frac{2x-2y}{4x-4y+5} \Rightarrow f(x-y) = \frac{2(x-y)}{4(x-y)+5}$$

Examples:-

$$f(x+y+z) = (x+y+z)^3 + 1$$

$$f(x+y) = \sin(x+y) + 1$$

$$f(x-y) = \frac{2x-2y+1}{x-y}$$

$$= \frac{2(x-y)+1}{x-y}$$

Example solve the Following ODEs

① $$\frac{dy}{dx} = (x+y+2)^2$$

$$u = x + y + 2$$

$$u' = 1 + y'$$

$$y' = u' - 1$$

$$u' - 1 = u^2$$

$$\frac{du}{dx} - 1 = u^2$$

$$\frac{du}{dx} = u^2 + 1$$

$$\int \frac{du}{u^2+1} = \int dx$$

$$\tan^{-1}(u) = x + c$$

$$\tan^{-1}(x+y+2) = x + C$$

$$(x+y+2) = \tan(x+c)$$

② $\frac{dy}{dx} = \sqrt{x+y} - 1$

$u = x+y$

$u' = 1+y'$

$y' = u'-1$

$u' \cancel{-1} = \sqrt{u} \cancel{-1}$

$\frac{du}{dx} = \sqrt{u}$

$\int \frac{du}{\sqrt{u}} = \int dx$

$\int u^{-\frac{1}{2}} du = \int dx$

$2u^{\frac{1}{2}} = x + C$

$2\sqrt{x+y} = x + C$

⑧ $\frac{dy}{dx} = \sin(x-y)$

$u = x-y$

$u' = 1-y'$

$y' = 1-u'$

$1-u' = \sin(u)$

$1 - \frac{du}{dx} = \sin(u)$
$-1 \qquad\qquad -1$

$\frac{du}{dx} = 1 - \sin(u)$

$\int \frac{du}{1-\sin u} \cdot \frac{1+\sin u}{1+\sin u} = \int dx$

$\int \frac{1+\sin u}{\cos^2 u} = \int dx$

$\int \frac{1}{\cos^2 u} + \int \frac{\sin u}{\cos^2 u} = \int dx$

$\tan u + \sec u = x$

$\int \frac{1}{\cos^2 u} + \int \frac{\sin u}{\cos^2 u} = \int dx$

$\int \sec^2 u + \int \tan u \sec u = \int dx$

$\tan u + \sec u = x + C$

$\tan(x-y) + \sec(x-y) = x + C$

④ $y' = (2x+y)^2 - 2$

$u = 2x+y$

$u' = 2+y'$

$-2 \quad -2$

$y' = u'-2$

$u' \cancel{-2} = (u)^2 \cancel{-2}$

$\frac{du}{dx} = u^2$

$\int \frac{du}{u^2} = \int dx$

$\int u^{-2} du = \int dx$

$\frac{u^{-1}}{-1} = x + C$

$\frac{-1}{u} = x + C$

بنز طا شلها

$u = \frac{-1}{x+C}$

$2x+y = \frac{-1}{x+C}$

بدي أعتبرو هيك

5 $y' = \frac{(x+y)+2}{(x+y)}$

$u = x+y$

$u' = 1 + y'$

$y' = u' - 1$

$u' - 1 = \frac{u+2}{u}$

$u' - 1 = 1 + \frac{2}{u}$

$u' = 2 + \frac{2}{u}$

$\frac{du}{dx} = 2 + \frac{2}{u}$

$\int \frac{du}{2+\frac{2}{u}} = \int dx$

$\int \frac{du}{\frac{2u+2}{u}} = \int dx$

طريقة بالحل

$\int \frac{(u+1)-1}{2u+2} du = dx$

$\int \frac{u+1-1}{2(u+1)} = dx$

$\int \frac{u+1-1}{u+1} = 2dx$

$\int 1 - \frac{1}{u+1} du = 2dx$

$u - \ln|u+1| = 2x + C$

$(x+y) - \ln(x+y+1) = 2x + C$

Revision for partial Derivatives

Ex: If $f(x,y) = x^3 + 2y^2$, find

$\frac{\delta f}{\delta y} = 4y \qquad \frac{\delta f}{\delta x} = 3x^2$

Ex: If $f(x,y) = x^2 + xy + \sin y$

find

1 $\frac{\delta f}{\delta x} = 2x + y$

2 $\frac{df}{dy} = x + \cos y$

Example: $f(x,y) = x\cos y - 2y$, find $\frac{\delta f}{\delta x}, \frac{\delta f}{\delta y}$

$\frac{\delta f}{\delta x} = \cos y \quad , \frac{\delta f}{\delta y} = -x\sin y - 2$

Example: If $\frac{\delta f}{\delta y} = x\cos y - 2y$, find f.

$f(x,y) = x\sin y - y^2 + g(x)$

ل هاد طلعوص التكامل

constant.x أو constant الي بالأصل

إذا كاملنا بالنسبة ل (y) كان حطينا g(y)

Exact DE

$M(x,y)\,dx + N(x,y)\,dy = 0$

Ex $(x-2y)\,dx + (y-2x)\,dy = 0$

هاد الشكل هو معناها المعادلة exact

هو بجهزني عشان أختبر إذا المعادلة exact ولا.

to be exact

Must be

$\frac{\delta M}{\delta y} = \frac{\delta N}{\delta x}$

$\boxed{u(x,y) = C}$ solution

$\frac{\delta u}{\delta x} = M \qquad \frac{\delta u}{\delta y} = N$

↳ to find u integrate by respect to (x)

$u = \int M\,dx + g(y) \qquad , u = \int N\,dy + h(x)$

Example: Solve:-

$(e^x + y)\,dx + (x - \sin y)\,dy = 0 \qquad , y(0) = 0$

$\frac{\delta M}{\delta y} = 1 \quad , \frac{\delta N}{\delta x} = 1 \Rightarrow$ Exact

$\therefore$ the solution is $u(x,y) = C$

$\frac{\delta u}{\delta x} = M$

$\frac{\delta u}{\delta x} = e^x + y$

$\int du = \int (e^x + y)\,dx \rightarrow$

$u = e^x + xy + f(y)$

$u(x,y) = e^x + xy + f(y)$

we must find this

Differentiate with respect to y this equation

$$u(x,y) = e^x + xy + f(y)$$

$$\frac{du}{\delta y} = x + f'(y)$$

Is this is N

$$\cancel{x} - \sin y = \cancel{x} + f'(y)$$

$$\int f'(y) = \int -\sin y$$

$$\boxed{f(y) = \cos y + C_1}$$

now substitute it in the previous equation

$$u(x,y) = e^x + xy + \cos y + C_1$$

$\therefore$ The solution is $u(x,y) = C$

$$u(x,y) = e^x + xy + \cos y + C$$

$$C = e^x + xy + \cos y + C_1$$

$\Rightarrow e^x + xy + \cos y = C$

this is the solution

$$1 + 0 + 1 = C$$
$$C = 2$$

$$e^x + xy + \cos y = 2$$

Solve:

$$(2x\cos y + 3x^2 y)dx + (x^3 - x^2 \sin y - y)dy = 0$$

$$\underbrace{\frac{\delta M}{\delta y} = -2x\sin y + 3x^2 \quad \frac{\delta N}{dx} = 3x^2 - 2x\sin y}_{\text{Exact}}$$

$$\frac{du}{dx} = M$$

$$\frac{\delta u}{\delta x} = 2x\cos y + 3x^2 y$$

$$\int du = \int (2x\cos y + 3x^2 y)\,dx$$

$$u = \frac{2x^2}{2}\cos y + \frac{3x^3}{3}y + f(y)$$

$$u = x^2 \cos y + x^3 y + f(y)$$

now we will differentiate it with respect to (y)

$$\frac{du}{\delta y} = -x^2 \sin y + x^3 + f'(y)$$

$$\cancel{x^3} - \cancel{x^2 \sin y} - y = \cancel{-x^2 \sin y} + \cancel{x^3} + f'(y)$$

$$\int f'(y) = \int -y$$

$$f(y) = -\frac{y^2}{2}$$

$$f(y) = -\frac{1}{2}y^2 + C$$

$\therefore u(x,y) = C$

$$x^2 \cos y + x^3 y - \frac{1}{2}y^2 = C$$

H.W: Solve : $\frac{dy}{dx} = \frac{-\sin y + y\cos x}{\sin x + x\cos y}$

$(\sin x + x\cos y)\,dy + (\sin y + y\cos x)\,dx = 0$

M يرجع يرتبهم N

$(\sin y + y\cos x)\,dx + (\sin x + x\cos y)\,dy = 0$

$\frac{\partial M}{\partial y} = \cos y + \cos x \quad , \quad \frac{\partial N}{\partial x} = \cos x + \cos y$

Exact

$\frac{du}{dx} = M$

$\frac{du}{dx} = \sin y + y\cos x$

$\int du = \int (\sin y + y\cos x)\,dx$

$u = x\sin y + y\sin x + f(y)$

$\frac{\partial u}{\partial y} = x\cos y + \sin x + f'(y)$

$\cancel{\sin x} + \cancel{x\cos y} = \cancel{x\cos y} + \cancel{\sin x} + f'(y)$

$f'(y) = 0$

$\therefore$ solution is $u(x,y) = C$

$x\sin y + y\sin x = C$

Example: If

$(2x+6x^2y^2)dx+(4x^ny-12y^3)dy=0$

$\therefore$ exact Find n.

$\frac{\partial M}{\partial y}=12x^2y \quad , \frac{\partial N}{\partial x}=4ynx^{n-1}$

$\cancel{12}x^2y=\cancel{4}ynx^{n-1}$

$3x^2y=ynx^{n-1}$

$\boxed{n=3}$

$n-1=2$
$+1 \quad +1$

$\boxed{n=3}$

Non-Exact DE

$M(x,y)dx+N(x,y)dy=0$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ non Exact

* we will search to function called

Ex (μ) may be $\rightarrow \mu(x)$

$\rightarrow \mu(x,y)$

$\rightarrow \mu(y)$

إذا ضربت هاد ال (μ) بالمعادلة وطلعت Exact

بسميها integrating factor

$\mu M dx+\mu N dy=0 \Rightarrow$ exact

μ: integrating factor

we only have to calculate integrating factor as a function of (x) or (y) alone only

$M(x,y)dx+N(x,y)dy=0$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\left.\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}\right\}=R(x)$

بشرط انه يطلع معي جواب هاد، function of (x)

$\mu(x)=e^{\int R(x)dx}$

إذا طلع معي R(x) فيه (y)؟ ما بقبلو بترجع:

$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{-M}=R(y) \Rightarrow \mu(y)=e^{\int R(y)dy}$

Ex: solve the Following ODEs

1. $(y^2-3xy-2x^2)dx+(xy-x^2)dy=0$

$\frac{\partial M}{\partial y}=2y-3x \quad , \frac{\partial N}{\partial x}=y-2x$

not exact

$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{(2y-3x)-(y-2x)}{x^2-xy}$

$=\frac{2y-3x-y+2x}{x^2-xy}=\frac{y-x}{x^2-xy}=\frac{\cancel{y-x}}{x(\cancel{x-y})}=\frac{1}{x}$

$\mu=e^{\int \frac{1}{x}}=\cancel{e}^{\ln x}=x$

Multiply the D.E by (x)

$\underbrace{(xy^2-3x^2y-2x^3)dx}+\underbrace{(x^2y-x^3)dy}=0$

يعني أصلاً ضربت x بكاملها بالنتيجة ... بكاملها بالنتيجة لـ x

& this equation is exact (integrating factor)

$\frac{1}{2}x^2y^2-x^3y-\frac{1}{2}x^4=C$

وبعدها كملوا الحل
ما وضحو

② $(3x^2y+y^2)dx+(2x^3+3xy)dy=0$

$$\frac{\partial M}{\partial y}=3x^2+2y \quad , \quad \frac{\partial N}{\partial x}=6x^2+3y$$

not exact

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{(3x^2+2y)-(6x^2+3y)}{3x^3+3xy}$$

$$=\frac{3x^2+2y-6x^2-3y}{3x^3+3xy}$$

$$=\underline{-3x^2-y} \Rightarrow$$ ما زبطت

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{-M}=\frac{-3x^2-y}{-(3x^2y+y^2)}$$

$$=\frac{-(3x^2+y)}{-(y)(3x^2+y)}=\frac{1}{y}$$

$$\mu=e^{\int\frac{1}{y}}=e^{\ln y}=y$$

$$(3x^2y^2+y^3)dx+(2x^3y+3xy^2)dy=0$$

$$\frac{\cancel{3}x^3y^2}{\cancel{3}}+y^3x=c$$

$$x^3y^2+y^3x=C$$

Homework: Find the integrating factor

① $(3x^2y+2xy+y^3)dx+(x^2+y^2)dy=0$

$$\frac{\partial M}{\partial y}=3x^2+2x+3y^2 \quad , \quad \frac{\partial N}{\partial x}=2x$$

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{-M}=\frac{3x^2+\cancel{2x}+3y^2-\cancel{2x}}{-(x^2+y^2)}$$

$$=\frac{3(\cancel{x^2+y^2})}{-(\cancel{x^2+y^2})}=-3$$

$$\mu=e^{\int R(y)}=e^{\int -3}=e^{-3y}$$

② $y\,dx+(2xy-e^{-2y})dy=0$

$$\frac{\partial M}{\partial y}=1 \quad , \quad \frac{\partial N}{\partial x}=2y$$

Not exact

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{-M}=\frac{1-2y}{-y}$$

$$=\frac{2y-1}{y}=2-\frac{1}{y}$$

$$\mu=e^{\int 2-\frac{1}{y}}=e^{2y-\ln y}$$

$$=e^{2y}\cdot e^{-\ln y}$$

$$=e^{2y}\cdot \cancel{e}^{\ln y^{-1}}$$

$$=\frac{e^{2y}}{y}$$

Home work:-

Show that $\mu(x,y) = xy^2$ is an interating factor of:-

$$(2y - 6x)\,dx + (3x - 4x^2y^{-1})\,dy = 0$$

$$(2y^3x - 6x^2y^2)\,dx + (3x^2y^2 - 4x^3y)\,dy = 0$$

$$\frac{\partial M}{\partial y} = 6xy^2 - 12x^2y$$

$$\frac{\partial N}{\partial x} = 6xy^2 - 12x^2y$$

Exact

so that μ is integrating factor

Chapter 2 Second order ODEs

second order Linear D.E

$$y'' + p(x)\,y' + q\,y = f(x)$$

second order أما بال هيك homoال

when $f(x)=0 \Rightarrow$ homo

$f(x)\neq 0 \Rightarrow$ non homo

$y' = f\left(\frac{y}{x}\right)$ first order لل homo قبل هيك كان ال

we have 2 solutions y_1, y_2

$$y = c_1 y_1 + c_2 y_2$$

يعني إذا أخذت ال c_1 وال c_2 أي رقم وعوضت راح يكونوا solutions

$\boxed{y_1 + y_2}$ $\boxed{y_1 - y_2}$ $\boxed{3y_1}$

solutions

\# if we have two soltions وكان أحدهم Multible للآخر

$$y_1 = c\,y_2 \qquad \frac{y_1}{y_2} = c \Rightarrow y_1, y_2 \Rightarrow \text{linearly dependent}$$

شو يعني؟ يعني هدول ال 2 solutions بعتبرو solution واحد

other wise $y_1 \neq c\,y_2 \Rightarrow$ are linearly independent

\# Ex: are the following solutions independent or dependent?

1) $f(x) = \sin x$, $g(x) = \cos x$, $x \in (0, \frac{\pi}{2})$

$$\frac{f(x)}{g(x)} = \frac{\sin x}{\cos x} = \tan x \neq \text{constant} \Rightarrow \text{independent}$$

2) $f(x) = x^2$, $g(x) = 3x^2$

$$\frac{f(x)}{g(x)} = \frac{x^2}{3x^2} = \frac{1}{3} \to \text{dependent}$$

\# if I have 2 solutions f & g

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} \quad fg' - f'g$$

$$W(g,f) = \begin{vmatrix} g & f \\ g' & f' \end{vmatrix} = gf' - fg'$$

$$W(f,g) = -W(g,f)$$

$$W(f,g,h) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix} \rightarrow$$ وبنكملها حل دا2 نقلي عنو بعرفها

\# if we have f, g solutions for homo D.E

if we calculat $W_{(f,g)} \neq 0 \iff f, g$ independent

$W(f,g) = 0 \rightarrow f, g$ dependent so we will consider them as one solution

\# in second order D.E we are searching for 2 independent solutions

Ex: is $\cos x$, $\sin x$ independent?

$$W_{(\cos x, \sin x)} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} \quad \cos^2 x + \sin^2 x = 1 \neq 0$$

so they are independent

for $(x^2, 3x^2)$
$$W_{(,)} \begin{vmatrix} x^2 & 3x^2 \\ 2x & 6x \end{vmatrix} = 6x^3 - 6x^3 = 0$$

so they are dependent

$$\underbrace{\{y_1, y_2\}}_{\text{2 independent solution}} \Rightarrow \overset{\text{group}}{\text{fundamental}} \text{ set of solutions}$$
Basis

2 independent solution المجموعة الي فيها

\# we call $\boxed{y = c_1 y_1 + c_2 y_2} \rightarrow$ general solution

\# Abel's theorem

$$1\, y'' + p(x) y' + q(x) y = 0$$

لشرط يكون معامل ال y'' هو 1

$$W_{(y_1, y_2)}(x) = C e^{-\int p(x)\,dx}$$

So this means that $W_{(y_1,y_2)}(x)$ either is zero for all $x \in$ interval or else is never zero for the interval.

$$t y'' + 2y' + t e^{t} y = 0$$

$W_{(y_1, y_2)}(2) = 3$, find $W_{(y_1, y_2)}(5)$

طريقة الحل هاي خاصة ب لما
يعطيك W عند نقطة ويطلب منك W عند نقطة

$$y'' + \frac{2}{t} y' + e^{t} y = 0$$

$$e^{-\int \frac{2}{t}} = e^{-2\ln t} = t^{-2} = \frac{1}{t^2}$$

$$W_{(y_1, y_2)}(t) = \frac{C}{t^2}$$

$$W(y_1, y_2)(2) = 3$$

$$3 = \frac{C}{4}$$

$$C = 12$$

$$W_{(y_1, y_2)}(t) = \frac{12}{t^2}$$

$$W(y_1, y_2)(5) = \frac{12}{25}$$

\# Ex: if $W_{(f,g)} = 3e^{4t}$ and $f(t) = e^{2t}$ find $g(t)$

$$W(f, g) = 3e^{4t}$$

$$f \cdot g' - g \cdot f' = 3e^{4t}$$

$$e^{2t} \cdot g' - g \cdot 2e^{2t} = 3e^{4t}$$

& this is first order linear D.E

$$g' - 2g = 3e^{2t}$$

$$e^{-\int 2} = e^{-2t}$$

$$\int (g \cdot e^{-2t})' = \int 3$$

$$g \cdot e^{-2t} = 3t + c$$

$$\boxed{g(t) = \frac{3t + c}{e^{-2t}}}$$

Ex: If $W(f,g)=2$, find $w(f, f+2g)$

$W(f, f+2g) = f(f+2g)' - (f+2g).f'$

$f(f'+2g') - (f+2g)f'$
$\cancel{ff'} + 2fg' - \cancel{ff'} - 2f'g$

$= 2fg' - 2f'g$

$= 2(\underbrace{fg' - f'g}_{w(f,g)})$

$= 2 \times 2 = 4$

لو كانت معطيني $w(g,f)$

بوخذ $\to (-2)$

X-y missing

$F(x, y', y'') = 0 \to y$ missed

Ex: $y'' = x^2 + y'^2$ — y missed

بطلع معنا تكون هاي ال equation linear ولا non liner

الشرط الوحيد عليها إنها تكون y missed

$y'' = F(x, y')$

$u = y' \to u' = y''$

$u' = F(x, u)$

$\hookrightarrow$ so it is now first order D.E in term of (x, y)

