FUZZY LOGIC

Selected Topics
In
Manufacturing

موضوعات مختارة
في
التصنيع

Written by
El Faouri - الفاعوري

Chapter 1
Introduction

Fuzzy: “not sharp, unclear, imprecise, approximate”.
Fuzziness occurs when the boundary of a piece of information is not clear-cut.

Fuzzy VS Vague
Fuzzy: Approximate (ex: There is a quiz in about 10 minutes)
Vague: Not specific (ex: There is a quiz)

Crisp Set VS Fuzzy Set

Crisp (classical) set	Fuzzy set
It is defined by **crisp (exact) boundaries** (i.e. no uncertainty about the location of the set boundaries).	It is defined by **ambiguous boundaries** (i.e. uncertainty about the location of the set boundaries).
Either an element belongs to the set or it does not.	It contains objects that satisfy **imprecise properties of membership (degree).**
Used in **digital systems** *(as in computer either one or zero)*	Used in **fuzzy controllers**

Example:

Crisp: Is the water colorless?
Answer: Yes, No

Fuzzy: Is he honest?
Answer: Extremely honest (0.8), Very honest (0.6), Sometimes (0.5), Dishonest (0.2)

Examples of Fuzzy Applications:

1- Fuzzy Washing Machine

The first major consumer product to use ‘fuzzy systems’ (Matsushita Electric Industrial Company in Japan in 1991). The fuzzy system included 3 main input variables (the extent of dirt, the dirt type and the load size), which were measured using optical sensors, and one output (choice of the correct cycle).

2- Fuzzy Control of Subway Train (Sendai Subway in Japan)

Two controllers: Constant speed controller (Starts the train and keeps speed below its safety limit) and Automatic stopping controller (Regulates the speed limit to stop at a target position).

Chapter 2
Crisp & Fuzzy Sets

2.1 Crisp Sets

2.1.1 Definitions:

Universe of discourse (X): A collection of objects having the same characteristics.

Cardinal number (CN): The total number of elements in a universe.

Note: Discrete universes have a finite cardinal number, whereas continuous universes have an infinite cardinality

Set: Collections of elements within a UNIVERSE.

Subset: Collections of elements within SETS.

The whole set: collection of all elements in the set.

A⊂B:	A is **fully contained** in B
A⊆BA:	A is **contained in or is equivalent to** B
A↔BA:	A **is equivalent to B**
Null set (Ø):	The set that contains no elements.
Power set (P(X)):	All possible sets of X (CN of power set: 2^n)

Example:
X= {1, 2, 3, 4}

Find CN:	CN=	4
Power Set:	P(X)=	{∅, {1},{2},{3},{4}, {1,2},{1,3},{1,4},{2,3},{2,4},{3,4}, {1,2,3},{1,2,4},{1,3,4},{2,3,4}, {1,2,3,4}}
CN of Power Set:		16

2.1.2 Crisp Sets Operations:

Union (A ∪ B)
Intersection (A∩B)
Difference (A|B)
Symmetric Difference (AΔB)

2.1.3 Crisp Set Properties

Property	Rule	Property	Rule
Commutativity	$A \cup B = B \cup A$ $A \cap B = B \cap A$	Transitivity	If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$
Associativity	$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Involution	$\overline{\overline{A}} = A$
Distributivity	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Excluded Middle	$A \cup \overline{A} = X$
Idempotency	$A \cup A = A$ $A \cap A = A$	Contradiction	$A \cap \overline{A} = \emptyset$
Identity	$A \cup \emptyset = A$ $A \cap X = A$ $A \cap \emptyset = \emptyset$ $A \cup X = X$	De Morgan's Laws	$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Example:
The Universe "X":
$X = \{\text{black, white, pink, blue, red, green}, \ldots\}$

Sets:
$A = \{\text{black, white, red, green}\}$
$B = \{\text{black, red, blue, pink}\}$

Find:

Union:
$A \cup B = \{\text{black, white, red, blue, green, pink}\}$

Intersection:
$A \cap B = \{\text{black, red}\}$

Complement of A:
$\overline{A} = \{\text{pink, blue}, \ldots\}$

Complement of B:
$\overline{B} = \{\text{white, green}, \ldots\}$

A \ B (A minus B):
$A \setminus B = \{\text{white, green}\}$

B \ A (B minus A):
$B \setminus A = \{\text{blue, pink}\}$

2.2 Fuzzy Sets

Fuzzy sets contain elements that have varying degrees of membership.
All the operations for classical sets are valid for fuzzy sets EXCEPT for the excluded middle axioms:

For fuzzy sets: $A \cup \overline{A} \neq X$, $A \cap \overline{A} \neq \emptyset$

Fuzzy intersections and unions can be represented as t-norms and t-conorms, respectively

For a collection of fuzzy sets and subsets on a universe, what is:
• The fuzzy power set? Answer: ∞
• The cardinal number of the fuzzy power set? Answer: ∞