$\frac{du}{dx} = F(x, u)$

* solve $y'' + \frac{2}{x} y' = \frac{1}{x^2}$

$\hookrightarrow$ y missed

$u = y' \Rightarrow u' = y''$

$u' + \frac{2}{x} u = \frac{1}{x^2}$

$e^{\int \frac{2}{x}} = e^{2\ln|x|} = x^2$

$x^2 u' + 2xu = 1$

$\int (u.x^2)' = \int 1$

$u.x^2 = x + C$

$u = \frac{x+C}{x^2}$

$y' = \frac{x+C}{x^2} \Rightarrow$ again we have D.E first order

$\frac{dy}{dx} = \frac{x+C}{x^2}$

$\int dy = \int \frac{x+C}{x^2} dx$

$y = \int \frac{1}{x} dx + \int \frac{C}{x^2} dx$

$y = \ln|x| + C \int x^{-2} dx$

$y = \ln|x| + C \frac{x^{-1}}{-1}$

$\boxed{y = \ln|x| - \frac{c_1}{x} + c_2}$ Solution

Second order D.E always the solution has 2 constants

H.W: Solve the following ODEs:-

① $xy''+y'=1$

$u=y' \Rightarrow u'=y''$

$$\frac{xu'+u=1}{x}$$

$u'+\frac{1}{x}u=1$

$e^{\int\frac{1}{x}}=\cancel{e}^{\ln x}=x$

$\int(x.u)'=\int x$

$xu=\frac{x^2}{2}+c_1$

$xy'=\frac{x^2}{2}+c_1$

$\frac{1}{x}.x\frac{dy}{dx}=\frac{x^2}{2}+c_1.\frac{1}{x}$

$\int dy=\int\frac{x}{2}+\frac{c_1}{x}dx$

$$\boxed{y=\frac{x^2}{4}+c_1\ln|x|+c_2}$$

③ $y''+(y')^2+1=0$

$u=y' \rightarrow u'=y''$

$u'+u^2+1=0$

$u'+u^2=-1$

$e^{\int 1}=e^x$

$\int(e^x.u)'=\int -e^x$

$e^x.u=-e^x+c_1$

$u=\frac{-e^x}{e^x}+\frac{c_1}{e^x}$

$u=-1+\frac{c_1}{e^x}$

$\frac{dy}{dx}=-1+\frac{c_1}{e^x}$

$\int dy=\int -1dx+\int\frac{c_1}{e^x}$

$y=-x+\int c_1e^{-x}$

$y=-x+\frac{c_1e^{-x}}{-1}+c_2 \Rightarrow \boxed{y=-x-c_1e^{-x}+c_2}$

now we are going to discuss x_missed

$y''=F(y,y'')$ x_missed

let $u=y' \rightarrow u'=y''$

$u'=F(y,u)$
↳ first order D.E

$\frac{du}{dx}=F(y,u)$

هاي معادلة first order ordinary D.E

بس عنا 3 متغيرات ماتعاملنا معها قبل
هيك لـ (x,y,u)

فعندي 3 متغيرات بدي أخلص من واحد أكتب بدلالة
متغير وبصير عندي متغيرين

$\frac{du}{dx}=\frac{du}{dy}.\frac{dy}{dx}$ $y'=u$

$\frac{du}{dx}=\boxed{u.\frac{du}{dy}}$

هاد بنحطها بدل ال $\frac{du}{dx}$

$u\frac{du}{dy}=F(y,u)$

Ex: Solve : $yy''+(y')^2=0$

x_missed

$u=y' \rightarrow u'=y''$

$yu'+u^2=0$

$u'=\frac{-u^2}{y}$

$\frac{du}{dx}=\frac{-u^2}{y}$

$u.\frac{du}{dy}=\frac{-u^2}{y}$

$\frac{u\,du}{-u^2}=\frac{dy}{y} \Rightarrow \int\frac{-1}{u}du=\int\frac{dy}{y}$

$-\ln|u|=\ln|y|+c$

$\ln|u|+\ln|y|=c$

$e^{\ln|uy|}=e^{c_1}$

$\parallel$

$\Downarrow$

تكملة الحل

$uy = e^{c_1}$ or $uy = c_1$

$uy = c_1$

$\frac{dy}{dx}\, y = c_1$

$\int dy\, y = \int c_1\, dx$

$\frac{y^2}{2} = c_1 x + c_2$

Solve: $(y+1)y'' = y'^2$

x-missed

$u = y' \rightarrow u' = y''$

$(y+1)u' = u^2$

$u' - \frac{u^2}{y+1} = 0$

$\frac{du}{dy} \cdot u - \frac{u^2}{y+1} = 0$

$\frac{du}{dy} \cdot u = \frac{u^2}{y+1}$

$\frac{u \cdot du}{u^2} = \frac{dy}{y+1}$

$\frac{du}{u} = \frac{dy}{y+1}$

$e^{\cancel{\ln}|u|} = e^{\cancel{\ln}|y+1| + c_1}$

$u = e^{\ln|y+1|} \cdot e^{c_1}$

$u = |y+1| \cdot c_1$

$\frac{dy}{dx} = |y+1|\, c_1$

$\int \frac{dy}{y+1} = \int dx\, c_1$

$e^{\ln|y+1|} = e^{c_1 x + c_2}$

$y+1 = e^{c_1 x + c_2}$

$y+1 = e^{c_1 x} \cdot e^{c_2}$

$y+1 = e^{c_1 x} \cdot c_2$

$y = c_2 e^{c_1 x} - 1$

H.w: solve

① $y'' + 2y(y')^3 = 0$

↳ x-missed

$u = y' \rightarrow u' = y''$

$u' + 2y u^3 = 0$

$u \cdot \frac{du}{dy} = 2y u^3$

$\frac{u\, du}{u^3} = 2y\, dy$

$\int \frac{du}{u^2} = \int 2y\, dy$

$-1 u^{-1} = \frac{2y^2}{2} + c_1$

$\frac{-1}{u} = y^2 + c_1$

$\frac{-1}{\frac{dy}{dx}} = y^2 + c_1$

$\frac{-dx}{dy} = y^2 + c_1$

$\int -dx = \int (y^2 + c_1)\, dy$

$-x = \frac{y^3}{3} + c_1 y + c_2$

$-x = \frac{y^3}{3} + c_1 y + c_2$

Reduction of order

Part1 : Having equasion second order linear homogenuous

وعندي y_1 و بطلع y_2

$$y'' + p(x)y' + q(x)y = 0$$

المعاملات مش شرط تكون أرقام عادي هيا متغيرات

إذا أعطاني y_1 كيف بقدر أوجد second linearly independent solution

بشرط معامل ال y'' هو 1 و متعابتكو دائماً معامل ال y'

$$y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx$$

* Example: Given $y_1 = x$ a solution of:-

$$(x^2 - x)y'' - xy' + y = 0$$

find second-linearly independent solution y_2

$$y'' - \frac{x}{x^2 - x}y' + \frac{1}{x^2 - x}y = 0$$

$$y_1 = x$$

$$e^{+\int \frac{x}{x^2 - x}} = e^{+\int \frac{\cancel{x}}{\cancel{x}(x-1)}} = e^{+\int \frac{1}{x-1}} = e^{+\ln|x-1|} = x - 1$$

$$y_2 = x \int \frac{x-1}{x^2}\,dx$$

$$y_2 = x \int \frac{1}{x} - \frac{1}{x^2}$$

$$y_2 = x\left(\ln x + \frac{1}{x}\right)$$

$$\boxed{y_2 = x\ln x + 1}$$

* Consider $xy'' + 2y' + xy = 0$, Given $y_1 = \frac{\cos x}{x}$ a solution, find the 2^{nd}-linearly independent solution

$$y_1 = \frac{\cos x}{x}$$

$$\frac{xy'' + 2y' + xy = 0}{x}$$

$$y'' + \frac{2}{x}y' + y = 0$$

$$e^{-\int \frac{2}{x}} = e^{-2\ln x} = x^{-2} = \frac{1}{x^2}$$

$$y_2 = y_1 \int \frac{\frac{1}{x^2}}{\left(\frac{\cos x}{x}\right)^2}\,dx$$

$$y_2 = \frac{\cos x}{x} \int \frac{1}{x^2} \cdot \frac{x^2}{\cos^2 x}\,dx$$

$$y_2 = \frac{\cos x}{x} \int \sec^2 x\,dx$$

$\Rightarrow$ continuoud

$$y_2 = \frac{\cos x}{x} \cdot \tan x$$

$$\frac{1}{x}\,\cancel{\cos x} \cdot \frac{\sin x}{\cancel{\cos x}} = \frac{\sin x}{x}$$

$$y_2 = \frac{\sin x}{x}$$

H.W : consider:-

$$(1-x^2)y''-2xy'+2y=0 \quad , y_1=x$$

Find a 2^{nd}-independent solution y_2

Hint: $\dfrac{1}{x^2(1-x^2)} = \dfrac{1}{x^2} + \dfrac{(1/2)}{x+1} - \dfrac{(1/2)}{x-1}$

$$y'' - \frac{2x}{1-x^2}y' + \frac{2}{1-x^2}y = 0$$

$$e^{-\int p(x)} = e^{\int \frac{2x}{1-x^2}} = e^{\ln|1-x^2|} = 1-x^2$$

$$y_2 = x \int \frac{1-x^2}{x^2}\,dx$$

$$y_2 = x \int \frac{1}{x^2} - 1\,dx$$

$$y_2 = x\left(\frac{-1}{x} - x\right)$$

$$\boxed{y_2 = -1 - x^2}$$

Part 2: Homo linear DEs with constant coefficients

$$y''+ay'+by=0 \rightarrow$$

Step1: بوخذ ال

$$y'+ay=0$$

$$\frac{dy}{dx} = -ay$$

$$\int \frac{dy}{y} = \int a\,dx$$

$$e^{\ln|y|} = e^{-ax+c}$$

$$y = e^{-ax}.e^{c}$$

$$y = c\,e^{-ax}$$

So now we search for solution in the form $y=e^{rx}$ as a solution for #

$$r^2e^{rx}+are^{rx}+be^{rx}=0$$

$$e^{rx}(\underbrace{r^2+ar+b}_{\Downarrow})=0$$

this equation is called characteristic equation

r^2+ar+b

this may has three solutions:-

1. two different solutions $y_1=e^{r_1x}$, $y_2=e^{r_2x}$
   $\hookrightarrow \Delta = +$
2. two similar solutions $y_1=e^{rx} \rightarrow$ وبجيب y_2 بنفس طريقة take solution [illegible]
   $\hookrightarrow \Delta = 0$
   constant مش c بالحل بدل ما يكون ← x بس solution [illegible]
3. $\Delta = -$ then we only have 2 complex solutions

دوسني constants بتعطوا
هاي

*** Ex:** Solve $y''+y'-2y=0$

$$r^2+r-2=0$$

$$(r+2)(r-1)=0 \Rightarrow \boxed{r=-2}, \boxed{r=1}$$

$$y_1=e^{-2x}, \quad y_2=e^{x}$$

∴ the general solution

$$y(x)=c_1e^{-2x}+c_2e^{x}$$

$\{e^{-2x}, e^{x}\}$ is called fundamentall set of solutions or Basis of solutions

*** Ex:** Solve : $2y''+3y'=0$

$$2r^2+3r=0$$
$$r(2r+3)=0$$
$$r=0, \quad 2r+3=0$$
$$r=\frac{-3}{2}$$

$$y_1=e^{0x}=1 \qquad y_2=e^{-3/2\,x}$$

$$y_1=1, \quad y_2=e^{-3/2\,x}$$

$$y=c_1+c_2e^{-3/2\,x}$$

*** Ex:** Solve: $y''+4y'+2y=0$

$$r^2+4r+2=0$$

$$a=1, \quad b=4, \quad c=2$$

$$\Delta=b^2-4ac = 16-4(1)(2)$$
$$=16-8$$
$$=8$$

$$\frac{-b\pm\sqrt{\Delta}}{2a} = \frac{-4\pm\sqrt{8}}{2} = \frac{-4\pm 2\sqrt{2}}{2} = \begin{matrix}-2+\sqrt{2}\\ -2-\sqrt{2}\end{matrix}$$

$$y_1=e^{-2+\sqrt{2}}, \quad y_2=e^{-2-\sqrt{2}}$$

$$y(x)=c_1e^{-2+\sqrt{2}}+c_2e^{-2-\sqrt{2}}$$

\# Ex: Find a 2^{nd}-order linear homo D.E whose solution is : $y = c_1 e^{2x} + c_2 e^{3x}$

$$r=2 \quad r=3$$
$$(r-2)(r-3)$$
$$r^2-3r-2r+6$$
$$r^2-5r+6$$
$$y''-5y'+6y=0$$

\# If the characteristic equation $r^2+ar+b=0$ has equal roots $r_1=r_2=r$, then

$$y_1=e^{rx}, \quad y_2=xe^{rx}$$

\# Example : solve : $y''-6y'+9y=0$

$$r^2-6r+9=0$$
$$(r-3)(r-3)=0$$
$$r=3$$
$$y_1=e^{3x}, \quad y_2=xe^{3x}$$

\# Example: solve $y''+4\pi y'+4\pi^2 y=0$

$$r^2+4\pi r+4\pi^2=0$$
$$(r+2\pi)(r+2\pi)=0$$
$$r=-2\pi$$
$$y_1=e^{-2\pi x}, \quad y_2=xe^{-2\pi x}$$

\# Example : Find a 2^{nd}-order linear homo D.E whose general solution is : $y=(c_1+c_2x)e^{-2x}$

$$y=c_1e^{-2x}+c_2xe^{-2x}$$
$$(r+2)(r+2)=0$$
$$r^2+2r+2r+4=0$$
$$r^2+4r+4=0$$
$$y''+4y'+4y=0$$

\# Example: solve IVP

$$y''-9y=0, \quad y(0)=-2, \; y'(0)=-12$$

$$r^2-9=0$$
$$(r-3)(r+3)=0$$
$$r=3, \quad r=-3$$

$$y_{(x)}=c_1e^{3x}+c_2e^{-3x}$$
$$-2=c_1+c_2$$
$$c_1=-2-c_2$$
$$c_1=-2-1=-3$$

$$y'(x)=3c_1e^{3x}-3c_2e^{-3x}$$
$$-12=3c_1-3c_2$$
$$-12=3(-2-c_2)-3c_2$$
$$-12=-6-3c_2-3c_2$$
$$-12=-6-6c_2$$

$$\underset{+6}{-12}=\cancel{-6}\underset{+6}{}-6c_2$$
$$\frac{-6}{-6}=\frac{-6c_2}{-6}$$
$$\boxed{c_2=1}$$

Homo DEs Complex Roots

$\sqrt{-1} = i$

$(\sqrt{-1})^2 = -1$

* $\sqrt{-5} = \sqrt{5}\, i$ → Form for complex num $a + bi$

Example: solve $r^2 + 2r + 2 = 0$

$a = 1,\ b = 2,\ c = 2$

$\Delta = b^2 - 4ac = 4 - 4(1)(2) = 4 - 8 = -4$

$$\frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = \begin{array}{l} -1 + i \\ \underbrace{-1}_{\lambda} \underbrace{-1}_{\mu} i \end{array}$$

we will call this λ

complex Roots

$r = \lambda + \mu i$

$y_1 = e^{(\lambda + \mu i)x}$, $y_2 = e^{(\lambda - \mu i)x}$ ⇒ & these are complex solutions

↳ From these two complex solutions we can obtain the following two real solutions

$y_1 = e^{\lambda x} \cos(\mu x)$

$y_2 = e^{\lambda x} \sin(\mu x)$

Example: solve: $y'' + 2y' + 5y = 0$

$r^2 + 2r + 5 = 0$

$a = 1,\ b = 2,\ c = 5$

$\Delta = b^2 - 4ac = 4 - 4(1)(5) = 4 - 20 = -16$

$$\frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = \begin{array}{l} -1 + 2i \\ -1 - 2i \end{array}$$

$y_1 = e^{-x} \cos(2x)$ بوخذ بس الي الا ن
فيه +

$y_2 = e^{-x} \sin(+2x)$

$y(x) = c_1 e^{-x} \cos(2x) + c_2 e^{-x} \sin(2x)$

$y(x) = e^{-x}(c_1 \cos(2x) + c_2 \sin(2x))$

$y(x) = c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)$

Example: solve $y'' + 9y = 0$

$r^2 + 9 = 0$

$r^2 = -3$

$r = \sqrt{-3}$

$r = \pm\sqrt{3}\, i$

$y_1 = e^0 \cos(\sqrt{3}x) \to y_1 = \cos(\sqrt{3}x)$

$y_2 = e^0 \sin(\sqrt{3}x) \to y_2 = \sin(\sqrt{3}x)$

* Ex: Solve: $y'' + 16y = 0$

$$r^2 + 16 = 0$$
$$\sqrt{r^2} = \sqrt{-16}$$
$$r = \pm 4i$$

$y_1 = \cos(4x)$

$y_2 = \sin(4x)$

$y(x) = c_1 \cos(4x) + \sin(4x)$

* قبل هيك أخذنا أسئلة يعطينا فيها الـ solutions وإحنا نجيب المعادلة 6 هسا بالـ complex راح نعمل نفس الاشي
بس الطريقة أصعب فراح نمشي عحسب الخطوات

we have two roots r_1, r_2

$$(r - r_1)(r - r_2) = 0$$

then $r^2 - (r_1 + r_2) r + r_1 r_2 = 0$

* Find a 2nd-order linear homogenuous DE, whose solution is:-

$$y = e^{2x}[c_1 \cos 3x + c_2 \sin 3x]$$

مجرد ماشفنا sin, cos بنعرف إنو ال roots complex

$$y = c_1 e^{2x} \cos 3x + e^{2x} c_2 \sin 3x$$

يابستعمل y_1 يابستعمل y_2

$\lambda = 2, \mu = 3$

$\lambda \pm \mu i = 2 \pm 3i$

$r_1 = 2 + 3i, \quad r_2 = 2 - 3i$

$r_1 \times r_2 = (2+3i)(2-3i) = 4 - \cancel{6i} + \cancel{6i} - 9i^2$
$= 4 + 9 = 13$