2.2.1 Fuzzy Sets Operations

Operation	Rule	Meaning
Union	$\mu_{A\cup B}(x) = \max(\mu_A(x), \mu_B(x))$	Take the maximum
Intersection	$\mu_{A\cap B}(x) = \min(\mu_A(x), \mu_B(x))$	Take the minimum
Compliment	$\mu_{\overline{A}}(x) = 1 - \mu_A(x)$	Subtract from 1
Containment	$\mu_A(x) \leq \mu_B(x)$	A contained B if A(x) ≤ B(x)

Example: Calculate: Complement, union, intersection and difference.

$$\tilde{A} = \frac{1}{2} + \frac{0.5}{3} + \frac{0.3}{4} + \frac{0.2}{5} \qquad \tilde{B} = \frac{0.5}{2} + \frac{0.7}{3} + \frac{0.2}{4} + \frac{0.4}{5}$$

Note that the membership function of 1 is Zero.

Solution:

Operation	Solution	Operation	Solution
Complement of A	$\overline{\tilde{A}} = \left\{\frac{1}{1} + \frac{0}{2} + \frac{0.5}{3} + \frac{0.7}{4} + \frac{0.8}{5}\right\}$	Complement of B	$\overline{\tilde{B}} = \left\{\frac{1}{1} + \frac{0.5}{2} + \frac{0.3}{3} + \frac{0.8}{4} + \frac{0.6}{5}\right\}$
Union A ∪ B	$\tilde{A} \cup \tilde{B} = \left\{\frac{1}{2} + \frac{0.7}{3} + \frac{0.3}{4} + \frac{0.4}{5}\right\}$	Intersection A ∩ B	$\tilde{A} \cap \tilde{B} = \left\{\frac{0.5}{2} + \frac{0.5}{3} + \frac{0.2}{4} + \frac{0.2}{5}\right\}$
A | B (A minus B)	$\tilde{A} \mid \tilde{B} = \left\{\frac{0.5}{2} + \frac{0.3}{3} + \frac{0.3}{4} + \frac{0.2}{5}\right\}$	B | A (B minus A)	$\tilde{B} \mid \tilde{A} = \left\{\frac{0}{2} + \frac{0.5}{3} + \frac{0.2}{4} + \frac{0.4}{5}\right\}$

Chapter 3
Fuzzy Relations

Relations

1- Cartesian Product (Minimum)
AxB

$$\tilde{A} = \frac{a}{x_1} + \frac{b}{x_2} + \frac{c}{x_3}$$
$$\tilde{B} = \frac{Q}{y_1} + \frac{R}{y_2}$$

Solution

$$\tilde{A} \times \tilde{B} = \begin{pmatrix} \min(a,Q) & \min(a,R) \\ \min(b,Q) & \min(b,R) \\ \min(c,Q) & \min(c,R) \end{pmatrix}$$

Steps:

1. Compare the **first element** of Set A with the **first element** of Set B and take the minimum.
2. Compare the **first element** of Set A with the **second element** of Set B and take the minimum.
3. Compare the **second element** of Set A with the **first element** of Set B and take the minimum.
4. Compare the **second element** of Set A with the **second element** of Set B and take the minimum.
5. Compare the **third element** of Set A with the **first element** of Set B and take the minimum.
6. Compare the **third element** of Set A with the **second element** of Set B and take the minimum.

Example: Find the Cartesian Product of these two fuzzy sets

$$\tilde{A} = \frac{0.2}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3} \qquad \tilde{B} = \frac{0.3}{y_1} + \frac{0.9}{y_2}$$

Sol: $$\tilde{A} \times \tilde{B} = \begin{pmatrix} 0.2 & 0.2 \\ 0.3 & 0.5 \\ 0.3 & 0.9 \end{pmatrix}$$

2- Composition $\tilde{R} \circ \tilde{S}$

2.1 Max-Min Composition (Take min then take max)

$$\mu_{\tilde{T}}(x,z) = \bigvee_{y \in Y} \left(\mu_{\tilde{R}}(x,y) \wedge \mu_{\tilde{S}}(y,z)\right)$$

Example 1:

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad S = \begin{pmatrix} e & f & g \\ h & i & j \end{pmatrix}$$

R1C1	**Max [min(a,e) | min(b,h)]**
R1C2	**Max [min(a,f) | min(b,i)]**
R1C3	**Max [min(a,g) | min(b,j)]**
R2C1	**Max [min(c,e) | min(d,h)]**
R2C2	**Max [min(c,f) | min(d,i)]**
R2C3	**Max [min(c,g) | min(d,j)]**

$$R \circ S = \begin{pmatrix} \mathrm{Max}[\min(a,e), \min(b,h)] & \mathrm{Max}[\min(a,f), \min(b,i)] & \mathrm{Max}[\min(a,g), \min(b,j)] \\ \mathrm{Max}[\min(c,e), \min(d,h)] & \mathrm{Max}[\min(c,f), \min(d,i)] & \mathrm{Max}[\min(c,g), \min(d,j)] \end{pmatrix}$$

Example 2: Find the Max-Min Composition of the following:

$$\tilde{R} = \begin{pmatrix} 0.2 & 0.2 \\ 0.3 & 0.5 \\ 0.3 & 0.9 \end{pmatrix} \qquad \tilde{S} = \begin{pmatrix} 0.1 & 0.2 & 0.3 & 0.4 \\ 0.2 & 0.3 & 0.4 & 0.5 \end{pmatrix}$$

Sol:

$$\tilde{R} \circ \tilde{S} = \begin{pmatrix} 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.5 \\ 0.2 & 0.3 & 0.4 & 0.5 \end{pmatrix}$$

2.2 Max-Product Composition (Multiply then take max)

$$\mu_{\tilde{T}}(x,z) = \bigvee_{y \in Y} \left(\mu_{\tilde{R}}(x,y) \cdot \mu_{\tilde{S}}(y,z)\right)$$

Example 1:

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad S = \begin{pmatrix} e & f & g \\ h & i & j \end{pmatrix}$$

R1C1	**Max [(a•e) | (b•h)]**
R1C2	**Max [(a•f) | (b•i)]**
R1C3	**Max [(a•g) | (b•j)]**
R2C1	**Max [(c•e) | (d•h)]**
R2C2	**Max [(c•f) | (d•i)]**
R2C3	**Max [(c•g) | (d•j)]**

$$R \circ S = \begin{pmatrix} \max[a \cdot e,\ b \cdot h] & \max[a \cdot f,\ b \cdot i] & \max[a \cdot g,\ b \cdot j] \\ \max[c \cdot e,\ d \cdot h] & \max[c \cdot f,\ d \cdot i] & \max[c \cdot g,\ d \cdot j] \end{pmatrix}$$

Example 2: Find the Max-Product Composition of the following :

$$\tilde{R} = \begin{pmatrix} 0.2 & 0.2 \\ 0.3 & 0.5 \\ 0.3 & 0.9 \end{pmatrix} \qquad \tilde{S} = \begin{pmatrix} 0.1 & 0.2 & 0.3 & 0.4 \\ 0.2 & 0.3 & 0.4 & 0.5 \end{pmatrix}$$

Sol:

$$\tilde{R} \circ \tilde{S} = \begin{pmatrix} 0.04 & 0.06 & 0.08 & 0.10 \\ 0.10 & 0.15 & 0.20 & 0.25 \\ 0.18 & 0.27 & 0.36 & 0.45 \end{pmatrix}$$

Tolerance and Equivalence

A fuzzy relation is considered as a fuzzy equivalence relation if it has the following properties:

1- Reflexivity $\mu_{\tilde{R}}(x_i, x_i) = 1$
2- Symmetry $\mu_{\tilde{R}}(x_i, x_j) = \mu_{\tilde{R}}(x_j, x_i)$
3- Transitivity $\mu_{\tilde{R}}(x_i, x_j) = \lambda_1 \quad \text{and} \quad \mu_{\tilde{R}}(x_j, x_k) = \lambda_2 \ \rightarrow \ \mu_{\tilde{R}}(x_i, x_k) = \lambda, \ \text{where } \lambda \geq \min[\lambda_1, \lambda_2]$

Value Assignment Methods

- **Cartesian product,**
- Closed-form expression,
- Lookup table,
- Linguistic rules of knowledge,
- Classification
- Automated methods from input/output data
- **Similarity methods in data manipulation.**

Similarity methods

Suppose we have the following data

Regions	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$
$\mathbf{x}_{11}$	**a**	**d**	**g**
$\mathbf{x}_{12}$	**b**	**e**	**h**
$\mathbf{x}_{13}$	**c**	**f**	**i**

1- Cosine Amplitude Method

$$r_{12} = \frac{|\,(a \cdot d) + (b \cdot e) + (c \cdot f)\,|}{\sqrt{(a^2 + b^2 + c^2)(d^2 + e^2 + f^2)}}$$

$$r_{13} = \frac{|\,(a \cdot g) + (b \cdot h) + (c \cdot i)\,|}{\sqrt{(a^2 + b^2 + c^2)(g^2 + h^2 + i^2)}}$$

2- Max-Min Method

$$r_{12} = \frac{\min(a, d) + \min(b, e) + \min(c, f)}{\max(a, d) + \max(b, e) + \max(c, f)}$$

$$r_{13} = \frac{\min(a, g) + \min(b, h) + \min(c, i)}{\max(a, g) + \max(b, h) + \max(c, i)}$$

Example: Using Cosine Amplitude Method and Max-Min Method, express the following data as fuzzy relations

Regions	x_1	x_2	x_3	x_4	x_5
x_{i1} – Ratio with no damage	0.3	0.2	0.1	0.7	0.4
x_{i2} – Ratio with medium damage	0.6	0.4	0.6	0.2	0.6
x_{i3} – Ratio with serious damage	0.1	0.4	0.3	0.1	0.0