$r_1 + r_2 = 2 + 3i + 2 - 3i$
$\boxed{r_1 + r_2 = 4}$

$r^2 - 4r + 13 = 0$

$y'' - 4y' + 13y = 0$

* Home work:-

1. if $\{e^x, e^{2x}\}$ is basis of $y'' + ay' + by = 0$, find a, b

$(r-1)(r-2)$
$r^2 - 2r - r + 2$
$r^2 - 3r + 2$ $\quad a = -3$
$\quad b = 2$

(2) if $\{e^{(-1+i)x}, e^{(-1-i)x}\}$

$r_1 = -1+i \quad , r_2 = -1-i$

$r_1 \times r_2 = (-1+i)(-1-i) = 1 + \cancel{i} - \cancel{i} - i^2 = 1+1 = 2$

$r_1 + r_2 = -1 + \cancel{i} - 1 - \cancel{i} = -2$

$r^2 - (r_1+r_2)r + r_1 r_2$

$r^2 - 2r + 2$

$y'' - 2y' + 2y = 0$

Cauchy Euler DEs

$x^2 y'' + a x y' + b y = 0 \quad , x > 0$ → second order

$f(x)$ نفس الي قبل بس المعاملات

$y = x^r$

method

$r(r-1) + ar + b = 0$

↳ & this has three cases

(1) Different Real Root $r_1 \neq r_2$

$\Delta > 0 \quad y_1 = x^{r_1} \quad , y_2 = x^{r_2}$

$\Delta = 0$ (2) equal Roots $r_1 = r_2 = r$

$y_1 = x^r \quad , y_2 = \ln x \, x^r$

$\Delta < 0$

(3) $\lambda \pm \mu i \Rightarrow$ complex roots

لا يوضع سالب الموجب هنا

$y_1 = x^{\lambda} \cos(\mu \ln x)$

$y_2 = x^{\lambda} \sin(\mu \ln x)$

Solve the following ODE :-

(1) $2x^2 y'' + 3xy' - y = 0$

$2r(r-1) + 3r - 1 = 0$

$2r^2 - 2r + 3r - 1 = 0$

$2r^2 + r - 1 = 0$

$(2r-1)(r+1) = 0$

$r = -1 \quad , r = \frac{1}{2}$

$y_1 = x^{-1}$

$y_2 = x^{\frac{1}{2}}$

$y(x) = c_1 x^{-1} + c_2 \sqrt{x}$

② $x^2y'' - 5xy' + 4y = 0$

$r(r-1) - 5r + 4 = 0$

$r^2 - r - 5r + 4 = 0$

$r^2 - 6r + 4 = 0$

$(r-3)\ (r-3)$

$r = 3$

$y_1 = x^3 \ , \ y_2 = x^3 \ln x$

$y(x) = c_1 x^3 + c_2 x^3 \ln x$

③ $x^2y'' - 5xy' + 13y = 0$

$r(r-1) - 5r + 13 = 0$

$r^2 - r - 5r + 13 = 0$

$r^2 - 6r + 13 = 0$

$a = 1, \ b = -6, \ c = 13$

$b^2 - 4ac = 36 - 4(1)(13)$

$= 36 - 52 = -16$

$\dfrac{6 \pm \sqrt{-16}}{2} = \dfrac{6 \pm 4i}{2} = \begin{matrix} 3 + 2i \\ 3 - 2i \end{matrix}$

$y_1 = x^3 \cos(2 \ln x)$

$y_2 = x^3 \sin(2 \ln x)$

$y(x) = c_1 x^3 \cos(2 \ln x) + c_2 x^3 \sin(2 \ln x)$

④ $xy'' + 4y' = 0 \quad , x > 0$

بنضرب المعادلة في x

$x^2y'' + 4xy' = 0$

$r(r-1) + 4r = 0$

$r^2 - r + 4r = 0$

$r^2 + 3r = 0$

$r(r+3) = 0$

$r = 0 \quad r = -3$

$y_1 = x^0 = 1 \qquad y_2 = x^{-3}$

$$\boxed{y(x) = c_1 + x^{-3} c_2}$$

* Given : $y = c_1 x^2 + c_2 x^2 \ln x$

$(r-2)(r-2)$

$r^2 - 2r - 2r + 4$

$r^2 - 4r + 4$

Cauchy Euler بهاي

$r(r-1) - 3r + 4$

$x^2 y'' - 3x y' + 4y = 0$

* Example: If $\{x^2 \cos(\ln x), x^2 \sin(\ln x)\}$ is a fundamental set of solutions of

$x^2 y'' + a x y' + b y = 0$, find a, b

sol :-

$r_1 = 2 + i \quad , \quad r_2 = 2 - i$

$r_1 \times r_2 = (2+i)(2-i) = 4 - 2i + 2i - i^2 = 4 + 1 = 5$

$r_1 + r_2 = 2 + \cancel{i} + 2 \cancel{- i} = 4$

$r^2 - (r_1 + r_2) r + r_1 r_2 = 0$

$r^2 - 4r + 5 = 0$

$r(r-1) - 3r + 5 = 0$

$x^2 y'' - 3x y' + 5y = 0$

$a = -3, \quad b = 5$

* Homework: If $\{\cos(2 \ln x), \sin(2 \ln x)\}$ is a basis of solutions of:

$a x^2 y'' + b x y' + c y = 0$

find a, b, and c

$r_1 = +2i \quad , \quad r_2 = -2i$

$r_1 \times r_2 = 2i \times -2i = -4i^2 = 4$

$r_1 + r_2 = 0$

$r^2 - (r_1 + r_2) r + (r_1 \times r_2)$

$r^2 - 0 + 4$

$r^2 + 4 = 0$

$x^2 y'' + 4y = 0$

* Homework: Solve the following ODEs

① $4x^2y''+y=0$

$4r(r-1)+1=0$
$4r^2-4r+1=0$

$(2r-1)\quad(2r-1)=0$

$2r-1=0$
$\boxed{r=\frac{1}{2}}$

$y_1=x^{\frac{1}{2}}\qquad y_2=x^{\frac{1}{2}}\ln x$

② $4x^2y''+4xy'-y=0$

$4r(r-1)+4r-1=0$
$4r^2-4r+4r-1=0$
$4r^2-1=0$

$\sqrt{r^2}=\sqrt{\frac{1}{4}}$

$r=\pm\frac{1}{2}$

$y_1=x^{-\frac{1}{2}}\quad , y_2=x^{\frac{1}{2}}$

$\boxed{y(x)=c_1x^{-\frac{1}{2}}+c_2x^{\frac{1}{2}}}$

③ $x^2y''-3xy'+6y=0$

$r(r-1)-3r+6=0$
$r^2-r-3r+6=0$
$r^2-4r+6=0$
$a=1\quad , b=-4, c=6$

$b^2-4ac=16-4(6)=-8$

$\frac{4\pm\sqrt{-8}}{2}=\frac{4\pm2\sqrt{2}i}{2}=2\pm\sqrt{2}i$

$\lambda=2,\ \mu=\sqrt{2}$

$y_1=x^2\cos(\sqrt{2}\ln x)$

$y_2=x^2\sin(\sqrt{2}\ln x)$

$y(x)=c_1\ x^2\cos(\sqrt{2}\ln x)+c_2\ x^2\sin(\sqrt{2}\ln x)$

④ $y'' = \frac{y}{x^2}$

$x^2 y'' = y$

$x^2 y'' - y = 0$

$r(r-1) - 1 = 0$

$r^2 - r - 1 = 0$

$a = 1 ,\ b = -1 ,\ c = -1$

$\Delta = b^2 - 4ac = 1 - 4(1)(-1) = 1 + 4 = 5$

$$\frac{1 \pm \sqrt{5}}{2} = \frac{1}{2} + \frac{\sqrt{5}}{2}$$

$$\frac{1}{2} - \frac{\sqrt{5}}{2}$$

$y_1 = x^{\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)}$

$y_2 = x^{\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)}$

$y(x) = c_1 \, x^{\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)} + c_2 \, x^{\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)}$

Undetermined coefficients

non-hom linear D.E راح نبلش بال

$y'' + p(x)\, y' + q(x)\, y = r(x)$

↳ non homo second order D E

Step one :→ بجيب ال homo و بحلها وبسمي حلها

$y_h = c_1 y_1 + c_2 y_2$

Step two : $y_{p(x)}$ → we use undetermined coeff to → find it

بنجيب ال y_p بشرط أن يكون ال non-homo term أحد ثلاث أشكال:-

① poly
② e^{ax}
③ $\sin ax$, $\cos x$

أو Compination بينهم

step three :- the general solution for non homo second order

$y = y_{p(x)} + y_h$

* Example : Solve : $y'' + y = 2x^2 + 6$ $\quad y(0) = 0$, $y'(0) = 1$

$y'' + y = 0$ $r^2 + 1 = 0$ $\sqrt{r^2} = \sqrt{-1}$ $r = \pm i$	$r_1 = i$, $r_2 = -i$ $y_1 = \cos(x)$ $y_2 = \sin(x)$	$y_h = C_1 \cos x + C_2 \sin x$

$y_p(x) = Ax^2 + bx + C$

$y' = 2Ax + b$

$y'' = 2A$

$2A + Ax^2 + bx + c = 2x^2 + 6$

$Ax^2 + bx + 2A + c = 2x^2 + 6$

$b = 0, \quad A = 2$

$2A + c = 6$

$4 + c = 6 \Rightarrow \boxed{c = 2}$
$-4 \quad -4$

$y_p = 2x^2 + 2$

$y = y_h + y_p$

$= c_1 \cos x + c_2 \sin x + 2x^2 + 2$

$y(0) = 0$

$y'(0) = 1$

$0 = c_1 + 2$

$c_1 = -2$

$y' = -c_1 \sin x + c_2 \cos x + 4x$

$1 = 0 + c_2 + 0$

$c_2 = 1$

$$\boxed{y(x) = -2\cos x + \sin x + 2x^2 + 2}$$

Example : solve : $y'' - 2y' = e^{3x}$

$\Rightarrow$ findin y_h

$y'' - 2y' = 0$

$r^2 - 2r = 0$

$r(r-2) = 0$

$r = 0, r = 2$

$y_1 = e^0 = 1$

$y_2 = e^{2x} = e^{2x}$

$y_h = c_1 + c_2 e^{2x}$

$\rightarrow$ finding y_p

$y_p = Ae^{3x}$

$$y' = 3Ae^{3x}$$
$$y'' = 4Ae^{3x}$$

$$4Ae^{3x} - 2(3Ae^{3x}) = e^{3x}$$

$$4Ae^{3x} - 6Ae^{3x} = e^{3x}$$

$$\cancel{e^{3x}}(4A - 6A) = \cancel{e^{3x}}$$

$$\frac{3A}{3} = \frac{1}{3} \Rightarrow A = \frac{1}{3}$$

$$y_p(x) = \frac{1}{3}e^{3x}$$

$$y = y_p + y_h$$

$$c_1 + c_2 e^{2x} + \frac{1}{3}e^{3x}$$

* Example: solve : $y'' + 2y' = 12\sin x$

$\Rightarrow$ finding y_h

$$y'' + 2y' = 0$$

$$r^2 + 2r = 0$$
$$r(r+2) = 0$$
$$r = 0,\ r = -2$$

$$y_1 = e^0 = 1$$
$$y_2 = e^{-2x}$$

$$y_h = c_1 + c_2 e^{-2x}$$

$\Rightarrow y_p = A\sin x + B\cos x$

$$y' = A\cos x - B\sin x$$

$$y'' = -A\sin x - B\cos x$$

$$-A\sin x - B\cos x + 2(A\cos x - B\sin x) = 12\sin x$$

$$-A\sin x - B\cos x + 2A\cos x - 2B\sin x = 12\sin x$$

$$(-A - 2B)\sin x + (-B + 2A)\cos x = 12\sin x$$

$$2(-A - 2B = 12)$$
$$-B + 2\cancel{A} = 0$$
$$-4B - 2A = 24$$
$$-5B = 24$$
$$B = \frac{24}{-5}$$

$$-B + 2A = 0$$
$$+\frac{24}{5} + 2A = 0$$
$$\frac{1}{2} \times 2A = \frac{-24}{5} \times \frac{1}{2}$$
$$A = \frac{-24}{10} = \frac{-12}{5}$$

$$y_p = \frac{-12}{5}\sin x - \frac{24}{5}\cos x$$

$$y = C_1 + C_2 e^{-2} - \frac{12}{5}\sin x - \frac{24}{5}\cos x$$

general solution

لو أعطاني مثلاً $y(0)=1$ ها بطبقها فوق بطريقة هاد

* Example: solve : $y'' - 2y' = 2e^{2x}$

$$y'' - 2y' = 0$$

$$r^2 - 2r = 0$$

$$r(r-2) = 0$$

$$\boxed{r=0} \quad \boxed{r=2} \qquad y_1 = 1, \quad y_2 = e^{2x}$$

so that

$$y_h = c_1 + c_2 e^{2x}$$

$$\Rightarrow y_p = Axe^{2x}$$

$$y' = Ax \cdot 2e^{2x} + e^{2x} \cdot A$$

$$y' = 2Axe^{2x} + Ae^{2x}$$

$$y' = e^{2x}(2Ax + A)$$

$$y'' = e^{2x} \cdot (2A) + 2(2Ax + A)e^{2x}$$

$$y'' = 2Ae^{2x} + 4Ax e^{2x} + 2Ae^{2x}$$

$$2Ae^{2x} + 4Axe^{2x} + 2Ae^{2x} - 2(2e^{2x}xA + Ae^{2x})$$

$$2Ae^{2x} + \cancel{4Axe^{2x}} + 2Ae^{2x} - \cancel{4xe^{2x}A} - 2Ae^{2x} = 2e^{2x}$$

$$4Ae^{2x} - 2Ae^{2x} = 2e^{2x}$$

$$2A\cancel{e^{2x}} = 2\cancel{e^{2x}}$$

$$\frac{2A}{2} = \frac{2}{2} \Rightarrow A = 2$$

$$y_p = 2e^{2x}$$

$$y(x) = y_p + y_h$$

$$= 2e^{2x} + C_1 + C_2 e^{2x}$$

Variation of parameters

applied on non-homo DEs

$$y'' + p(x)y' + q(x)y = r(x) \quad , r(x) \neq 0$$

$y = y_h + y_p \rightarrow$ undet-coeff

كان عليها شروط

اليوم راح نتعلم طريقة جديدة لحساب ال $y_p(x)$ وماعليها

أي قيود لا عال $r(x)$ ولا على عوامل ال y', y''

① $y_h = c_1 y_1 + c_2 y_2$

② W

بدون ال c_2, c_1

③ $y_p(x) = -y_1 \int \frac{y_2}{W} r + y_2 \int \frac{y_1}{W} r$

بس نعوض ال r تأكد إنو معامل ال y'' هو 1

Ex: Solve: $y'' + y = \sec x$

$y'' + y = 0$

$r^2 + 1 = 0$

$r = \pm i$

$y_1 = \cos x$

$y_2 = \sin x$

$y_h = c_1 \cos x + c_2 \sin x$

$W(\cos x, \sin x) = \cos^2 x - \sin x \cdot -\sin x$

$= \cos^2 x + \sin^2 x = 1$

$y_p = -\cos x \int \sin x \cdot \sec x - \sin x \int \cos x \cdot \sec x$

$y_p = -\cos x \int \frac{\sin x}{\cos x} - \sin x \int \frac{\cos x}{\cos x}$

$y_p = -\cos x \int \tan x - \sin x \int 1$

$y_p = -\cos x\,(-\ln|\cos x|) - x \sin x$

$$\boxed{y_p = \cos x \ \ln \cos x - x \sin x}$$

$y = c_1 \cos x + c_2 \sin x + \cos x \ \ln \cos x - x \sin x$

Example: Solve: $x^2 y'' - 2x y' - 4y = 12x^{-3}$

$\Rightarrow y_h$

$y_h \Rightarrow x^2 y'' - 2x y' - 4y = 0$

$r(r-1) - 2r - 4 = 0$

$r^2 - r - 2r - 4 = 0$

$r - 3r - 4 = 0$

$(r-4)(r+1) = 0$

$r = 4, \ r = -1$

$y_1 = x^4, \ y_2 = x^{-1}$

$y_h = c_1 x^4 + c_2 x^{-1}$

$$W(x^4, x^{-1}) = x^4 \cdot -1x^{-2} - x^{-1} \cdot 4x^3$$
$$= -x^2 - 4x^2$$
$$= -5x^2$$

$$y_p = -y_1 \int \frac{y_2}{W} \, r + y_2 \int \frac{y_1}{W} \, r$$

$$y_p = -x^4 \int \frac{x^{-1}}{-5x^2} \times 12x^{-5} + x^{-1} \int \frac{x^4}{-5x^2} \times 12x^{-5}$$

$$y_p = \frac{12x^4}{5} \int x^{-6} \cdot x^{-5} + \frac{-1x^{-1}}{5} \int x^2 \times 12x^{-5}$$

$$y_p = \frac{12x^4}{5} \int x^{-11} - \frac{12}{5} x^{-1} \int x^{-3}$$

⇐ هاد تخبيص بطلع الحل تحت

$$y_p = \frac{12x^4}{5} \frac{x^{-10}}{-10} - \frac{12}{5} x^{-1} \left(\frac{x^{-2}}{-2}\right)$$

$$y_p = \frac{-12}{50} x^{-10} + \frac{12}{10} x^{-3}$$

the general solution

$$y = \frac{c_1}{x} + c_2 x^4 + \frac{6}{7} x^{-3}$$

* H.w: $y'' + 2y' + y = 4e^{-x} \ln x$

فيه ln لم ما بقدر اطبق ال undetermined coeff

اجباري أروح عال variation of parameters

Reversion

consider:

r^3-2r^2-5r+6 ؟كيف بنوجد حلول هاي المعادلة

قبل كنا نوخذ قواسم العدد 6 : $1, 2, 3, 6$

ونجرب السالب لكل واحد كمان له

after trying $(r-1)(\_\ \_\ \_\ \_\ \_\ \_)$ معادلة

ل $r=1$ is a solution المعادلة الي فوقه

①

	1	-2	-5	6
		1	-1	-6
	1	-1	-6	0
	r^2	r	c	

① بنزل الواحد

② نضرب الواحد بالواحد الي فوقه ونحطها تحت التين وبعدين بنجمعهم

so the solutions are

$$(r-1)(r^2-r-6)=0$$

$$(r-1)(r-3)(r+2)=0$$

Ex: solving the following ODEs

① $y'''-2y''-5y'+6y=0$

$$r^3-2r^2-5r+6=0$$

$$(r-1)(r-3)(r+2)=0$$

$$r=1, 3, -2$$

$$y_1=e^x,\ y_2=e^{3x},\ y_3=e^{-2x}$$

$$y(x)=c_1e^x+c_2e^{3x}+c_3e^{-2x}$$

c_1 أول solution

وهكذا.