1- Cosine Amplitude Method

$$r_{12} = \frac{(0.3 \cdot 0.2) + (0.6 \cdot 0.4) + (0.1 \cdot 0.4)}{\sqrt{(0.3^2 + 0.6^2 + 0.1^2)(0.2^2 + 0.4^2 + 0.4^2)}} = \frac{0.34}{\sqrt{0.46 \cdot 0.36}} = 0.836$$

$$r_{13} = \frac{(0.3 \cdot 1) + (0.6 \cdot 0.6) + (0.1 \cdot 0.3)}{\sqrt{(0.3^2 + 0.6^2 + 0.1^2)(1^2 + 0.6^2 + 0.3^2)}} = \frac{0.57}{\sqrt{0.46 \cdot 0.386}} = 0.914$$

$$\tilde{R}_1 = \begin{bmatrix} 1 & & & & \\ 0.836 & 1 & & & \\ 0.914 & 0.934 & 1 & & \\ 0.682 & 0.6 & 0.441 & 1 & \\ 0.982 & 0.74 & 0.818 & 0.774 & 1 \end{bmatrix}$$

1- Max-Min Method

$$r_{12} = \frac{0.2 + 0.4 + 0.1}{0.3 + 0.6 + 0.4} = 0.5385$$

$$r_{13} = \frac{0.1 + 0.6 + 0.1}{0.3 + 0.6 + 0.3} = 0.667$$

$$\tilde{R}_1 = \begin{bmatrix} 1 & & & & \\ 0.538 & 1 & & & \\ 0.667 & 0.667 & 1 & & \\ 0.429 & 0.333 & 0.250 & 1 & \\ 0.818 & 0.429 & 0.538 & 0.429 & 1 \end{bmatrix}$$

Chapter 4

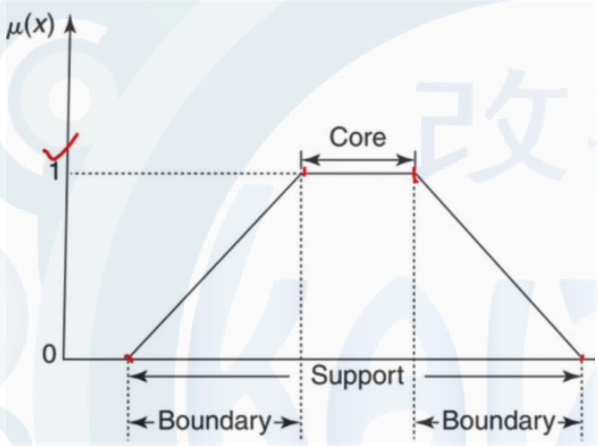
Properties of Membership Functions, Fuzzification and Defuzzification

A membership function: The values assigned to elements of a universal set fall within a specified range

Note: Larger values indicate higher degrees of membership.

The core: The region (element) of a universe that is characterized by full membership in a set.

The support: The region (element) of a universe that is characterized by nonzero membership in a set.



A normal fuzzy set: A set whose membership function has at least one element whose membership value is unity.

Prototype: It is an element (only one element) that has a membership value that is equal to one

A convex fuzzy set: It is a set which is described by a membership function whose values are (1) strictly monotonically increasing, (2) strictly monotonically decreasing, or (3) monotonically increasing then decreasing with increasing the elements values.

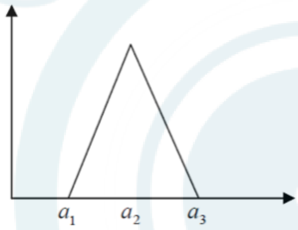
The **height** of a fuzzy set is the **maximum** value of the membership function.

Membership functions can be **symmetrical** or **asymmetrical**. • They can also be defined as **1D** or **nD** membership functions.

4.1 Fuzzification

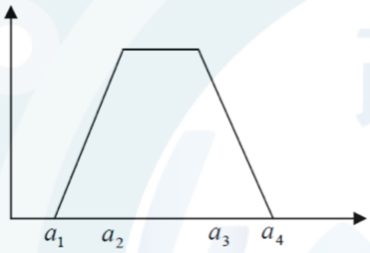
Fuzzification: the process of mapping crisp values to fuzzy sets.

a) Triangular Membership Function:



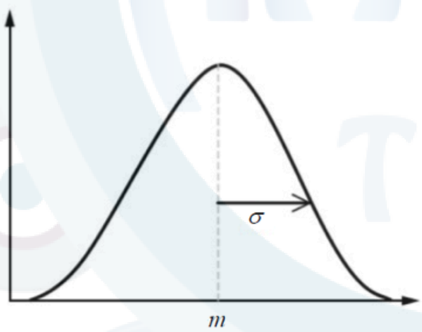
$$\mu_A(u) = \begin{cases} 0 & u < a_1 \\ \dfrac{u - a_1}{a_2 - a_1} & a_1 \leq u < a_2 \\ \dfrac{a_3 - u}{a_3 - a_2} & a_2 \leq u < a_3 \\ 0 & u \geq a_3 \end{cases}$$

b) Trapezoidal Membership Function:



$$\mu_A(u) = \begin{cases} 0 & u < a_1 \\ \dfrac{u - a_1}{a_2 - a_1} & a_1 \leq u < a_2 \\ 1 & a_2 \leq u < a_3 \\ \dfrac{a_4 - u}{a_4 - a_3} & a_3 \leq u < a_4 \\ 0 & u \geq a_4 \end{cases}$$

c) Gaussian Membership Function:



$$\mu_A(u) = e^{-\left(\frac{u-m}{\sigma}\right)^2}$$

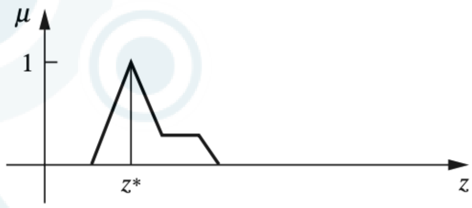
4.2 Defuzzification

Defuzzification: the process of mapping fuzzy values to crisp ones.

Defuzzification Methods

A) Max-Membership Method:

$\mu(z^*) \geq \mu(z)$



Only works when there is a peak.
We take the highest peak.

B) Centroid Method:

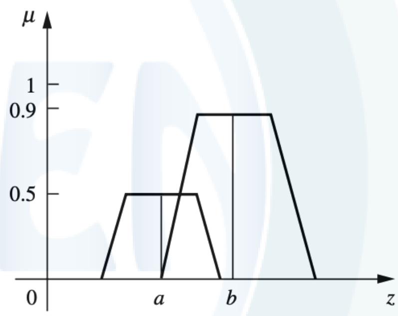
$$z^* = \frac{\int z \cdot \mu(z)\, dz}{\int \mu(z)\, dz}$$

Steps:
1) If more than one shape is given, we draw the union
2) We divide the shape into parts
3) We find the equation for each part
4) Solve!

C) Weighted Average Method:

$$z^* = \frac{\sum \mu(\bar{z}_i) \cdot \bar{z}_i}{\sum \mu(\bar{z}_i)}$$

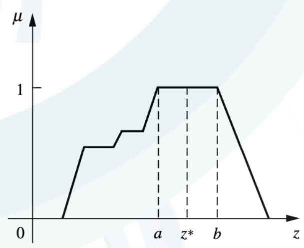
$$z^* = \frac{\mu(a) \cdot a + \mu(b) \cdot b + \dots}{\mu(a) + \mu(b) + \dots}$$



Note: This solution is used only
When the shapes are symmetrical

D) Mean of Maxima Method:

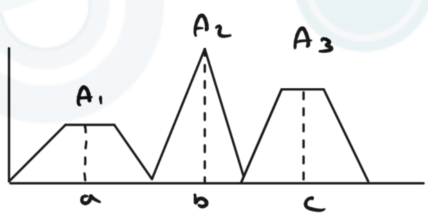
$$z^* = \frac{a+b}{2}$$



E) Center of Sums Method:

$$z^* = \frac{\sum A_i \cdot \bar{z}_i}{\sum A_i}$$

$$z^* = \frac{A_1 \cdot a + A_2 \cdot b + A_3 \cdot c}{A_1 + A_2 + A_3}$$



F) First/Last Maxima

Chapter 5
Logic & Fuzzy Systems

Definitions:

Logic is a small part of the human capacity to reason.

Fuzzy logic is a method to formalize the human capacity of imprecise (approximate) reasoning.

Reasoning: represents the human ability to judge under uncertainty.

Interpolative reasoning: the process of interpolating between the binary extremes of true and false is represented by the ability of fuzzy logic to encapsulate partial truths.

Proposition is associated with the concepts of truth sets, tautologies, and contradictions.

Crisp Logic

A proposition (P) is a linguistic (declarative) statement contained within a universe of elements (X) that can be identified as being a collection of elements in X, which are strictly true or strictly false. • The veracity of an element in the proposition P can be assigned a binary (Boolean) truth value, T (P).

$$T(p) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

Example:

$$U = \{\text{black, blue, red, white, green}\}$$

$$P \to \text{red} \in U, \qquad \therefore\ T(P) = 1$$
$$P \to \text{yellow} \notin U, \qquad \therefore\ T(P) = 0$$

• Truth set, T(P): all elements u in U that are true for proposition P.
• Falsity set: all elements u in U that are false for proposition P

$$T(U) = 1 \qquad T(\emptyset) = 0$$

Connectives

disjunction ($\vee$) conjunction ($\wedge$) negation ($\neg$) implication ($\rightarrow$) equivalence ($\leftrightarrow$)

1- Disjunction (MAX): $P \vee Q : x \in A$ or $x \in B$; hence, $T(P \vee Q) = \max(T(P), T(Q))$

2- Conjunction (MIN): $P \wedge Q : x \in A$ and $x \in B$; hence, $T(P \wedge Q) = \min(T(P), T(Q))$

3- Negation (Opposite): If $T(P) = 1$, then $T(\overline{P}) = 0$; if $T(P) = 0$, then $T(\overline{P}) = 1$.