② $y^{(5)}-3y^{(4)}+3y'''-y''=0$

$$r^5-3r^4+3r^3-r^2=0$$

$$r^2(r^3-3r^2+3r-1)=0$$

$$r^2(r-1)(\_\ \_\ \_\ \_)$$

1	1	-3	3	-1
		1	-2	1
	1	-2	1	0
	r^2	r	c	

$$r^2(r-1)(r^2-2r+1)$$

$$r^2(r-1)(r-1)(r-1)$$

$$r=0, 0, 1, 1, 1$$

then the solutions are

$y_1=e^{0x}=1$

$y_2=xe^{0x}=x \Rightarrow$ لانو مكرر ضربته بـ x

$y_3=e^x=e^x$

$y_4=xe^x=xe^x$

$y_5=x^2e^x=x^2e^x$

$$y(x)=c_1+c_2x+c_3e^x+c_4xe^x+x^2e^x$$

Remark

we can also do

$$\underbrace{r^3}\underbrace{-3r^2+3r}\underbrace{-1}$$

$$(r^3-1)\ (-3r^2+3r)=0$$

$$(r-1)(r^2+r+1)-3r(r-1)$$

$$(r-1)(r^2+r+1-3r) \qquad r-1=0,\ r=1$$

$$(r-1)(r^2-2r+1)=0 \qquad r^2-2r+1 \Rightarrow r=1,1$$

$y^{(4)} + 2y'' + y = 0$

$r^4 + 2r^2 + 1 = 0$

$(r^2)^2 + 2(r^2) + 1 = 0$

$(r^2+1)\ (r^2+1) = 0$

$r^2+1=0\ ,\ r^2+1=0$

$\sqrt{r^2} = \sqrt{-1}$ $\quad r^2 = -1$

$r = \pm i$ $\quad r = \pm i$

so the solutions are

عشان تكرار

$y_1 = \cos x$ $\quad y_3 = \underline{x}\cos x$

$y_2 = \sin x$ $\quad y_4 = x \sin x$

↳ $e^{\lambda x}\cos(\mu x)$

$y(x) = c_1\cos x + c_2 \sin x + c_3 x\cos x + c_4\ x \sin x$

---

④ $x^3y''' - 3x^2y'' + 6xy' - 6y = 0$

How to solve Cauchy-Euler has a second order

$$\underset{r(r-1)(r-2)}{x^3y'''} + \underset{r(r-1)}{x^2y''} + \underset{r}{bxy'} + \underset{1}{cy} = 0$$

بيكون هذا (ماله) عشان زي ال y'''

وهكذا الكل مافيهمش ال y'''

↳ we don't make a solutions as

$x\ \underline{e^{rx}} \Rightarrow$ it's $\underline{x^r}$

& if we have repeated

بنضرب ب $\ln x$ مش بال $\underline{x}$

وبعدها sin

if we have complex instead of $e^{\lambda x}\cos(\mu x)$ we use

$x^{\lambda}\cos(\mu \ln x)$

---

④ $x^3y''' - 3x^2y'' + 6xy' - 6y = 0$

$r(r-1)(r-2) - 3r(r-1) + 6r - 6 = 0$

$r(r-1)(r-2) - 3r(r-1) + 6(r-1) = 0$

$(r-1)\,(r(r-2) - 3r + 6) = 0$

$(r-1)\,(r^2 - 2r - 3r + 6) = 0$

$(r-1)(r^2 - 5r + 6) = 0$

$(r-1)\ \ (r-3)\ \ (r-2) = 0$

$r = 1, 3, 2$

$r = 1, 3, 2$

$y_1 = x^1\ ,\ y_2 = x^2\ ,\ y_3 = x^3$

$y(x) = c_1 x + c_2 x^2 + c_3 x^3$

لنفرض صار عنا بحل هاي المعادلة تكرار مثلًا

$r = 1, 1, 3$

$y_1 = x\ ,\ y_2 = x\ln x\ ,\ y_3 = x^3$

Ex: find a 3^{rd}-order linear homogeneous D.E whose solution is :-

$y(x) = c_1 + c_2x + c_3e^{2x}$

Sol:

$r = 0, 0, 2$

$r^2(r-2) = 0$

$r^3 - 2r^2 = 0$

$\therefore\ y''' - 2y'' = 0$

so this is the solution

Ex: solve: $y''' + y'' = 2x+1$ ⇒ this part is not (0)

this is non-homogeneous

first step نفرض ال Homogenous ونحلها أساسية

$y''' + y'' = 0$

$r^3 + r^2 = 0$

$r^2(r+1) = 0$

$r = 0, 0, -1$

$y_1 = 1\ ,\ y_2 = x\ ,\ y_3 = e^{-x}$

$y_h(x) = c_1 + c_2x + c_3e^{-x}$ we find the homo---- solution now we want to find the particular solution

to find the particular solution:

method 1: the undetermined coefficients ← الها شروط

method 2: variation of parameters methods

له ما الها شروط لازم ان الطرف التاني تكون continous

$2x+1 \rightarrow$ continuous

we will use the first method & discuss the others next

الشكل تبع شرط الطريقة الأولى $\Leftarrow 2x+1$

$y_p(x) = Ax+B$

بنحط بدل الـ A

والـ $B \Leftarrow 1$

ونشوف إذا متكررين

بحلول الـhomogenous فنضرب بـ $\underline{x}$

$y_p(x) = x(Ax+B)$

برجع بقارن بحلول

الـhomogeneous بجربها أعوض بدل الـA والـ B واحد

برضو حلو موجود ✗ لأن الـ $B=x$ وقتها بضرب

$y_p(x) = x^2(Ax+B)$

or $y_p(x) = Ax^3+Bx^2 \rightarrow$ sotable form

بنشتق عدد ما مطلوب بالمعادلة الأصلية

$y''' + y'' = 2x+1$ --- (*)   (هريته $3 \Leftarrow$ مرات)

$y_p(x) = Ax^3+Bx^2$

$y_p'(x) = 3Ax^2+2Bx$

$y_p''(x) = 6Ax+2B$

$y'''(x) = 6A$

By substitution in (*)

$6A + 6Ax + 2B = 2x+1$

$6Ax + (6A+2B) = 2x+1$

$6A\cancel{x} = 2\cancel{x}$	$6A+2B = 1$
$\frac{6A}{6} = \frac{2}{6}$	$6(\frac{1}{3}) + 2B = 1$
$A = \frac{1}{3}$	$2+2B = 1$, $-2 \quad -2$

$\frac{2B}{2} = \frac{-1}{2} \Rightarrow B = \frac{-1}{2}$

so that

$y_p(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2$

The general solution of (*) is given by:-

$y = y_h + y_p$

$= C_1 + C_2x + C_3e^{-x} + \frac{1}{3}x^3 - \frac{1}{2}x^2$

Ex: Consider the ODE

$$y^{(4)} + 2y''' + 2y'' = x^2 + 3 + 2e^{-x} + xe^x \sin x$$

Find the suitable form for $y_p(x)$ if the undetermined coefficient is to be used

$y^{4} + 2y''' + 2y'' = 0$

$r^4 + 2r^3 + 2r^2 = 0$

$r^2(r^2+2r+2) = 0$

$r=0, r=0$

$r^2+2r+2=0$

$a=1, b=2, c=2$

$b^2 - 4ac =$

$4 - 4(1)(2)$

$= 4 - 8 = -4$

$\frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-2 \pm \sqrt{-4}}{2}$

$-1 \pm i$

so $r = 0, 0, -1 \pm i$ $\quad e^{\lambda x}\cos(\mu x)$

$y_1 = 1 \qquad y_1 = e^{-x}\cos x$

$y_2 = x \qquad y_2 = e^{-x}\sin x$

now taking the general form to

#غلطتي فيه

مهم ركزي

$Ax^2+3x+C \qquad pe^{-x}$

$x^2 + 3 + 2e^{-x} + xe^x \sin x \rightarrow$ بوخذ هاد لحاله

وال general form لهاد

$(Ex+F)e^x \sin x$

$y_p(x) = (A_2x^2 + A_1x + A_3)\, x^2$

بدل حالي عن الـx^2 و الـx والواحد هل في واحد

$+ Be^{-x}$ | موجود بضرب بالـ x وهكذا

$+ (Cx+D)e^x \sin x + (Ex+F)e^x \cos x$

this is the general form

لازم نحط الـ(cos)

Revision

$w(y_1, y_2) = y_1 * y_2' - y_2 * y_1'$

$$w(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

كيف نحسب الdeterminant لمصفوفة ثلاثية

$$\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & 1 & 4 \end{vmatrix}$$

بنختار أي صف أو عامود حابين نشتغل فيه ويفضل إذا في أصفار نختار أكبر صف/عامود فيه عدد من الأصفار و نشتغل فيه

هون إحنا اخترنا ال ① فبنشيل الصف مع عامود ال ①

positive, negative, positive

$$(1)\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} + (1)\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$$

$1(4-2) - (-1)(4-1) + 1(2-1)$

$2 - (-1)(3) + (1)$

$2 + 3 + 1 = 6$

ال 1 كيف بدي أعرف إشارتو عن طريقه رتبة الصف + رتبة العامود

$2 + 3 = 5$ odd number $\Rightarrow$ negative

even $\Rightarrow$ positive

$$\begin{bmatrix} 4 & 8 \\ 2 & 6 \end{bmatrix} \Rightarrow$$ بنربط أطلع منها عامل مشترك $$= 2 \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$$ إذا بدنا ندخل ال 2 عال (Matrix) $\Rightarrow$ بدنا نجبر فندخل ال 2 ونضربها برقم رقم بال Matrix

لم تكن اجباري من كل ال Matrix

find the determenant

$(4 \times 6) - (2 \times 8)$

$24 - 16 = 8$

هسا بربط أستفيد من خصائص ال determenant وأطلع منه عامل مشترك بس ال determenant التفاضل مع خاصيته اطلع من (صف/صف) أو (عامود/عامود)

$$\begin{bmatrix} 4 & 8 \\ 2 & 6 \end{bmatrix} = 4 \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix} = 4 \cdot 2 \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

له بدي أوخذ العامل المشترك من العامود

$8(3-2) = 8(1)$

$= 8$

& this is important to be used in this type of :-

$w(e^{-x}, e^x, e^{2x}) =$

$$\begin{array}{ccc} e^{-x} & e^{x} & e^{2x} \end{array}$$

$$\begin{bmatrix} e^{-x} & e^{x} & e^{2x} \\ -e^{-x} & e^{x} & 2e^{2x} \\ e^{-x} & e^{x} & 4e^{2x} \end{bmatrix} = e^{-x} \cdot e^{x} \cdot e^{2x} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & 1 & 4 \end{bmatrix}$$

له إذا بدنا نحسب ال determenant بالطريقة التقليدية ممكن تلخبطو ممكن نوخذ عامل مشترك من كل عامود لحالو

$e^{-x} \cdot e^{x} \cdot e^{2x},$

$\{(4-2) - (-1)(4-1) + (2-1)\}$

$= e^{-x} \cdot e^{x} \cdot e^{2x} (6)$

$= 6e^{2x}$

Higher order Non homogeneous ODEs "variation of parameters"

Consider the 3^{rd} - order linear ODE :-

ما راح نتجاوز ال third مشتقة

$$1y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = r(x)$$

1 اجباري أو نقسم عليه

where $p_2(x), p_1(x), p_0(x)$ and $r(x)$ are continuous

on the same open interval

① step 1: solve the Homogenuous eq,n

بخلي ال $\boxed{0 = r(x)}$

② step 2: find the particular solution

→ ① using undetermine coficiants

الي حلينا عليها الهرة الماضية وما بتحل الا 3 أشكال

Ⓐ polynomial

Ⓑ exponential

Ⓒ sin/cos

& it will be prefered that the left hand side with constant coffecient

→ $y_p(x) = y_1 \int \frac{w_1}{w} r + y_2 \int \frac{w_2}{w} r + y_3 \int \frac{w_3}{w} r$

y_1, y_2, y_3 are independent solutions of the homogeneous part.

$w = w(y_1, y_2, y_3) =$

$\cancel{y_1}$ 0	y_2	y_3
$\cancel{y_1'}$ 0	y_2'	y_3'
$\cancel{y_1''}$ 1	y_2''	y_3''

w_1 بنستعمل عوض
0 نتبه نستبدل ب
0
1 زي بالأخير
وبحسبها
$w_2 \rightarrow$ عالعمود الثاني

first voice

Ex: Find the following :-

1. $w(x, x^2, x^3)$
2. $w_1(x, x^2, x^3)$
3. $w_2(x, x^2, x^3)$
4. $w_3(x, x^2, x^3)$

$w(x, x^2, x^3)$

x	x^2	x^3
1	$2x$	$3x^2$
0	2	$6x$

$$x(12x^2 - 6x^2) - 1(6x^3 - 2x^3) + \cancel{0(3x^4 - 2x^4)}$$

$$6x^3 - 4x^3 = 2x^3$$

$w_1(x, x^2, x^3)$

0	x^2	x^3
0	$2x$	$3x^2$
1	2	$6x$

$$0(12x^2 - 6x^2) - 0(6x^3 - 2x^3) + 1(3x^4 - 2x^4)$$

$$w_1 = x^4$$

$w_2(x, x^2, x^3)$

x	0	x^3
1	0	$3x^2$
0	1	$6x$

$$-1(3x^3 - x^3)$$

$$-2x^3$$

$w_3 = (x, x^2, x^3)$

x	x^2	0
1	$2x$	0
0	2	1

$$+1(2x^2 - x^2)$$

$$= x^2$$

now returning to the subject

Ex: Solve the ODE :-

$$x^3 y''' - 3x^2 y'' + 6x y' - 6y = 24x^4, \quad x > 0$$

$$r(r-1)(r-2) - 3r(r-1) + 6r - 6 = 0$$

$$r(r-1)(r-2) - 3r(r-1) + 6(r-1) = 0$$

$$(r-1)(r(r-2) - 3r + 6) = 0$$

$$(r-1)(r^2 - 2r - 3r + 6) = 0$$

$$(r-1)(r^2 - 5r + 6) = 0$$

$$(r-1)(r-2)(r-3) = 0$$

$$r = 1, 2, 3$$

$$y_1 = x, \quad y_2 = x^2, \quad y_3 = x^3$$

$$y_h(x) = c_1 x + c_2 x^2 + c_3 x^3$$

\# we can not use undetermined coefficient here because the معاملات must be constants

continued

$$y_p(x) = y_1 \int \frac{w_1}{w} r + y_2 \int \frac{w_2}{w} r + y_3 \int \frac{w_3}{w} r$$

هيك برتبهم
$w(x, x^2, x^3)$ وحسبناه فوقه

$y_1 = x$
$y_2 = x^2$
$y_3 = x^3$

$$w(x, x^2, x^3) = 2x^3$$

$$w_1 = x^4 \qquad w_3 = x^2$$

$$w_2 = -2x^3$$

now I only have to calculate the (r)

⇐ الحالة الوحيدة الي بقسم فيها على معامل ال y''' هي عند تكوين ال r ال r هي ال (wright hand side) المعادلة

$\frac{24x^4}{x^3}$ $24x^4 \rightarrow$ write hand side
$x^3 \rightarrow$ معامل ال y'''

$$r(x) = 24x$$

$$y_p(x) = x\int \frac{x^4}{2x^3}\cdot 24x + x^2\int \frac{-2x^3}{2x^3}\cdot 24x + x^3\int \frac{x^2}{2x^3}\cdot 24x$$

$$y_p(x) = (x \cdot 4x^3) - 12x^4 + 12x^4$$

$$= 4x^4 - 12x^4 + 12x^4$$

$$= 4x^4$$

$$y(x) = y_h + y_p$$

$$= c_1 x + c_2 x^2 + c_3 x^3 + 4x^4$$

Home work: Solve the ODE solution

$$y''' + y' = \sec x$$

Homogeneous linear system with constant coefficients

- consider the linear system:-

$$a y_1 + b y_2 = f_1$$

$$c y_1 + d y_2 = f_2$$

this this system can be written

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$A * y = f$$

$$\begin{bmatrix} a y_1 + b y_2 \\ c y_1 + d y_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

& that means that
the colored that makes us go again to the first eqns

also it can be written as

$$Ay = f$$

non homogenonous

$$A y = \vec{0} \quad \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

homogenuous

A^{-1} exists $\equiv |A| \neq 0$ (determinant $\neq 0$)

if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $|A| \neq 0$

then $A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

بنغير الإشارة

بنبدل

Ex: if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then $|A|$?? & $|A|\ A^{-1}$

لا المعكوس
مش يطلع غلط

$$|A| = (4-6) = -2$$

$$A^{-1} = \frac{-1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}$$

- $A A^{-1} = A^{-1}A = I$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

ال I إذا ضربتها بأي vector
بعطيني نفس ال vector

to day we will talk about homogenous system

$$Ay = 0$$

how to solve it?

if the $|A| \neq 0$
so the only solution is $y = 0$

$$Ay = 0$$

بدنا نضرب الطرفين بال A^{-1}

$$A^{-1}Ay = A^{-1}0$$

$$Iy = 0$$

$$\boxed{y = 0}$$

but if the $Ay = 0$
$|A| = 0$
↳ so this system has infinitly many solutions & one of them is zero

so
if we have

$$Ay = 0$$

only two solutions

- $|A| = 0$: infinitly many solutions & the zero is one of them
- $|A| \neq 0$: $y = 0$ only one solution

now we will talk about eigen value & eigenvector

$A\acute{S} = \lambda \acute{S}$

$Ay = \lambda y$

y: eigen vector

λ: corresponding eigen value

إذا ضربنا ال A بالـ vector وطلعنا ال Multible كأنو ال

but $y \neq 0$

if $y \neq 0$

$Ay = \lambda y$

$Ay = \lambda I y$

$Ay - \lambda I y = 0$

$[A - \lambda I] y = 0$ to find the non zero eigen vector the dector must يحقق هاي المعادلة

إذا مستحيل تكون Zero y لازم إذا الإجباري ما بساوي Zero =

$By = 0$

$|B| = 0$

what is B شو

$[A - \lambda I] = 0$

So to find eigen vectors must أولا تحقق هاي المعادلة

Summary

if we have this system $Ay = 0$ which is homogeneous

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

وعشان نلاقي ال eigen value لما ال A

determinant form

$|A - \lambda I| = 0$

إذا طلب مني eigenvectors

matrix form

$[A - \lambda I] y = 0$

what is $A - \lambda I$??

$A - \lambda I = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$\begin{bmatrix} a-\lambda & b-0 \\ c-0 & d-\lambda \end{bmatrix}$

$= \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$

so it is the same of A

$\lambda \in$ main diagonal بس مطروح من ال

$\acute{S} \neq 0$ vector so that

$A\acute{S} = \lambda \acute{S}$

then $(\acute{S})$ is called an eigen vector of A corresponding to eigen value (λ)

Ex: find the eigen values & the corresponding eigen vectors to

$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

هلأ بحسب ال eigen values

$[A - \lambda I] = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$= \begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} = 0$

وحلها زي ما كنا نحل ال Matrix قبل

$[(1-\lambda)(1-\lambda) - 4] = 0$

$[1 - \lambda - \lambda + \lambda^2 - 4] = 0$

$[\lambda^2 - 2\lambda - 3] = 0$

$(\lambda - 3)(\lambda + 1) = 0$

$\lambda = 3 \quad , \quad \lambda = -1$

هدول ال eigen values

continued in the second page

corresponding eigen vectors يدو ال

$[A-\lambda I]S^{\varepsilon}=0$

$$\begin{bmatrix}1-\lambda & 1\\ 4 & 1-\lambda\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\lambda=-1\quad\begin{bmatrix}2 & 1\\ 4 & 2\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$2y_1+y_2=0$$

$$4y_1+2y_2=0$$

if the determinant for the matrix is zero, that means:-

① there is no inverse

② so the rows are dependent شو يعني؟ المعادلات يعني أحدهم multible للآخر

& that mean the the eqns are the same that means that we will only solve one eqn

$2y_1+y_2=0 \Rightarrow y_2=-2y_1 \Rightarrow$

بنعوض اي رقم الا الصفر وبنعتبر جواب كا طيب ليش ما بوخد الصفر؟ لانو بدو يعطيني eigen vector ناو بدو على $S^{\varepsilon}\neq 0$

how many solutions did we have? infinitly many solutions

$S^{\varepsilon(1)}=\begin{bmatrix}1\\ -2\end{bmatrix}$ → dependant طيب بربط أوخد هاي $y_1=2,3$ 6 بزبط بس الا تكون

$[A-\lambda I]S^{\varepsilon}=0$

بنرجع نوخد نفس المعادلة وزي ما عملنا الا - بنعوض ال 3

$$\begin{bmatrix}1-\lambda & 1\\ 4 & 1-\lambda\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\lambda=3\quad\begin{bmatrix}-2 & 1\\ 4 & -2\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$-2y_1+y_2=0$

$4y_1-2y_2=0$ $\Rightarrow$ نفس الحكي وحدة multible عن التانية

$S^{\varepsilon(2)}=\begin{bmatrix}1\\ 2\end{bmatrix}$ $y_2=2y_1$

now we are going to discuss First order Differential systems

so consider this system

$$y_1'=ay_1+by_2$$

$$y_2'=cy_1+dy_2$$

as a vectors y_1' & y_2' are y' which is the vector

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix}'=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}$$

$y'=Ay$ | أول خطوة: نوجد ال eigen values لل $\underline{A}$ λ_1,λ_2

كيف شكل ال solution | وبعدين ال eigen vector بحسب ال eigen value corresponding

$y^{(1)}=e^{\lambda_1 t}S^{\varepsilon(1)} \Rightarrow$ إذا بشبعو بعض | $S^{\varepsilon(1)}, S^{\varepsilon(2)}$ real + هيك الأجوبة