4- Implication (Therefore): $(\mathrm{P} \rightarrow \mathrm{Q}) : x \notin A$ or $x \in B$; hence, $T(\mathrm{P} \rightarrow \mathrm{Q}) = T(\overline{\mathrm{P}} \cup \mathrm{Q})$.

5- Equivalence (Equal): $(\mathrm{P} \leftrightarrow \mathrm{Q}) : T(\mathrm{P} \leftrightarrow \mathrm{Q}) = \begin{cases} 1, & \text{for } T(\mathrm{P}) = T(\mathrm{Q}) \\ 0, & \text{for } T(\mathrm{P}) \neq T(\mathrm{Q}) \end{cases}$

If $T(\mathrm{P}) \cap T(\mathrm{Q}) = \emptyset$ the truth of P always implies the falsity of Q and vice versa, then P and Q are mutually exclusive propositions.

P	**Q**	$\overline{\mathbf{P}}$	**P ∨ Q**	**P ∧ Q**	**P → Q**	**P ↔ Q**
T (1)	T (1)	F (0)	T (1)	T (1)	T (1)	T (1)
T (1)	F (0)	F (0)	T (1)	F (0)	F (0)	F (0)
F (0)	T (1)	T (1)	T (1)	F (0)	T (1)	F (0)
F (0)	F (0)	T (1)	F (0)	F (0)	T (1)	T (1)

Example:

If $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{3, 4, 6, 8, 9, 10\}$

$P \rightarrow 3 \in A \quad Q \rightarrow 1 \in B$

Find: $P \vee Q$

Sol: $T(P) \vee T(Q) = 1 \vee 0 = 1$

Tautologies

Tautologies are compound propositions that are always true irrespective of the truth values of the individual simple propositions.

Tautologies are useful for reasoning, proving theorems, and making deductive inferences.

"All humans are mammals"

"Prime numbers are not divisible by 6"

Truth Table for a Tautology: (It's Always TRUE!)

P	Q	T
1	1	1
1	0	1
0	1	1
0	0	1

A valid argument is a list of premises from which the conclusion follows.

Modus ponens: is a very common scheme used in **forward-chaining** rule based expert systems. It is an operation to find the truth value of a consequent given the truth value of the antecedent in a rule.

FORWARD CHAINING
If A, then B. A.
Therefore, B.

$(A \wedge (A \longrightarrow B))$

A	B	A → B	(A ∧ (A → B))
0	0	1	0
0	1	1	0
1	0	0	0
1	1	1	1

Modus Tollens: is a very common scheme used in **backward-chaining** expert systems.

BACKWARD CHAINING
If A, then B.A.
Therefore, B.

$(A \wedge (A \longrightarrow B))$

A	B	A → B	(A ∧ (A → B))
0	0	1	0
0	1	1	0
1	0	0	0
1	1	1	1

	Modus Ponens	Modus Tollens
Explanation	If A then B	If B didn't happen, so A didn't happen
Form	$A \wedge (A \rightarrow B)$	$\overline{B} \wedge (A \rightarrow B)$
Fuzzy Equivalent	$A \wedge (\overline{A} \vee B)$	$\overline{B} \wedge (\overline{A} \vee B)$
Operation	$\min(A, \max(\overline{A}, B))$	$\min(\overline{B}, \max(\overline{A}, B))$

Contradictions and Equivalence

Contradictions are compound propositions that are always false, regardless of the truth value of the individual propositions constituting the compound proposition.

Example: "Prime numbers are a multiple of 4"

Truth table for Tautologies and (Always True)
Compared with Contradictions (Always False)

T; Tautologies, C: Contradictions

P	Q	T	C
1	1	1	0
1	0	1	0
0	1	1	0
0	0	1	0

Example Suppose we consider the universe of positive integers, $X = \{1 \leq n \leq 8\}$. Let P $=$ "n is an even number" and let Q $=$ "$(3 \leq n \leq 7) \wedge (n \neq 6)$." Then $T(P) = \{2, 4, 6, 8\}$ and $T(Q) = \{3, 4, 5, 7\}$. The equivalence $P \leftrightarrow Q$ has the truth set

$$T(\mathrm{P} \longleftrightarrow \mathrm{Q}) = (T(P) \cap T(Q)) \cup (\overline{T(P)} \cap \overline{T(Q)}) = \{4\} \cup \{1\} = \{1, 4\}$$

XOR vs XNOR

Truth table comparing XOR and XNOR

P	Q	XOR	XNOR
0	0	0	1
0	1	1	0
1	0	1	0
1	1	0	1

Inference: the process of making certain conclusions from some given hypotheses.
How?
The linguistic statement (compound proposition) is made.
The statement is decomposed into its respective single propositions.
The statement is expressed algebraically with logical connectives.
A truth table is used to establish the veracity of the statement.