$y^{(2)}=e^{\lambda_2 t}S^{\varepsilon(2)}$

Ex: solve : $y_1'=y_1+y_2$
$y_2'=4y_1+y_2$

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix}'=\begin{bmatrix}1 & 1\\ 4 & 1\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}$$

$$y'=A\quad y$$

أهم شي ال matrix A وبنحسبلها " eigen values & eigen vector

هون ال Matrix A نفس أرقام السؤال الي قبل فبوخدهم زي ما هم

$\lambda_1=-1,\ \lambda_2=3$

$S^{\varepsilon(1)}=\begin{bmatrix}1\\ -2\end{bmatrix}$ $S^{\varepsilon(2)}=\begin{bmatrix}1\\ 2\end{bmatrix}$

$$y^{(1)}=e^{\lambda_1 t}S^{\varepsilon(1)}=e^{-t}\begin{bmatrix}1\\ -2\end{bmatrix}$$

$$y^{(2)}=e^{\lambda_2 t}S^{\varepsilon(2)}=e^{3t}\begin{bmatrix}1\\ 2\end{bmatrix}$$

the general solution

$$y=c_1y^{(1)}+c_2y^{(2)}$$

Homogeneous system II

Ex: solve $y' = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} y$

جلية ضروري

اصنافها موصفينا لهاي

sol

بحسب eigen value

$$[A - \lambda I] = \begin{bmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{bmatrix}$$

$$|A - \lambda I| = 0$$

اذا أخذت ال determinant وساويتها بالصفر ضمناها إنا قاعدة بحسب eigen value

$$\begin{bmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)(-4-\lambda) - -6 = 0$$

$$(\lambda - 1)(\lambda + 4) + 6 = 0$$

$$\lambda^2 + 4\lambda - \lambda - 4 + 6 = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda + 1) \quad (\lambda + 2) = 0$$

$$\lambda = -1, -2$$

eigen values

إذا أخذت ال $|A - \lambda I|$ ع هيك إنا بوجد ال vector

$$\begin{bmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ بعوض

eigen vector هسا بدي أوجد ال

$$\lambda = -1 \Rightarrow \begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2y_1 - 2y_2 = 0 \qquad y_1 = y_2$$

$$3y_1 - 3y_2 = 0 \qquad \text{taking } 1$$

one of them is the multible for another

$$\xi^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$y^{(1)} = \xi^{(1)} \cdot e^{\lambda_1 t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

$\lambda = -2$

$$\begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3y_1 - 2y_2 = 0 \Rightarrow y_2 = \frac{3}{2} y_1$$

$$3y_1 - 2y_2 = 0$$

taking $y_1 = 2$ عشان نتجنب الكسر

$$\xi^{(2)} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$y^{(2)} = \xi^{(2)} e^{\lambda_2 t}$$

$$y^{(2)} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{-2t}$$

the general solution is:-

$$y = c_1 y^{(1)} + c_2 y^{(2)}$$

$$= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{-2t}$$

Complex eigen values

consider:-

$$y' = Ay$$

① راح نحسب ال eigen function راح يطلعو two (complex)

$$y(t) = \mu(t) + i\,\nu(t)$$

the real part $\nu(t)$ & the imaginary part $\mu(t)$ are imaginary solutions so we will recieve two real solutions from one complex solution

there is a question followed to this in the next page

Ex: solve: $y_1' = -y_1 + y_2$

$y_2' = -y_1 - y_2$

sol:-

أول شي بنكتبها بال Matrix form

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$y' = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} y$$

و ذكي بنعوض بال Matrix form هو Matrixform هاي

$$y' = Ay$$

* to find the eigen values we set

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = 0$$

$$(-1-\lambda)(-1-\lambda) - -1 = 0$$

$$(1+\lambda)(1+\lambda) + 1 = 0$$

$$\sqrt{(1+\lambda)^2} = \sqrt{-1}$$

$$1 + \lambda = \pm i$$

$$\lambda = \pm i - 1$$

$$\lambda = +i - 1 \ , \ -i - 1$$

we have two eigen values

Choose one of them

& corresponding eigen vector بنجيب

$$\lambda = i - 1 \Rightarrow \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

لأنه أولى Multiple EV مو مضاعفة

$$-iy_1 + y_2 = 0$$

$$y_2 = iy_1$$

$y_1 = 1$

$$S^{(1)} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$y^{(1)} = e^{(-1+i)t} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\lambda = -i - 1 \Rightarrow \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

لما بختار complex واحد من الي طلعولي ما بفرقة ع واحد

$$iy_1 + 1y_2 = 0$$

$$y_2 = iy_1$$

taking $y_1 = 0$

$y_2 = i$

$$S^{(2)} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$y^{(2)} = e^{(i-1)t} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

but we don't have to solve $y^{(2)}$ to find it we can use only the first solution $y^{(1)}$ to find $y^{(2)}$

$$y^{(1)} = e^{(-1+i)t} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

إذا الصعن كان في (-)

$$e^{-t} e^{it} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

بنحط هون (-)

$$e^{-t} [\cos t + i \sin t] \begin{bmatrix} 1 \\ i \end{bmatrix}$$

بنضرب هاد القوس بال Matrix تحتها

$$e^{-t} \begin{bmatrix} \cos t + i \sin t \\ -\sin t + i \cos t \end{bmatrix}$$

$$e^{-t} \left(e^{-t} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + i e^{-t} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} \right)$$

real + imaginary

هسا بنزل ال e^{-t}

$$y^{(1)}(t) = e^{-t} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + i e^{-t} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

$$y^{(1)}(t) = e^{-t} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} , \quad y^{(2)}(t) = e^{-t} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

the general solution

$$y(x) = c_1 e^{-t} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

Solve:

$$y' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} y$$

sol

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$y' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} y$$

$$(A - \lambda I) = \begin{bmatrix} -1-\lambda & -4 \\ 1 & -1-\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(-1-\lambda)(-1-\lambda) - -4 = 0$$

$$(-1-\lambda)^2 + 4 = 0$$

$$\sqrt{(-1-\lambda)^2} = \sqrt{-4}$$

$$\begin{aligned} -1-\lambda &= \pm 2i \\ +1 \quad & \quad +1 \end{aligned}$$

$$\begin{aligned} \frac{-\lambda}{-} &= +2i + 1 \quad \times -1 \\ & -2i - 1 \end{aligned}$$

$$\lambda = -2i - 1$$
$$2i + 1$$

$$\lambda = -2i-1 \Rightarrow \begin{bmatrix} 2i & -4 \\ 1 & 2i \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2iy_1 - 4y_2 = 0$$

$$\frac{4y_2}{2} = \frac{2iy_1}{2}$$

$$y_2 = iy_1 \qquad \begin{bmatrix} 1 \\ i \end{bmatrix}$$

so that when $y_1 = 1$

$$S^{(1)} = e^{(2i-1)t} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\Downarrow$$

$$= e^{-2it} e^{-1t} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$e^{-t} [\cos(2t) - i\sin(2t)] \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$e^{-t} \begin{bmatrix} \cos(2t) - i\sin(2t) \\ \sin 2t + i\cos 2t \end{bmatrix}$$

$$e^{-t} \left(\begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix} + i \begin{bmatrix} -\sin 2t \\ \cos 2t \end{bmatrix} \right)$$

$$\left(e^{-t} \begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix} + e^{-t} i \begin{bmatrix} -\sin 2t \\ \cos 2t \end{bmatrix} \right)$$

$$y^{(1)} = e^{-t} \begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix}$$

$$y^{(2)} = e^{-t} \begin{bmatrix} -\sin 2t \\ \cos 2t \end{bmatrix}$$

general solution is

$$y(x) = c_1 e^{-t} \begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -\sin 2t \\ \cos 2t \end{bmatrix}$$

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we discuss the cases when we have:-

$\lambda = 1, 3 \Rightarrow$ discussed

$\lambda = -1 \pm i \Rightarrow$ discussed

$\lambda = 3, 3 \Rightarrow$ repeated
we are going to discuss this

equal eigen values

Ex: solve: $y' = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} y$

$$[A - \lambda I] = \begin{bmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix} = 0$$

$$(4-\lambda)(2-\lambda) - -1 = 0$$

$$(\lambda-4)(\lambda-2) + 1 = 0$$

$$\lambda^2 - 2\lambda - 4\lambda + 8 + 1 = 0$$
$$\lambda^2 - 6\lambda + 9 = 0$$
$$(\lambda-3)(\lambda-3) = 0$$

$$\lambda = 3, \quad \lambda = 3$$

$\lambda = 3, 3$
repeated

$\lambda = 3$

$[A - \lambda I]\,\overset{\epsilon}{S} = 0$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

بوخذ
قيمة
وحدة
كيفي
صفر

$y_1 + y_2 = 0$

$y_2 = -y_1$

note

the eigen vector ممنوع أخليها $y_1 = 0$ لأنو بصير $\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ وهاد ممنوع

$$\overset{\epsilon}{S} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

$$y^{(1)} = e^{3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

to find the second solution

$$y^{(2)} = t e^{\lambda t}\overset{\epsilon}{S} + \underbrace{\eta}_{\text{generalized eigen vector}} e^{\lambda t}$$

بوخذ نفس المعادلة الي جبتها
ال $\overset{\epsilon}{S}$ بس بدل ما عالجهة الثانية $\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ بعوض ال $\overset{\epsilon}{S}$ $\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$

to find η

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$y_1 + y_2 = 1$

$y_1 = 1 - y_2$

$\eta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

هون بنوخذ ال $y_1 = 0$

$$y^{(2)}(t) = \overset{\epsilon}{S}\, t e^{\lambda t} + \eta\, e^{3t}$$

$$t e^{3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{3t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$e^{3t}\left(t\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

the general solution is

$$y(t) = c_1 e^{3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{3t}\left(t\begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

solve: $y' \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} y$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(1-\lambda)(3-\lambda) - -1 = 0$

$(\lambda - 1)(\lambda - 3) + 1 = 0$

$\lambda^2 - 3\lambda - \lambda + 3 + 1 = 0$

$\lambda^2 - 4\lambda + 4 = 0$

$(\lambda - 2)(\lambda - 2) = 0$

في هعا (-1)
طلعت كا!

$\lambda = 2, 2$

$$\lambda = 2 \Rightarrow \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-y_1 - y_2 = 0$

$y_2 = -y_1$

$y_1 = 1$

$y_2 = -1$

$$\overset{\epsilon}{S} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$y^{(1)} = e^{2\lambda}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$y^{(2)} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$-y_1 - y_2 = 1$

$-y_2 = 1 - y_1$

$y_2 = y_1 - 1$

$y_1 = 0$

$y_2 = -1$

$\eta = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$$y^{(2)} = t e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{2t}\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$y(t) = c_1 e^{2\lambda}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{2t}\left(t\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$$

eigen vector من شروطها ال

non-zero vector انو

Non Homogenuous system

$$y' = \underbrace{Ay}_{\text{homo لحالهم}} + \underbrace{F(t)}$$

إذا صيغة أي vector لـ
سواء يعتمد عال t و x x
بصير non-homogenuous

How to use undetermined coefficients
في حالة الـ system

Ex: solve: $y' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} y + \begin{bmatrix} -6 \\ 2 \end{bmatrix}$

راح نوجد الحل للـ homo عادي
نجيب الـ particular solution
ومجموعهم راح يكون الـ general solution

بنوخذ الـ homo بنحلو

$$y' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} y$$

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{bmatrix} = 0$$

$$(-3-\lambda)^2 - 1 = 0$$

$$\sqrt{(-3-\lambda)^2} = \sqrt{1}$$

$$-3 - \lambda = 1 \quad , \quad -3 - \lambda = -1$$
$$+3 \qquad +3 \qquad\qquad +3 \qquad +3$$

$$-\lambda = 4 \qquad\qquad -\lambda = 2$$

$$\boxed{\lambda = -4} \qquad\qquad \boxed{\lambda = -2}$$

$$\lambda = -4 \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$y_1 + y_2 = 0$$

$$y_2 = -y_1 \qquad y_1 = 1, \; y_2 = -1 \qquad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$y^{(1)} = e^{-4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda = -2 \Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-y_1 + y_2 = 0 \qquad y_1 = 1$$
$$y_2 = y_1 \qquad y_2 = 1$$

$$y^{(2)} = e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$y_h(x) = c_1 e^{-4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$y^{(p)} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

مبدئياً حالياً هل
الـ constant vector هو
أحد حلول الـ homo? لا

راجعيه ضروري

suppose this is a solution

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

$$0 = -3a_1 + 1a_2 - 6$$
$$0 = a_1 - 3a_2 + 2$$

$$(3a_1 - a_2 = -6)\,3$$
$$3a_2 - a_1 = +2$$

$$9a_1 - 3\cancel{a_2} = -18$$
$$-a_1 + 3a_2 = +2$$

$$\frac{8a_1}{8} = \frac{-16}{8}$$

$$\boxed{a_1 = -2}$$

$$\boxed{a_2 = 0} \qquad \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

the general solution:-

$$y(t) = c_1 e^{-4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

Ex: consider $y' = \begin{bmatrix}-3 & 1\\ 1 & -3\end{bmatrix} y + \begin{bmatrix}-6\\ 2\end{bmatrix} e^{-2t}$

Determine the suitable form for the particular solution $y^{(p)}$

sol

في مقدمة ادرسها قبل الحل ⇒

المفروض أول خطوة إني نحل ال Homo

بس هو نفس أرقام السؤال اللي قبل فراح نوخذ الحل جاهز

$y^{(1)}(t) = c_1 e^{-4t}\begin{bmatrix}1\\ -1\end{bmatrix}$ $y^{(2)}(t) = c_2 e^{-2t}\begin{bmatrix}1\\ 1\end{bmatrix}$

continued من حل ال λ ذي تحت

$y^{(p)} = \begin{bmatrix}a_1\\ a_2\end{bmatrix} e^{-2t}$

$y^{(p)} = \underbrace{a}_{\text{vector}} e^{-2t}$ ⇒ سؤال حالي هل ال constant في e^{-2t} واحد من حلول ال homo؟ نعم.

ما بصير ب solution ب t

لنضرب ب n $(at+b)$

$y^{(p)}(t) = (at+b)e^{-2t}$

ممنوع أدخل ال e^{-2t}

$y^{(p)}(t) = \left(\begin{bmatrix}a_1\\ a_2\end{bmatrix} t + \begin{bmatrix}b_1\\ b_2\end{bmatrix}\right) e^{-2t}$

$y' = \begin{bmatrix}-3 & 1\\ 1 & -3\end{bmatrix} y + \begin{bmatrix}-6\\ 2\end{bmatrix} \cancel{e^{-2t}}$

إذا بدل ال e^{-2t} شي cost

بعط إنا طلعو نفس أجوبة ال homo

$a\cos t + b\sin t$

Ex: Determine a suitable form for the particular solution $y^{(p)}$ of the system

$y_1' = 5y_1 + 3y_2 - 2e^{-t} + 1$

$y_2' = -y_1 + y_2 + e^{-t} - 5t + 7$

sol

$y' = \begin{bmatrix}5 & 3\\ -1 & 1\end{bmatrix} y + e^{-t}\begin{bmatrix}-2\\ 1\end{bmatrix} + t\begin{bmatrix}0\\ -5\end{bmatrix} + \begin{bmatrix}1\\ 7\end{bmatrix}$

بنحسب ال homo solutions

$\begin{bmatrix}5-\lambda & 3\\ -1 & 1-\lambda\end{bmatrix} = 0$

$(5-\lambda)(1-\lambda) - -3 = 0$

$5 - 5\lambda - \lambda + \lambda^2 + 3 = 0$

$\lambda^2 - 6\lambda + 5 + 3 = 0$

$\lambda^2 - 6\lambda + 8 = 0$

$(\lambda - 4)(\lambda - 2) = 0$

$\lambda = 4, \quad \lambda = 2$

Since the eigen values are: $\lambda = 2, \lambda = 4$

when the $\lambda = 3, 3$

ركزنا على ال eigenvalue

$y^{(1)} = \xi e^{3t}$

$y^{(2)} = \xi t e^{3t} + \eta e^{3t}$

$\longleftarrow e^{3t}(\underbrace{\xi t + \eta}_{\text{constant ال}})$

---

so the solutions for the homo

$e^{2t}\xi^{(1)}$ $e^{4t}\xi^{(2)}$

ما في ولا أي منهم هو

constant e^{2t} & constant e^{4t}

$y' = \begin{bmatrix}5 & 3\\ -1 & 1\end{bmatrix} y + e^{-t}\begin{bmatrix}-2\\ 1\end{bmatrix} + t\begin{bmatrix}0\\ -5\end{bmatrix} + \begin{bmatrix}1\\ 7\end{bmatrix}$

$y^{(p)} = ae^{-t} + bt + c$

بتاخذ كل واحد لحالو.