Example:
Hypotheses: Engineers are mathematicians. Logical thinkers do not believe in magic. Mathematicians are logical thinkers.
Conclusion: Engineers do not believe in magic.
Decomposing the hypotheses:
P : A person is an engineer.
Q : A person is a mathematician.
R : A person is a logical thinker.
S : A person believes in magic.

$$((P \to Q) \wedge (R \to \overline{S}) \wedge (Q \to R)) \to (P \to \overline{S})$$

Mamdani Systems (2-input Mamdani Systems.)

1- Max-Min Inference Method

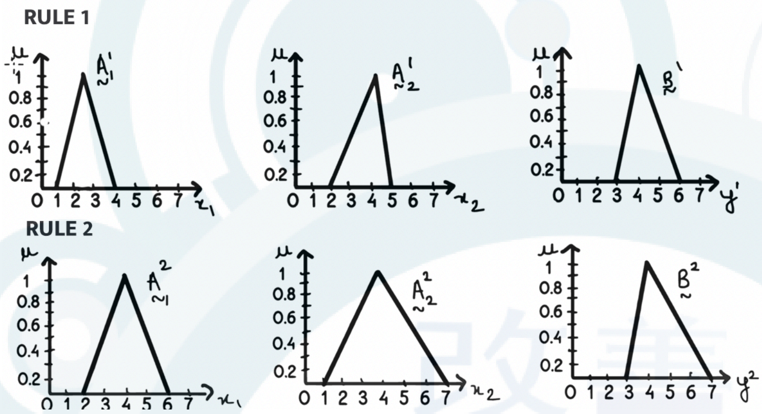
Example with steps:

Consider a simple 2 rule system where each rule has 2 antecedents and one consequent as follows:

Rule 1: IF x_1 is $\tilde{A}_1^1$ and x_2 is $\tilde{A}_2^1$ THEN $\tilde{y}$ is $\tilde{B}^1$

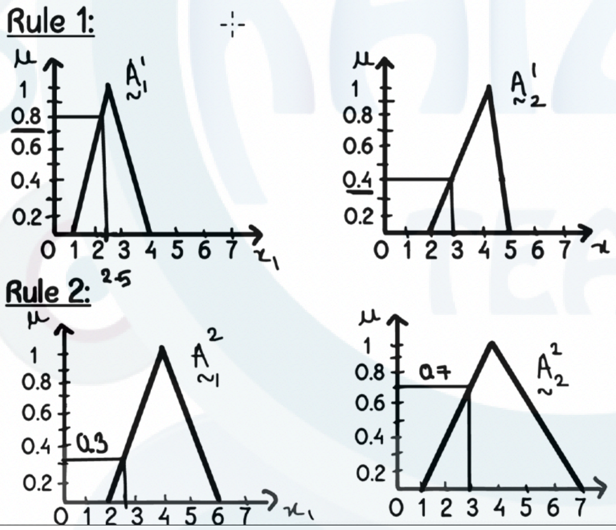
Rule 2: IF x_1 is $\tilde{A}_1^2$ or x_2 is $\tilde{A}_2^2$ THEN $\tilde{y}$ is $\tilde{B}^2$

Given: $x_1 = 2.5 \quad x_2 = 3$

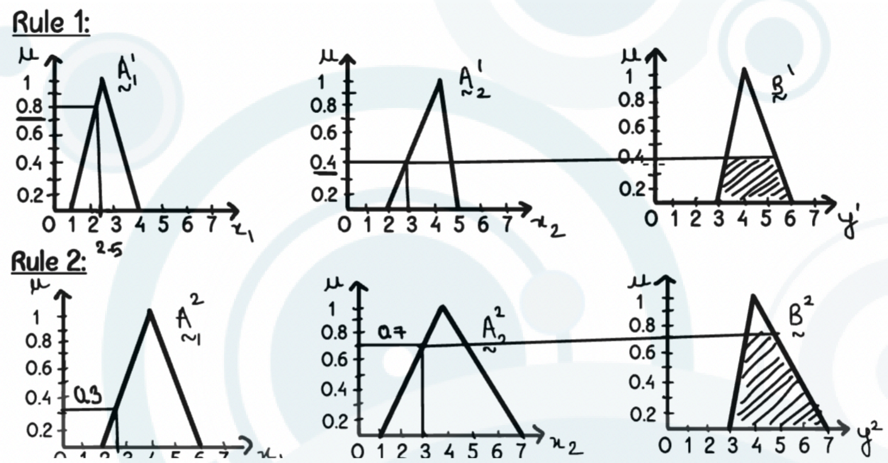


Steps:

1- Find corresponding values of x1 and x2 on the graph
And get the y coordinate from them