ولا واحد فيهم نفس ال homo إذا صح ال

suitable form اللي كتبتها

$y^{(p)} = ae^{-t} + bt + c$

as total

as vectors

$y^{(p)} = \begin{bmatrix}a_1\\ a_2\end{bmatrix} e^{-t} + \begin{bmatrix}b_1\\ b_2\end{bmatrix} t + \begin{bmatrix}c_1\\ c_2\end{bmatrix}$

لو كان عندي وحدة من ال λ تساوي $\lambda = -1$

so the solution of it will be

$e^{-t}\xi^{(1)}$

بصير في 2 بتشبه بعض وهو أحد

حلول ال homo فبدي أربط ال (suitable form)

$y^{(p)} = ae^{-t} + bt + c$

بدي أعدل هاي ولصير

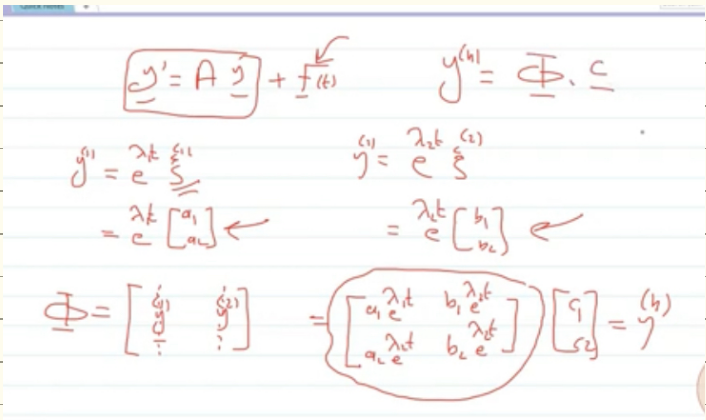
$(at+d)e^{-t}$

Variation of parameters

① بنحل ال (homo) بنطلعها $y^{(h)}$

② بنتعلم كيف نحسب ال $y^{(P)}$

in homo we find $y^{(1)}, y^{(2)}$
↳ solution 1, solution 2



voice 3

$$\Phi(t) = \begin{bmatrix} y_1 & y_2 \end{bmatrix}$$

$$y^{(h)} = \Phi . \underset{\text{vector}}{C}$$

same

$$c_1 y^{(1)} + c_2 y^{(2)}$$

هسا بدنا نحل ال non homo يلي مسمينها
وهو (*)

$$y' = Ay + F(x) \text{ ------ } (*)$$

$$y^{(p)} = \Phi(t) \int \Phi^{-1}(t) . F(t)\, dt$$

Ex: Solve the system :-

$$y' = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} y + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$$

finding $y^{(1)}, y^{(2)}$

$$\begin{bmatrix} -\lambda & 2 \\ -1 & 3-\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-\lambda(3-\lambda) - -2 = 0$$
$$-3\lambda + \lambda^2 + 2 = 0$$
$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 2)(\lambda - 1) = 0$$
$$\lambda = 2, \quad \lambda = 1$$

$$\lambda = 2 \Rightarrow \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2y_1 + 2y_2 = 0$$
$$2y_2 = 2y_1$$
$$y_2 = y_1$$
$$y_1 = 1, \quad y_2 = 1$$

$$y^{(1)} = e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = 1 \Rightarrow \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-y_1 + 2y_2 = 0$$
$$2y_2 = y_1$$
$$y_2 = \frac{y_1}{2}$$
$$y_1 = 2$$
$$y_2 = 1$$

$$y^{(2)} = e^t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$y^{(1)} = e^t\begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad y^{(2)} = e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$y^{(h)} = c_1 e^t\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

or

$$y^{(h)} = \begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$y^{(P)} = \Phi t \int \Phi^{-1}(t) . F(t)\, dt$$

$$\Phi(t) = \begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{bmatrix}$$

$$\Phi^{-1}(t) = \frac{1}{e^{3t}}\begin{bmatrix} e^{2t} & -e^{2t} \\ -e^t & 2e^t \end{bmatrix}$$

بنوزعها

$$\frac{1}{|\Phi|}$$

$$e^{-3t}\begin{bmatrix} e^{2t} & -e^{2t} \\ -e^t & 2e^t \end{bmatrix}$$

$$\Phi^{-1}(t) = \begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{bmatrix}$$

$\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$ لا بدنا نغيرها لل non homo

بندخل الاس e^t

$$\begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix}$$

$$\Phi^{-1} \cdot f = \begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{bmatrix} \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix}$$

مصفوفة X عمود

$$\begin{bmatrix} 1+1 \\ -e^{-t} - 2e^{-t} \end{bmatrix} = \begin{bmatrix} 2 \\ -3e^{-t} \end{bmatrix}$$

now $$\int \Phi^{-1} f = \int \begin{bmatrix} 2 \\ -3e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} 2t \\ 3e^{-t} \end{bmatrix}$$

$$\Phi * \int \Phi^{-1} f = \begin{bmatrix} 2e^{t} & e^{2t} \\ e^{t} & e^{2t} \end{bmatrix} \begin{bmatrix} 2t \\ 3e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} 4te^{t} + 3e^{t} \\ 2te^{t} + 3e^{t} \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 2 \end{bmatrix} te^{t} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} e^{t}$$

$$y^{(p)} = \left(\begin{bmatrix} 4 \\ 2 \end{bmatrix} te^{t} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} e^{t} \right.$$

or

$$e^{t} \left(\begin{bmatrix} 4 \\ 2 \end{bmatrix} t + \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right)$$

$$y = y^{(h)} + y^{(p)}$$

$$c_1 e^{t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{t} \left(\begin{bmatrix} 4 \\ 2 \end{bmatrix} t + \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right)$$

Chapter 5

power series Method

how to find my solution in form of series solution

$$y = \sum_{n=0}^{n} a_n (x - x_0)^n$$

\# what we mean a function be analytic

we said that function is analytic at x_0 if I can wright this function as taylor series

$$= f(x_0) + f'(x_0)(x - x_0) + \underline{f''(x)}\,(x - x_0)^2 + \dots$$

Ex:-

① all the polynomials are analytic every where (عند أي نقطة)

② $e^x, \sin x, \cos x$ are analytic every where

③ Rational function are analytic except عند أصفار المقام

Ordinary point : x_0 is called ordinary point of

$$A(x)\,y'' + B(x)\,y' + C(x)\,y = 0$$

if $\frac{B(x)}{A(x)}, \frac{C(x)}{A(x)}$ are analytic at x_0.

& $A(x), B(x), C(x) \to$ analytic

if $A(x) = 0 \to$ singular points وبسمي هاي النقاط ل بستثنيها

Ex: find the ordinary points for:-

① $y'' + xy' + (x^2+2)y = 0$

the ordinary points, are all $x \in \mathbb{R}$

② $\underline{(x-1)y'' + xy' + \frac{1}{x}y = 0}$

$x - 1$ يعني مش شرط أقسم عليه

$$y'' + \frac{x}{x-1}y' + \frac{1}{x(x-1)}y = 0$$

singular points: $x = 0$ & $x = 1$

ordinary points are: $\mathbb{R} - \{0, 1\}$

singular points $\to$ أصفار المقام وأصفار الـ $A(x) = 0$

Theorem: let x_0 be an ordinary point of $A(x)y'' + B(x)y' + C(x)y = 6$

لما تكون النقطة ordinary point بنضمن وجود 2 independents solution على شكل series

Remark : ① $y = \sum_{n=0} a_n (x-x_0)^n \rightarrow$ هاد فعلياً الـ solution للـ second order ordinary D.E

a_n ← هدول الي بدور عليهم

لما نشتق مرتين ونعوضهن بالمعادلة

بس نشتق مرة بنزل درجة

$$y' = \sum_{n=1} n\, a_n (x-x_0)^{n-1}$$

$$y'' = \sum_{n=2} n(n-1)\, a_n (x-x_0)^{n-2}$$

② مرات بنجبر أحد الـ powers كيف أعمل بالطريقة الرمج.

$$\sum_{n=2} n(n-1)(x-x_0)^{n-2} \rightarrow$$ بستبدل كل n بـ $(n+2)$

$$\sum_{n+2=2} (n+2)(n+2-1)(x-x_0)^{n+2-2}$$

$$\sum_{n=0} (n+2)(n+1)(x-x_0)^n$$

بنحاول الـ n ما تبلش من رقم سالب

in particular

if

$\sum_{n=0} a_n x^n = 0$, then $a_n = 0$, for all values of n $\quad n \geq 0$

إذا هون $n=2$ → for all $n \geq 2$

Series solution about ordinary point

$$\sum_{n=0} a_n x^n = 0$$

so that mean that $a_n = 0$ for all values of $n \geq 0$

$$2a_2 + 6a_2 x + \sum_{n=2} \boxed{(n+1)(n+2)a_{n+2} - a_{n-2}}\, x^n = 0$$

لو كانت

$2a_1 + 6a_2 = 0$

لو هيك لا

لازم يكون كل الـ constants 0

$(n+1)(n+2)a_{n+2} - a_{n-2} = 0 \quad , n \geq 2$

$6a_3 = 0$

$2a_2 = 0$

* I we asked to solve about $x_0=1$

$$y=\sum_{n=0} a_n (x-1)^n \quad {}^{x_0=1}$$

$$y'=\sum_{n=1} n a_n (x-1)^{n-1}$$

* Example: Find a power series solution of:-

$$A(x)y''+x^2y=0 \quad \text{about } x_0=0$$

ثابت

$a(0)\neq 0$

$$y=\sum_{n=0} a_n x^n$$

$$y'=\sum_{n=1} n a_n x^{n-1}$$

$$y''=\sum_{n=2} n(n-1)a_n x^{n-2}$$

$$\sum_{n=2} n(n-1)a_n x^{n-2}+x^2\sum_{n=0} a_n x^n$$

بوحد ال power

$$\sum_{n=2} n(n-1)a_n x^{n-2}+\sum_{n=0} a_n x^{n+2-2}$$

بدنا نوحد صحول

$$\sum_{n=0}(n+2)(n-1+2)a_{(n+2)}x^n+\sum_{n=2} a_{(n-2)}x^n$$

$$\sum_{n=0}(n+2)(n+1)a_{(n+2)}x^n+\sum_{n=2} a_{(n-2)}x^n$$

بخلي الصغير يصير كبير

بنطلع الحد الأول والحد الثاني وبخليهم يبين آمن 2

$$2a_2+6a_3x+\sum_{n=2}\left[(n+2)(n+1)a_{n+1}+a_{(n-2)}\right]x^n=0$$

$$2a_2=0$$

$$a_2=0$$

$$6a_3x=0$$

$$a_3=0$$

$$(n+2)(n+1)a_{n+2}+a_{n-2}=0 \quad , n\geq 2$$

↳ Recurrance relation

$$n+2 = \frac{-a_{n-2}}{(n+2)(n+1)}, \quad n \geq 2$$

$\boxed{n=2}$ $a_4 = \frac{a_0}{(4)(3)} = -\frac{a_0}{12}$

$\boxed{n=3}$ $a_5 = \frac{-a_1}{(5)(4)} = \frac{-a_1}{20}$

$\boxed{n=4}$ $a_6 = \frac{-a_2}{(6)(5)} = 0$

$\boxed{n=5}$ $a_7 = \frac{-a_3}{(7)(6)} = 0$

but I already calculated a_4

$\boxed{n=6}$ $a_8 = \frac{-a_4}{(7)(8)(12)} = \frac{a_0}{672}$

مسبقا

$\boxed{n=7}$ $a_9 = \frac{-a_5}{(9)(8)(20)} = \frac{a_1}{1440}$

The solution of the D.E

$y(x) = \sum_{n=0} a_n x^n$

$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots$

$= a_0 + a_1 x - \frac{a_0}{12} x^4 - \frac{a_1}{20} x^5 \dots$

$$= \underbrace{a_0 \left[1 - \frac{x^4}{12}\right]}_{y_1} + \underbrace{a_1 \left[x - \frac{x^5}{20}\right]}_{y_2} \dots$$

* Find a power series solution of

$y'' - xy = 0$ about $x_0 = 2$

$$y = \sum_{n=0} a_n (x-2)^n$$

$$y' = \sum_{n=1} n a_n (x-2)^{n-1}$$

$$y'' = \sum_{n=2} n(n-1) a_n (x-2)^{n-2}$$

$$\sum_{n=2} n(n-1) a_n (x-2)^{n-2} - \underbrace{x}_{[(x-2)+2]} \sum_{n=0} a_n (x-2)^n$$

$$\sum_{n=2} n(n-1)a_n (x-2)^{n-2+2} - \sum_{n=0} a_n (x-2)^{n+1-1} - \sum_{n=0} 2a_n (x-2)^n = 0$$

$$\sum_{n=0} (n+2)(n+1)\, a_{n+2} (x-2)^n - \sum_{n=1} a_{n-1} (x-2)^n - \sum_{n=0} 2a_n (x-2)^n = 0$$

$$2a_2 + \sum_{n=1} (n+2)(n+1) a_{n+2} (x-2)^n - \sum_{n=1} a_{n-1} (x-2)^n - 2a_0 - \sum_{n=1} 2a_n (x-2)^n = 0$$

$$2a_2 - 2a_0 + \sum_{n=1} \left[(n+2)(n+1)\, a_{n+2} - a_{n-1} - 2a_n\right](x-2)^n = 0$$

$$2a_2 - 2a_0 = 0$$
$$2a_2 = 2a_0$$
$$a_2 = a_0$$

$$(n+2)(n+1)\, a_{n+2} - a_{n-1} - 2a_n = 0$$

$$\rightarrow \quad a_{n+2} = \frac{a_{n-1} + 2a_n}{(n+1)(n+2)}, \quad n \geq 1$$

لما نجيب a_0
بنعوض ال n بواحد

$$y = a_0 + a_1(x-2) + a_2(x-2)^2 + a_3(x-2)^2 \text{ ----}$$

$$= a_0 + a_1(x-2) + a_0(x-2)^2 + a_3(x-2)^3 \text{----}$$

$$\frac{a_0 + 2a_1}{(2)(3)} = \frac{a_0}{6} + \frac{a_1}{3}$$

بعوّضيه
ال a_0
من آخر شي
ويعوضوها
من هون

$\boxed{n=3}$ محسوبة فوق

$$a_2 = a_0$$

$$n=2 \quad a_4 = \frac{a_1 + 2a_2}{(3)(4)} = \frac{a_1}{12} + \frac{a_0}{6}$$

The solution is:-

$$y = \sum_{n=0} a_n (x-2)^n$$

$$= a_0 + a_1(x-2) + a_0(x-2)^2 + \left(\frac{a_0}{6} + \frac{a_1}{3}\right)(x-2)^3 + \text{---}$$

$$= a_0 \underbrace{\left(1 + \frac{(x-2)^3}{6} + (x-2)^2\right)}_{y_1} + a_1 \left[\underbrace{(x-2) + \frac{(x-2)^3}{3}}_{y_2}\right]$$

Regular singular points

$$A(x)y'' + B(x)y' + C(x)y = 0$$

$B(x)$, $C(x)$ } analytic

$A(x_0) = 0 \Rightarrow x_0$ is singular

$A(x) = 0$ only

regular — irregular

متى بتكون الـ singular point ⇐ regular و أحسن أسميها regular singular

x_0 ? singular? عنا 2 limits

إذا لقيت إنو هاي exist ⇐ بدي أسميها p_0

① $\lim\limits_{x \to x_0} \frac{B(x)}{A(x)}(x-x_0) < \infty \to$ exist

② $\lim\limits_{x \to x_0} \frac{C(x)}{A(x)}(x-x_0)^2 < \infty$

إذا لقيت إنو هاي exist بدي أسميها q_0

if both of them is exist so the point can be classified as regular singular point

↳ doesn't exist
إذا وحدة منهم مش موجودة بتتحول الى Irrigular

if x_0: regular singular
وبطلع initial equation

inetial equation : $r(r-1)+p_0 r+q_0=0$

* Example: Find all regular singular points

① $(x^2-4x+3)y''+4xy'+2xy=0$

x^2-4x+3

$(x-3)(x-1)=0$

$x=3$, $x=1$

↳ singular points

at $x_0=1$

$\lim\limits_{x\to 1} \frac{4x}{x^2-4x+3}(x-1)$

$\lim\limits_{x\to 1} \frac{4x(x-1)}{(x-3)(x-1)}$

$\lim\limits_{x\to 1} = \frac{4x}{x-3} = \frac{4}{-2} = -2 = p_0$ exist

$\lim\limits_{x\to 1} \frac{2x}{x^2-4x+3}(x-1)^2 = \frac{2x\cancel{(x-1)^2}}{\cancel{(x-1)}(x-3)} = \frac{2x(x-1)}{x-3} = \frac{2(0)}{-3} = 0$ exist

these two are exist so the point $x_0=1$ is regular singular

$x_0=3$

$\lim\limits_{x\to 3} \frac{4x}{x^2-4x+3}(x-3) = \frac{4x}{x-1} = \frac{12}{-2} = -6$ exist

$\lim\limits_{x\to 3} \frac{2x}{x^2-4x+3}(x-3)^2 = \frac{2x}{(x-3)(x-1)}(x-3)^2 = \frac{2x(x-3)}{(x-1)} = 0$ exist

$$\lim_{x\to 3} \frac{2x}{x^2-4x+3} (x-1)^2$$

$$\lim_{x\to 2} \frac{2x}{(x-1)(x-3)} (x-1)^2$$

$$\lim_{x\to 2} \frac{2x(x-1)}{x-3} = \lim_{x\to 2} \frac{4(1)}{-1} = -4$$ exist.

so $x_0 = 3$ is also regular singular point

the set of all regular singular points

is $\{1, 3\}$

Find the inetical equation for each regular singular points

for $x_0 = 1$

$$r(r-1) + p_0 r + q_0 = 0$$

$$r(r-1) - 2r = 0$$

$$r^2 - r - 2r = 0$$

$$r^2 - 3r = 0$$

* Q: find the roots of the inetial equation

$$r^2 - 3r = 0$$

$$r(r-3) = 0$$

$$r = 0, 3$$

Find the regular singular points

$$x(x+2)^2 y'' + (x+1) y' + 2xy = 0$$

$$x(x+2)^2 = 0$$

$$x = 0, -2$$

At $x_0 = 0$

$$\lim_{x\to 0} \frac{(x+1)}{\cancel{x}(x+2)^2} \cdot \cancel{x} = \frac{x+1}{(x+2)^2} = \underbrace{\frac{1}{4}}_{p_0}$$

$$\lim_{x\to 0} \frac{2x}{x(x+2)^2} x^2 = \frac{2x}{(x+2)^2} = \underbrace{0}_{q_0}$$

set of regular singular points $\{0\}$

So $x_0 = 0$ is regular singular point

إذا طلب مني ال inetial equation الها :-

$$r(r-1) + p_0 r + q_0 = 0$$

$$r(r-1) + \frac{1}{4} r = 0$$

$$r^2 - r + \frac{1}{4} r = 0 \Rightarrow r^2 - \frac{3}{4} r = 0$$

At $x_0 = -2$

$$\lim_{x\to -2} \frac{(x+1)}{x(x+2)^2} (x+2) = \frac{x+1}{x(x+2)} = \frac{-4}{0}$$ D.N.E

راح نوخذ أمثلة كل حل D.E أما ايجاد ال Series solution arount a point & this point will be regular singular

Ex: solve:

$A(x) = 2x^2 \quad A(0) = 0$ singular regular

$$2x^2 y'' + (x^2 - x)y' + y = 0 \quad \text{near } x_0 = 0$$

لشي ما اختبرنا هو regular مطلوب تختار انه

$\Rightarrow x_0$ is regular singular

في هون $+r$ زيادة مهما

هاد ال regular singular

Let $y = \sum_{n=0} a_n x^{n+r}$, $a \neq 0$ be a solution

$$y' = \sum_{n=0} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0} (n+r)(n+r-1) a_n x^{n+r-2}$$

بعوض بالمعادلة

$$2x^2 \sum_{n=0} (n+r)(n+r-1) a_n x^{n+r-2} + (x^2 - x) \sum_{n=0} (n+r) a_n x^{n+r-1} + \sum_{n=0} a_n x^{n+r} = 0$$

$$\sum_{n=0} 2(n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0} (n+r) a_n x^{n+r+1-1} - \sum_{n=0} (n+r) a_n x^{n+r} + \sum_{n=0} a_n x^{n+r} = 0$$

بدنا نوحد ال powers

$$\sum_{n=0} 2(n+r)(n+r-1) a_n x^{n+r} + \sum_{n=1} (n+r-1) a_{n-1} x^{n+r} - \sum_{n=0} (n+r) a_n x^{n+r} + \sum_{n=0} a_n x^{n+r} = 0$$

هسا بدنا نوحد ال n

$$2(r)(r-1) a_0 x^r + \sum_{n=1} 2(n+r)(n+r-1) a_n x^{n+r} + \sum_{n=1} (n+r-1) a_{n-1} x^{n+r} - r a_0 x^r - \sum_{n=1} (n+r) a_n x^{n+r}$$

$$2r(r-1) a_0 x^r - r a_0 x^r + a_0 x^r \qquad + a_0 x^r + \sum_{n=1} a_n x^{n+r} = 0$$

$$\sum_{n=1} \left(2(n+r)(n+r-1) a_n + ((n+r-1) a_{n-1}) - (n+r) \times a_n \right) + a_n \Big) x^{n+r}$$

$$a_0 x^r \left(2r(r-1) - r + 1 \right)$$

$$+ \sum_{n=1} \Big[\big(2(n+r)(n+r-1) + -(n+r) + 1 \big) a_n + (n+r-1) a_{n-1} \Big] x^{n+r} = 0$$

$$(2r(r-1)-r+1)a_0=0 \Rightarrow 2r^2-3r+1=0$$

$$\left[2(n+r)(n+r-1)-(n+r)+1\right]a_n+(n+r-1)a_{n-1}=0 \quad , n\geq 1$$

$$2(n+r)(n+r-1)\underbrace{-(n+r)+1}_{\substack{\text{هاي عبارة عن} \\ -(n+r-1)}}\,a_n=-(n+r-1)a_{n-1}$$

$$\left[2(n+r)(n+r-1)-(n+r-1)\right]a_n=-(n+r-1)a_{n-1}$$

$$(n+r-1)\,\underbrace{(2(n+r)-1)}_{2n+2r-1}\,a_n=-(n+r-1)\,a_{n-1}$$

$$a_n=\frac{-a_{n-1}}{2(n+r)-1} \quad , n\geq 1$$

$2r^2-3r+1=0$ & this is another method to find the inetial equation

$(2r-1)(r-1)=0$

$r=\frac{1}{2} \qquad r=1$

why do we calculate the roots for this? because we have to substitute it in a_n

for $\boxed{r=1}$ $\quad a_n=\dfrac{-a_{n-1}}{2(n+1)-1}=\dfrac{-a_{n-1}}{2n+2-1}$

$$a_n=\frac{-a_{n-1}}{2n+1} \quad , n\geq 1$$

$\boxed{n=1}$ $\quad a_1=\dfrac{-a_0}{3}$

$\boxed{n=2}$ $\quad a_2=\dfrac{-a_1}{5}=\dfrac{-1}{5}\times\dfrac{-a_0}{3}=\dfrac{1}{15}a_0$

$\boxed{n_3=3}$ $\quad a_3=\dfrac{-a_2}{7}=\dfrac{-1}{7}\times\dfrac{1}{15}a_0=\dfrac{-1}{105}a_0$

$$y=\sum_{n=0} a_n x^{n+1}$$

$$=a_0x+a_1x^2+a_2x^3$$

$$=a_0x+\frac{1}{15}a_0x^2-\frac{1}{105}a_0x^3$$

$$y_1=a_0\underbrace{\left(x+\frac{x^2}{15}-\frac{x^3}{105}\right)}_{y_1}$$

طلعلي series solution واحد
وهادا هو ال y_1

to find the second solution substitute $r=\frac{1}{2}$

$$a_n = \frac{-a_{n-1}}{2(n+\frac{1}{2})-1}$$

$$a_n = \frac{-a_{n-1}}{2n+1-1}$$

$$a_n = \frac{-a_{n-1}}{2n} \quad , n \geq 1$$

$\boxed{n=1}$ $a_1 = \frac{-a_0}{2}$

$\boxed{n=2}$ $a_2 = \frac{-a_1}{4} = \frac{-1}{4} \times \frac{-a_0}{2} = \frac{a_0}{8}$

$$y_2 = \sum_{n=0} a_n x^{n+\frac{1}{2}}$$

$$= a_0 x^{\frac{1}{2}} + a_1 x^{3/2} + a_2 x^{5/2}$$

$$= a_0 x^{\frac{1}{2}} - \frac{a_0}{2} x^{3/2} + \frac{a_0}{8} x^{5/2} \Rightarrow$$

لهسا ال a_0 الي فوقه
غير هاي هالـ r
لما $r=\frac{1}{2}$

$$a_0 \Big[\underbrace{x^{\frac{1}{2}} - \frac{1}{2} x^{3/2} + \frac{1}{8} x^{5/2} \cdots}_{y_2} \Big]$$

Remark:- $a_1 = \frac{-a_0}{2.1}$

$$a_2 = \frac{(-a_1)}{2.2} = \frac{a_0}{2^2 (2)(1)}$$

$$a_3 = \frac{(-a_2)}{2.3} = \frac{-a_0}{2^3 2 \times (3)(1)}$$

so to find

$$a_n = \frac{(-1)^n a_0}{2^n n!}$$

هاي

لعطيني ال
an بالصيغة
العادية فبحكيلي
the general form
for an is?

when calculate the roots of inetial equation we have three cases :-

1 ⇒ $r_1 > r_2$ and $r_1 - r_2 \notin \mathbb{Z}$ الأعداد الصحيحة

there exist two linearly independent solutions :-

$$y_1(x) = \sum_{n=0} a_x x^{n+r_1} , a_0 \neq 0$$

$$y_2(x) = \sum_{n=0} a_x x^{n+r_2} , a_0 \neq 0$$

2 ⇒ $r_1 > r_2$ and $r_1 - r_2 \in \mathbb{Z}$

there exist two linearly independent solutions :-

الأكبر م

we will have ↳ $y_1(x) = \sum_{n=0} a_n x^{n+r_1} , a_0 \neq 0$

الأول ضامنين وجوده بس الثاني لا

يعتمد على أي إذا في ولا لا

$$y_2(x) = \underset{I c=0}{C}\, y_1(x) \ln x + \sum_{n=0} a_n x^{n+r_2} , a_0 \neq 0$$

C: a constant which may be zero

so the second solution may contain a logarithm ($\ln x$)

3 ⇒ $r_1 = r_2 = r$

$$y_1(x) = \sum_{n=0} a_n x^{n+r} , a_0 \neq 0$$

$$y_2(x) = y_1(x) \ln x + \sum_{n=1} a_n x^{n+r}$$

→ So the second solution will always contain a logarithm

Ex: consider $xy'' - xy' + y = 0$

1. Show that $x_0 = 0$ is regular singular
2. Determine the roots of the indicial equation at $x_0 = 0$
3. Find the Frobenius series solution corresponding to the larger root
4. what would be the form of the second linearly independent solution.

Sol. 1

$$\lim_{x \to 0} \frac{-x}{x} \cdot (x) = -x = 0 \quad P_0$$

$$\lim_{x \to 0} \frac{1}{x} \cdot (x)^2 = x = 0 \quad q_0$$

both are exist so X_0 is a regular singular point sol.2

$$r(r-1)+p_0 r+q_0=0$$

$$r(r-1)+0+0=0$$

$$r(r-1)=0$$

$$r_1=1 \quad , r_2=0$$

sol.3 $r_1>r_2 \quad r_1-r_2 \in \mathbb{Z}$

$$y_1(x)=\sum_{n=0} a_n x^{n+1}$$

$$y_1'(x)=\sum_{n=0}(n+1)a_n x^n$$

$\leftarrow$ ليش ما عملنا لـ
لأنو الحد الأول
مو صفر إلو قيمة

$$y''(x)=\sum_{n=1}(n+1)n a_n x^{n-1}$$

$n=1 \leftarrow$ لأنو أول حد صفر

$$x y''-xy'+y=0$$

$$x\sum_{n=1}(n+1)n a_n x^{n-1} - x\sum_{n=0}(n+1)a_n x^n + \sum_{n=0} a_n x^{n+1}$$

$$\sum_{n=1}(n+1)n a_n x^{n} - \sum_{n=0}(n+1)a_n x^{n+1} + \sum_{n=0} a_n x^{n+1}$$

$$\sum_{n=0}(n+2)(n+1)a_{n+1} x^{n+1} - \sum_{n=0}(n+1)a_n x^{n+1} + \sum_{n=0} a_n x^{n+1}$$

$$\sum_{n=0}\left((n+2)(n+1)a_{n+1} - (n+1)a_n + a_n\right)x^{n+1}$$

$$(n+2)(n+1)a_{n+1}-(n+1)a_n+a_n=0$$

$$a_{n+1}=\frac{(n+1)a_n-a_n}{(n+1)(n+2)} \quad , n\geq 0$$

$$a_{n+1}=\frac{na_n+a_n-a_n}{(n+1)(n+2)} \quad , n\geq 0$$

$$a_{n+1}=\frac{na_n}{(n+1)(n+2)}$$

$\boxed{n=0}$ $\quad a_1=0$

$\boxed{n=1}$ $\quad a_2=\frac{a_1}{(2)(3)}=\frac{a_1}{6}=\frac{1}{6}\times 0=0$

$\boxed{n=2}$ $a_3 = \dfrac{2a_2}{(3)(4)} = \dfrac{2}{12} a_2 = \dfrac{1}{6} \times 0 = 0$

الي هم هدول

$a_n = 0, \ n \geq 1$

$$y_1 = \sum_{n=0} a_n x^{n+1}$$

$$y_1 = a_0 x + a_2 x^2 + a_3 x^3 + \dots$$

$$y_1 = a_0 x + 0 + 0 + \dots$$

$$y_1 = a_0 x$$

Sol.4

$$y_2(x) = c\, y_1 \ln x + \sum_{n=0} a_n x^n$$

Chapter 6

Lablace transform 1

note $e^{-\infty} = 0$

$e^{\infty} = \infty$

$\mathcal{L} \rightarrow f(t) \rightarrow F(s)$

بحول هاد لهاد

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)\, e^{-st}\, dt = F(s)$$

Example: Find $\mathcal{L}\{1\}$:- → هاد تكامل للـ∞

بنوخذ بدل الـ∞ رمز معين

$$\mathcal{L}\{1\} = \int_0^{\infty} f(t) . e^{-st} = \int_0^{\infty} e^{-st} = \lim_{p\to\infty} \int_0^{p} e^{-st}\, dt = \frac{e^{-st}}{-s}\Big|_0^{p} = \frac{-1}{s}\left[e^{-pt} - e^{0}\right] = \frac{-1}{s}\left[e^{-pt} - 1\right]$$

وبنوخذ lim لهاد الرمز

هسا بوخذ الـ lim لهاد المقدار

$$\lim_{p\to\infty} \frac{-1}{s}\left[e^{-pt} - 1\right] = \frac{-1}{s}\left[0 - 1\right] = \frac{1}{s} \ , \ s > 0$$

هسا ممكن تكون الـ $s<0$ لا

بس وقتها راح تصير e^{∞} و بتطلع D.N.E

بس أنا بدي Laplace exist

حفظ

$\Rightarrow \mathcal{L}\{a\} = \dfrac{a}{s} \ , \ s > 0$

$\Rightarrow \mathcal{L}\{e^{at}\} = \dfrac{1}{s-a} \ , \ s > a$

$\Rightarrow \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s>0$

$\Rightarrow \mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}, \quad s>0$

$\Rightarrow \mathcal{L}\{\cosh(at)\} = \frac{s}{s^2+a^2}, \quad s>0$

$\Rightarrow \mathcal{L}\{\sinh(at)\} = \frac{a}{s^2-a^2}$

$\Rightarrow \mathcal{L}\{\cosh(at)\} = \frac{s}{s^2-a^2}$

\# Find :-

① $\mathcal{L}\{5\} = \frac{5}{s}$

② $\mathcal{L}\{t^2\} = \frac{2!}{s^3} = \frac{2}{s^3}$

③ $\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$

Remark : ال Laplace بتوزع كال الجمع وهاي الخاصية اسما linearity

So $\mathcal{L}$ is linear   ولكنها ما بتتوزع عالضرب و القسمة

$\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$

Ex: Find

① $\mathcal{L}\{2t+e^{3t}+\sin t\}$

طلعت برا

$2\mathcal{L}\{t\} + \mathcal{L}\{e^{3t}\} + \mathcal{L}\{\sin t\}$

$2\,\frac{1}{s^2} + \frac{1}{s-3} + \frac{1}{s^2+1}$

$= \frac{2}{s^2} + \frac{1}{s-3} + \frac{1}{s^2+1}$

② $\mathcal{L}\{\cos^2 t\}$ بيستعمل المتطابقة

$\mathcal{L}\{\frac{1}{2}(1-\cos 2t)\}$

$\frac{1}{2}\mathcal{L}\{(1-\cos 2t)$

$\frac{1}{2}\left\{\frac{1}{s} - \frac{s}{s^2+4}\right\}$

note

$\mathcal{L}\{f(t)\} = F(S)$

$\mathcal{L}\{g(t)\} = G(S)$

$\mathcal{L}\{y(t)\} = Y(S)$

Capital بيصيرو

$$\mathcal{L}\{f(t)e^{at}\} = F(s)\Big|_{s-a} = F(s-a)$$

بـ مقدار ال shift
الي نعملو

Ex:

① $\mathcal{L}\{\sin 3t\} = \dfrac{3}{s^2+9}$

بحسب $\mathcal{L}\{\sin 3t\}$ وبعمل shift بمقدار ال e^{at}
وببدل محل كل s الطقم كامل

② $\mathcal{L}\{e^{2t}\sin 3t\} = \dfrac{3}{s^2+9}\Big|_{s-2} = \dfrac{3}{(s-2)^2+9}$

③ $\mathcal{L}\{t^3e^{2t}\} = \dfrac{3!}{s^4}\Big|_{s-2} = \dfrac{6}{s^4}\Big|_{s-2} = \dfrac{6}{(s-2)^4}$

④ $\mathcal{L}\{e^{-3t}\cos 2t\}$

$$= \frac{s}{s^2+4}\Big|_{s+3} = \frac{s}{(s+3)^2+4}$$

طيب هسا بدو يعطيني

$F(s) \longrightarrow f(t)$

$\mathcal{L}^{-1}$ تبعتها

① $\frac{1}{2}\mathcal{L}^{-1}\left\{2\cdot\frac{1}{s^3}\right\} = \frac{1}{2}t^2$

② $\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} = \sin(3t)$

③ $\frac{2}{3}\mathcal{L}^{-1}\left\{\frac{2\cdot\frac{3}{3}}{s^2+9}\right\} = \frac{2}{3}\sin(3t)$

④ $\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t$

④ $\mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\} = \cos\sqrt{2}t$

(5) $\frac{1}{2!}\mathcal{L}^{-1}\left\{\frac{1.2!}{s^3}\right\} = \frac{1}{2!}t^2 = \frac{1}{2}t^2$

(6) $\mathcal{L}^{-1}\left\{\frac{5}{s^4}\right\} = 5\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = \frac{5}{3!}\mathcal{L}^{-1}\left\{\frac{1.3!}{s^4}\right\} = \frac{5}{6}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} = \frac{5}{6}t^3$

(7) $\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \sin 2t$

(8) $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$

(9) $\frac{1}{2!}\mathcal{L}^{-1}\left\{\frac{1.2!}{s^3}\right\} = \frac{1}{2}t^2$

(8) $\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \sin 2t$

(4) $\mathcal{L}^{-1}\left\{\frac{3}{s^3} + \frac{s}{s^2+9}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{3}{s^3}\right\} + \mathcal{L}\left\{\frac{s}{s^2+9}\right\}$

$= 3\mathcal{L}^{-1}\left\{\frac{1.2!}{s^3}\right\} + \cos 3t$

$= \frac{3}{2}t^2 + \cos 3t$

(10) $\mathcal{L}^{-1}\left\{\frac{2}{(s-4)^3}\right\} = t^2 e^{4t}$

(11) $\mathcal{L}^{-1}\left\{\frac{1.2}{(s-1)^2+4}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2+4}\right\} = \frac{1}{2}\sin 2t\, e^{t}$

(12) $\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{(s-2).3}{(s-2)^2+9}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{(s-2)}{(s-2)^2+9}\right\} = \frac{1}{3}\cos 3t\, e^{2t}$

(13) $\frac{1}{3!}\mathcal{L}^{-1}\left\{\frac{1.3!}{(s-2)^4}\right\} = \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{3!}{(s-2)^4}\right\} = \frac{1}{6}t^3 e^{2t}$

(14) $\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1.2}{(s-1)^2+4}\right\} = \frac{1}{2}\sin(2t)e^{t}$

(15) $\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+1}\right\} = \sin t\, e^{2t}$

(16) $\mathcal{L}^{-1}\left\{\frac{(s-2)+2}{(s-2)^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{(s-2)^2+9}\right\}$

$\cos 3t\, e^{2t} + \frac{2}{3} \mathcal{L}^{-1}\left\{\frac{1.3}{(s-2)^2+9}\right\}$

$\cos 3t\, e^{2t} + \frac{2}{3} \sin 3t\, e^{2t}$

Ex: $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+5}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1.3}{(s+2)^2+1}\right\} = \frac{1}{3} \sin(3t)\, e^{-2t}$

بدنا نعمل اكمال مربع

s^2+4s+5

$\left(\frac{4}{2}\right)^2 = 2^2 = 4$

$s^2+4s+4-4+5$

$(s+2)(s+2)-4$

② $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{(s+1+1)-1}{(s+2)+1}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)+1}\right\} + \mathcal{L}^{-1}\left\{\frac{-1}{(s+2)+1}\right\}$

$= e^{-2t}\cos(t) + -1 \sin(t)\, e^{-2t}$

$= e^{-2t}\left[\cos(t) - \sin(t)\right]$

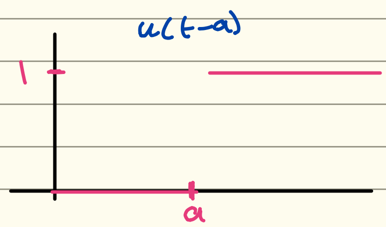
③ $\mathcal{L}^{-1}\left\{\frac{2s+1}{(s-1)^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{2(s-1)+3}{(s-1)^2+4}\right\}$

$= 2\mathcal{L}^{-1}\left\{\frac{(s-1)}{(s-1)^2+4}\right\} + \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{3.2}{(s-1)^2+4}\right\}$

$2\cos 2t\, e^{t} + \frac{3}{2} \sin(2t)\, e^{t}$

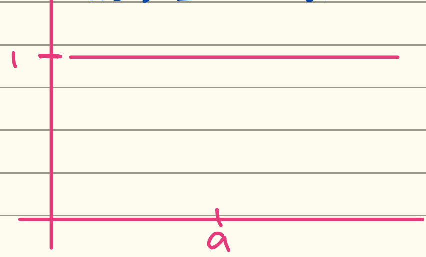
unit step function "Heavi side function"

$$u(t-a) = \begin{cases} 0 & 0 \le t < a \\ 1 & t \ge a \end{cases} \quad , a \ge 0$$



if $a = 0$ $u(t) \rightarrow$ where $u(t)$ is

$u(t) = 1$ constant function



Ex: If $f(t) = t^2 + u(t-1)$, find $f(2)$

$$f(2) = 2^2 + u(2-1) = 4 + 1 = 5$$

$2 > 1$, $t \downarrow$, $a \downarrow$, $= u(t) = 1$

$t < a$, $\frac{1}{2} < 1$

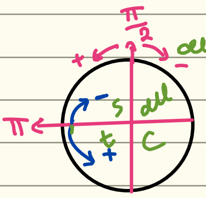
$$f\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{2}\right)^2 + 0 = \frac{1}{4}$$

* $\mathcal{L}\{u(t-a)\} = \dfrac{e^{-as}}{s}$

* $\mathcal{L}\{u(t-a)f(t)\} = e^{-as}\,\mathcal{L}\{f(t+a)\}$

\# Remark (حفظ)

this is π



all: sin, cos, tan

S: students: sin

لتسهيل الحفظ

t: take: tan

C: candies: cosx

$\sin(\pi + t) = -\sin t$

$\sin(\pi - t) = \sin t$

$\cos(\pi + t) = -\cos t$

$\cos(\pi - t) = -\cos t$

$\Rightarrow \pi$ تغير الاشارة فقط

(حسب الربع مش شرط دائمًا)

مش دائمًا حسب الربع

$\sin\left(\frac{\pi}{2} + t\right) = \cos t$

إشارة ال sin بالربع ↳ costx

$\sin\left(\frac{\pi}{2} - t\right) = \cos t$

$\Rightarrow \frac{\pi}{2}$ تغير الاشارة

والمقدار

$\cos t \rightarrow \sin t$

$\sin t \rightarrow \cos t$

$\cos\left(\frac{\pi}{2} + t\right) = -\sin t$

$\cos\left(\frac{\pi}{2} - t\right) = \sin t$

إشارة هاي بعاد الربع $x \nearrow$ $\sin t$

Ex: Find:-

① $\mathcal{L}\{u(t+2)\} = \dfrac{e^{-2s}}{s}$

② $\mathcal{L}\{t\,u(t-2)\} = e^{-2s}\,\mathcal{L}\{t+2\} = e^{-2s}\left[\frac{1}{s^2} + \frac{2}{s}\right]$

بعدين يستبدل

كل t بـ $t+2$

وبحسابها

③ $\mathcal{L}\{t^2\,u(t-1)\} = e^{-s}\,\mathcal{L}\{(t+1)^2\}$

$$= e^{-s}\,\mathcal{L}\{(t^2+2t+1)\}$$

$$e^{-s}\left[\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right]$$

④ $\mathcal{L}\{\sin t\, u(t-\frac{\pi}{2})\} = e^{-\frac{\pi}{2}s}\mathcal{L}\{\sin(t+\frac{\pi}{2})\}$

$= e^{-\frac{\pi}{2}s}\mathcal{L}\{\cos t\}$

$= e^{-\frac{\pi}{2}s} \cdot \frac{s}{s^2+1}$

⑤ $\mathcal{L}\{e^{2t}u(t-3)\} = e^{-3s}\mathcal{L}\{e^{2(t+3)}\}$

$= e^{-3s}\mathcal{L}\{e^{2t+6}\}$

$= e^{-3s}\mathcal{L}\{\underbrace{e^{6}}_{\text{constant}}.e^{2t}\}$ بنطلعوبرا

$= e^{-3s}.e^{6}\mathcal{L}\{e^{2t}\}$

$= e^{-3s}.e^{6} \cdot \frac{1}{s-2}$

$= e^{6-3s} \cdot \frac{1}{s-2}$

$= \frac{e^{3(2-s)}}{(s-2)}$

or $\mathcal{L}\{e^{2t}u(t-3)\} = \left.\frac{e^{-3s}}{s}\right|_{s-2} = \frac{e^{-3(s-2)}}{s}$

$= \frac{e^{-3s+6}}{s}$

$= \frac{e^{6-3s}}{s}$ نفس الجواب

⑥ $\mathcal{L}\{te^{5t}u(t-2)\} = \mathcal{L}\{t\,u(t-2)\}$

$= e^{-2s}\mathcal{L}\{t+2\}$

$= \left. e^{-2s}\left[\frac{1}{s^2} + \frac{2}{s}\right]\right|_{s-5}$

$= e^{-2(s-5)}\left[\frac{1}{(s-5)^2} + \frac{2}{(s-5)}\right]$

$= e^{-2s+10}\left[\frac{1}{(s-5)^2} + \frac{2}{(s-5)}\right]$

Remark: Let $f(t) = \begin{cases} f_1(t), & 0 \le t < a \\ f_2(t), & a \le t < b \\ f_3(t), & t \ge b \end{cases}$

كيفه بنقدر نحسب ال laplace ل لاقتران متشعب.

Then we can rewrite $f(t)$ as follows:-

$$f(t) = f_1(t)\,[u(t) - u(t-a)] + f_2(t)\,[u(t-a) - u(t-b)] + f_3(t)\,u(t-b)$$

Example: Let $f(t) = \begin{cases} t, & 0 \le t < 2 \\ 0, & t \ge 2 \end{cases}$

Capital لانه laplace يعني

Find $F(S)$

حكينا بأول المحاضرة انه $u(t)=1$

$$f(t) = t\,.\,[u(t) - u(t-2)] + 0\,u(t-2)$$

$$f(t) = t\,[1 - u(t-2)]$$

$$f(t) = t - t\,u(t-2)$$

$$\mathcal{L}\{t\} - \mathcal{L}\{t\,u(t-2)\}$$

$$\frac{1}{s^2} - e^{-2s}\,.\,\mathcal{L}\{t+2\}$$

$$\frac{1}{s^2} - e^{-2s}\,.\left[\frac{1}{s^2} + \frac{2}{s}\right]$$

هسا بدنا نحسب $\mathcal{L}^{-1}$ ونرجعو الى unit step function

كيفه؟

Step 1: $\mathcal{L}\{f(t)\,u(t-a)\} = e^{-as}\,\mathcal{L}\{f(t+a)\}$

بنرجع هاي أول إشي

بنطرح ال a هنا وبنرجعها

$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\} = u(t-2)\,\mathcal{L}^{-1}\left[\frac{1}{s^2}\right]$

$$= u(t-2)\; t'\Big|_{t-2}$$

$$= u(t-2)\,(t-2)$$

$\mathcal{L}^{-1}\left\{\frac{s\,e^{-s}}{s^2-1}\right\} = u(t-1)\,\mathcal{L}^{-1}\left\{\frac{s}{s^2-1}\right\}$

$$= u(t-1)\,(\cosh(t))\Big|_{t+1}$$

$$= u(t-1)\,\cosh(t+1)$$

② $\mathcal{L}^{-1}\left\{\frac{4e^{-2s}}{s-7}\right\} = u(t-2)\, 4\, \mathcal{L}^{-1}\left\{\frac{1}{s-7}\right\}$

$u(t-2) . 4 . e^{7t}\Big|_{t+2}$

$u(t-2) . 4 . e^{7(t+2)}$

$= u(t-2) . 4 . e^{7t+14}$

$= 4\, u(t-2)\, e^{7t+14}$

* $\mathcal{L}^{-1}\left\{\frac{e^{-2s}+s}{s^2}\right\} =$

$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2}\right\}$

$u(t-2) . \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$

$u(t-2)\ t\Big|_{t-2} + 1$

$= u(t-2)\,(t-2) + 1$

* Remark:-

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} = \frac{A(s+1)+Bs}{s(s+1)}$$

قاعدة :-
المقام = المقام
البسط = البسط
so that

$1 = A(s+1) + Bs$

بوخذ أصفار هدول
$s=0$
$s=-1$
وبعوضهم

$\boxed{s=0}$ $1 = A(s+1)$

$1 = A(1)$

$A = 1$

$\boxed{s=-1}$ $1 = Bs$

$\frac{1}{-1} = \frac{-B}{-1}$

$B = -1$

so that

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

طيب متى بستعملها؟ بس يكون عندي ضرب وبطلب أطلعهم الابتدائي
الطريقة وأكون بدي أطلع ال $\mathcal{L}^{-1}$

* Final :-

① $\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}$

$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} = \frac{1}{s} - \frac{1}{s+1}$

لما طلعناهم فوق

$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$

$= 1 - e^{-t}$ ← cosh, sinh, sin, cos إذا كان الجواب بدلالة ال
نعمل إكمال مربع
غير هيك بستعمل partial fraction ال
لهاي العلاقة

② $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s+1)}\right\}$

$= u(t-1) \cdot \boxed{\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}}_{t-1}$

$= u(t-1) \left(\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\}\right)$

ما بعرفه ليش طرح المفروض أجمع

$= u(t-1) \cdot (1 - e^{-t})\Big|_{t-1}$

$= u(t-1) \cdot (1 - e^{-(t-1)})$

* $\mathcal{L}\{y'(t)\} = s\,Y(s) - y(0)$

* $\mathcal{L}\{y''(t)\} = s^2\,Y(s) - s\,y(0) - y'(0)$

* Example: Solve: $y' + y = u(t-1)$, $y(0) = 0$

$\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{u(t-1)\}$

$s\,Y_s - y(0) + Y_s = \frac{e^{-s}}{s}$

$(s+1)\,Y_s = \frac{e^{-s}}{s}$

$\mathcal{L}^{-1}\, Y_s = \mathcal{L}^{-1}\, \frac{e^{-s}}{s(s+1)}$

solution ال ← هو $y(t)$

$y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s+1)s}\right\}$

$y(t) = u(t-1) \cdot \mathcal{L}^{-1}\left\{\frac{1}{(s+1)s}\right\}$

$y(t) = u(t-1) \cdot \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\}$

$y(t) = u(t-1) \cdot (1 - e^{t})$

Dirac Delta Function

↳ $\delta(t-a)$ بنرمزله ب is generalized function (distribution) which can be characterized by the following properties :-

① $\delta(t-a) = \begin{cases} \infty & , t=a \\ 0 & , t \neq a \end{cases}$

② $\int_{-\infty}^{\infty} \delta(t-a)\,dt = 1$; $a \geq 0$

لو وضعنا 0 نفس الحل

$\delta(t) = \begin{cases} \infty & , t=0 \\ 0 & , t \neq 0 \end{cases}$

$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

Theorem: if $g(t)$ is continuous function, then

$$\int_{-\infty}^{\infty} \delta(t-a)\, g(t)\,dt = g(a)$$

بدون وجود ال $g(t)$
هون بكون الجواب 1
مع وجود ال $g(t)$ بكون الجواب
هو صورة ال g عند ال a

* Example: Find:-

① $\int_{-\infty}^{\infty} \underbrace{\delta(t)}_{\delta(t-0)} \cos t\,dt = \cos 0 = 1$

② $\int_{0}^{\infty} (1+e^{-t})\,\delta(t-2)\,dt = 1+e^{-2}$

③ $\int_{0}^{\infty} \delta(t-\frac{\pi}{2}) \sin t\,dt = \sin\frac{\pi}{2} = 1$

Theorem: for $a \geq 0$ and $g(t)$ continuous :-

① $\mathcal{L}\{\delta(t-a)\} = e^{-as}$

② $\mathcal{L}\{g(t)\,\delta(t-a)\} = e^{-as}\, g(a)$

* Example:-

① $\mathcal{L}\{\delta(t-3)\} = e^{-3t}$

② $\mathcal{L}\{(2t+1)\,\delta(t-1)\} = e^{-s}\, g(1)$

$2\times1+1=3$

$= 3e^{-s}$

* Example: solve:

$$\mathcal{L}\quad y'' + \pi^2 y = \overset{\mathcal{L}}{\delta}(t-1)$$

$$y(0) = 1, \; y'(0) = 0$$

$$Y_s s^2 + s\,y(0) - y'(0) + \pi^2 Y_s = e^{-s}$$

$$s^2 Y_s + s + \pi^2 Y_s = e^{-s}$$

$$Y_s (s^2 + \pi^2) = e^{-s} - s$$

$$\mathcal{L}^{-1}\; X_s = \frac{\mathcal{L}^{-1} e^{-s} - s}{s^2 + \pi^2}$$

$$y_s = \mathcal{L}^{-1}\left\{\frac{e^{-s} - s}{s^2 + \pi^2}\right\}$$

$$y_s = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2 + \pi^2}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2 + \pi^2}\right\}$$

$$y_s = u(t-1) \cdot \mathcal{L}^{-1}\left\{\frac{1 \cdot \pi}{s^2 + \pi^2}\right\}\Bigg|_{t-1} + \cos(\pi t)$$

$$y(s) = u_{(t-1)} \cdot \frac{1}{\pi} \sin(\pi t)\Big|_{t-1} + \cos \pi t$$

$$y(s) = \frac{u_{(t-1)}}{\pi} \sin(\pi(t-1)) + \cos \pi t$$

* $\mathcal{L}\{f(t)\} = F(s)$
* $\mathcal{L}[\underline{t}\; \underline{f(t)}] = -F'(s)$
* $\mathcal{L}[t^2 f(t)] = +F''_s$
* $\mathcal{L}\{t^3 f(t)\} = -F'''_s$

* Find :-

① $\mathcal{L}\{t \sin 2t\}$

بنحل هاي لحالها

$$\frac{-d}{ds} \cdot \left[\frac{2}{s^2+4}\right] = \frac{-(-2)\,2s}{(s^2+4)^2} = \frac{4s}{(s^2+4)^2}$$

② $\mathcal{L}[6 e^{2t} \cos 2t]$

$$= \frac{s}{s^2+4}\Bigg|_{s-2} = \frac{s-2}{(s-2)^2+4} \; \frac{-d}{ds} = -\frac{((s-2)^2+4) - (s-2).2(s-2)}{[(s-2)^2+4]^2} = \frac{-(s-2)^2 + 4 - 2(s-2)^2}{((s-2)^2+4)^2}$$

$$= \frac{4 - 3(s-2)^2}{((s-2)^2+4)^2}$$

③ $\mathcal{L}\{t\sin^2 t\}$

$$= \mathcal{L}\left\{\frac{1}{2}(1-\cos 2t)\right\}$$

$$= \frac{1}{2}\mathcal{L}\{(1-\cos 2t)\}$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$= \frac{-d}{ds}\left[\frac{1}{2s} - \frac{s}{2s^2+8}\right]$$

$$= -\left[\frac{(-1)(2)}{4s^2} - \frac{(2s^2+8).(1) - s.(4s)}{(2s^2+8)^2}\right]$$

$$-\left[\frac{-1}{2s^2} - \frac{2s^2+8-4s^2}{(2s^2+8)^2}\right]$$

$$= \frac{1}{2s^2} + \frac{8-2s^2}{(2s^2+8)^2}$$

$$= \frac{1}{2}\left[\frac{1}{s^2} + \frac{4-s^2}{(s^2+4)^2}\right]$$

* $\mathcal{L}\{t\, f(t)\} = -F_s'$

لو ضربنا الطرفين بـ -1

$\mathcal{L}\{-t\, f(t)\} = F_s'$

$\mathcal{L}^{-1}\{F_s'\} = -t . f(t)$

\# Example:- Find $\mathcal{L}^{-1}(F(s)) = f(t)$

* إذا إجاني سؤال بالـ Laplace فيه $\ln, \cot^{-1}, \tan^{-1}$ بنطبق عليه زي هاد السؤال

① $\mathcal{L}^{-1}\left\{\ln\left(\frac{s^2+4}{s^2}\right)\right\}$

افرضناه

Let $F(s) = \ln\left(\frac{s^2+4}{s^2}\right)$

$F(s) = \ln(s^2+4) - \ln(s^2)$

$F(s) = \ln(s^2+4) - 2\ln s$

حسابها ناخذ الـ $\mathcal{L}^{-1}$

$\mathcal{L}^{-1}$ $F'(s) = \mathcal{L}^{-1}\frac{2s}{s^2+4} - 2\frac{1}{s}$

$$\mathcal{L}^{-1}\{F'(s)\} = \mathcal{L}^{-1}\left[\frac{2s}{s^2+4} - \frac{2}{s}\right]$$

$$\frac{-t\, f(t)}{-t} = \frac{2\cos(2t) - 2}{-t}$$

$$f(t) = \frac{2\cos(2t) - 2}{-t}$$

لـ وهو المطلوب

* $\mathcal{L}^{-1}\left\{\ln\left(\frac{s}{s-1}\right)\right\}$

$$F(s) = \ln(s) - \ln(s-1)$$

$$\mathcal{L}^{-1} F'(s) = \mathcal{L}^{-1}\frac{1}{s} - \frac{1}{s-1}$$

$$-t \cdot f(t) = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s-1}\right]$$

$$\frac{-t \cdot f(t)}{-t} = \frac{1 - e^{t}}{-t}$$

$$f(t) = \frac{1 - e^{t}}{-t}$$

* $\mathcal{L}^{-1}\left\{\ln\frac{s^2+1}{(s-1)^2}\right\}$

$$F(s) = \ln(s^2+1) - 2\ln(s-1)$$

$$\mathcal{L}^{-1} F'(s) = \mathcal{L}^{-1}\frac{2s}{s^2+1} - \frac{2}{s-1}$$

$$-t\, f(t) = \mathcal{L}^{-1}\left\{\frac{2s}{s^2+1} - \frac{2}{s-1}\right\}$$

$$-t\, f(t) = 2\cos(x) - 2e^{t}$$

$$f(t) = \frac{2\cos x - 2e^{t}}{-t}$$

③ $\mathcal{L}^{-1}\left\{\cot^{-1}\frac{s}{\pi}\right\}$

let $F(s) = \cot^{-1}\left(\frac{s}{\pi}\right)$

$\cot^{-1}(as)$

$= \frac{-\text{المشتقة}}{1+(as)^2}$

$\tan^{-1}(as)$

$\frac{\text{المشتقة}}{1+(as)^2}$

$$F'(s) = \left\{\frac{-\frac{1}{\pi}}{1+\frac{s^2}{\pi^2}}\right\} \times \pi^2$$

بدي أبسطو

$$\mathcal{L}^{-1} F'(s) = \mathcal{L}^{-1}\frac{-\pi}{\pi^2+s^2}$$

$$-t \cdot f(t) = \mathcal{L}^{-1}\left[\frac{-\pi}{\pi^2+s^2}\right]$$

$$-t \cdot f(t) = \sin \pi t$$

$$f(t) = \frac{\sin \pi t}{t}$$

④ $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+16)^2}\right\}$

Let $\mathcal{L}^{-1} F(s) = \mathcal{L}^{-1} \frac{1}{4} \cdot \frac{4}{(s^2+16)^2} \Rightarrow f(t) = \frac{1}{4}\sin 4t$

inverse الو المعروفة

شو علاقة هاد بالسؤال؟ مشتقة هاد بتعطيني المطلوب (بغض النظر عن ال constant)

$\mathcal{L}^{-1} F'(s) = \mathcal{L}^{-1} \frac{-2s}{(s^2+16)^2}$

$-t.f(t) = \mathcal{L}^{-1}\left[\frac{-2s}{(s^2+16)^2}\right]$

$-t\; f(t) = -2\,\mathcal{L}^{-1}\left[\frac{s}{(s^2+16)^2}\right]$

$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+16)^2}\right\} = \frac{-t\,f(t)}{-2} = \frac{t\,f(t)}{2} = \frac{1}{2}\; t \cdot \frac{1}{4}\sin 4t$

$= \frac{t}{8}\sin 4t$

Steps:-

1. بجيب $F(s)$ للي بعرفه
2. بشتقه $F(s)$
3. بجيب ال $\mathcal{L}^{-1}$ وبكون هي ال $f(t)$
4. برجع بجيب مشتقة $F'(s)$ لل $F(s)$
5. بجيب ال Laplace للطرفين.
6. وبعوض

$\# \; \mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(u).du$

وبنحسب ال Laplace لهاي

function

مقسومة

t

بدل القسمة

ع t بعطي تكامل

تذكير $\mathcal{L}\{t.f(t)\} = -F'_s$

$\tan^{-1}\infty = \frac{\pi}{2}$

note: $\frac{1}{ab} = \frac{\frac{1}{b}}{a} = \frac{\frac{1}{a}}{b}$

$\frac{1}{s(s+1)} = \frac{\frac{1}{s}}{s+1} = \frac{\frac{1}{(s+1)}}{s}$

Ex: Find $\mathcal{L}\left\{\frac{\sin t}{t}\right\}$

$= \int_s^{\infty} \frac{1}{u^2+1}\,du = \tan^{-1}u \Big|_s^{\infty}$

$= \tan^{-1}\infty - \tan^{-1}s$

$= \frac{\pi}{2} - \tan^{-1}s$

$\# \; \mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \frac{F(s)}{s}$

طب التكامل شو معملو؟ قسم النتيجة على s

بتروح لهاد ال function وبنحسبلو ال $(\mathcal{L})$

بنحسبلو $(\mathcal{L})$

$\Rightarrow \mathcal{L}^{-1}\left\{\frac{F(s)}{\boxed{s}}\right\} = \int_0^t f(u).du$

التكامل

* $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$

$$= \mathcal{L}^{-1}\,\frac{\frac{1}{(s^2+1)}}{s} = \int_0^t \sin u\,du$$

$$-\cos u \Big|_0^t$$

$$-[\cos t - 1]$$

$$= 1 - \cos t$$

لو اجاني السؤال

* $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}$, $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1-\cos t$

$$\mathcal{L}^{-1}\left\{\frac{\frac{1}{s(s^2+1)}}{s}\right\} = \int_0^t 1-\cos t = t - \sin t \Big|_0^t$$

$$t - \sin t - [0 - \sin 0]$$

$$= t - \sin t$$

قبل هيك حسبناها بطريقة ال Partial Fraction

* $\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{\frac{1}{s+1}}{s}\right\} = \int_0^t e^{-t} = -e^{-t}\Big|_0^t$

$$-e^{-t} - -e^{0}$$

$$= 1 - e^{-t}$$