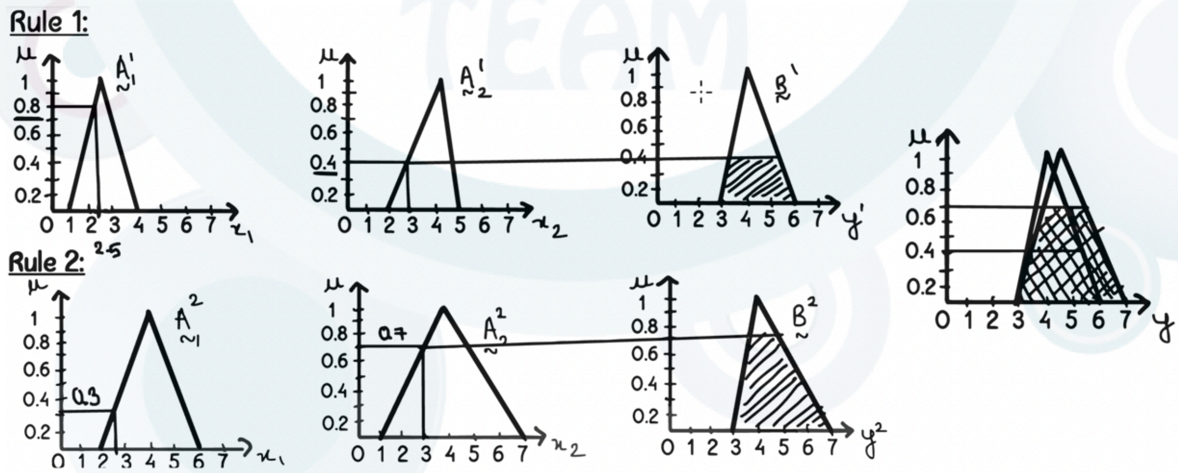
2- Extend a Line from the Rules to the Output B1 and B2
If the rule contains AND extend from the smaller value
If the rule contains OR extend from the bigger value



3- Draw the truncated final output

4- De-fuzzify the output using any of the defuzzification methods

Full overview:



2- Max-Product Inference Method

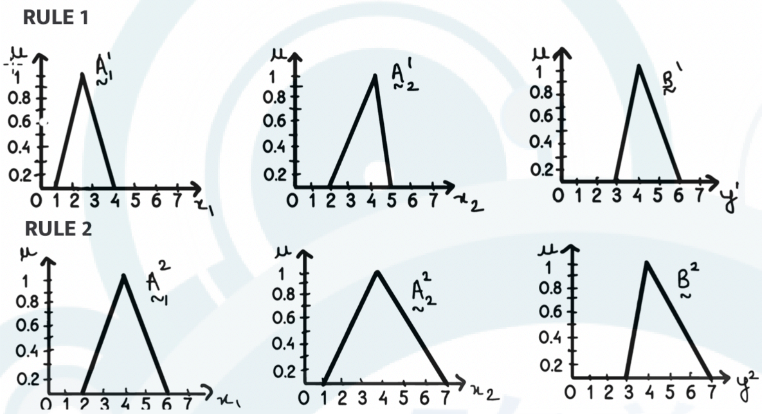
Example with steps:

Consider a simple 2 rule system where each rule has 2 antecedents and one consequent as follows:

Rule 1: IF x_1 is $\tilde{A}_1^1$ and x_2 is $\tilde{A}_2^1$ THEN $\tilde{y}$ is $\tilde{B}^1$

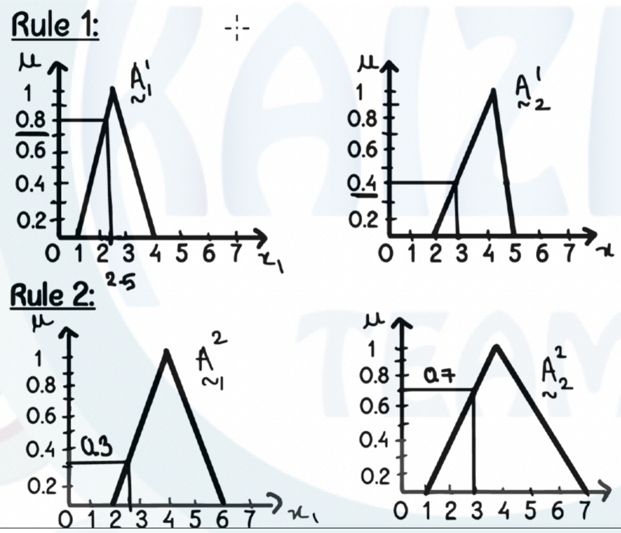
Rule 2: IF x_1 is $\tilde{A}_1^2$ or x_2 is $\tilde{A}_2^2$ THEN $\tilde{y}$ is $\tilde{B}^2$

Given: $x_1 = 2.5 \quad x_2 = 3$



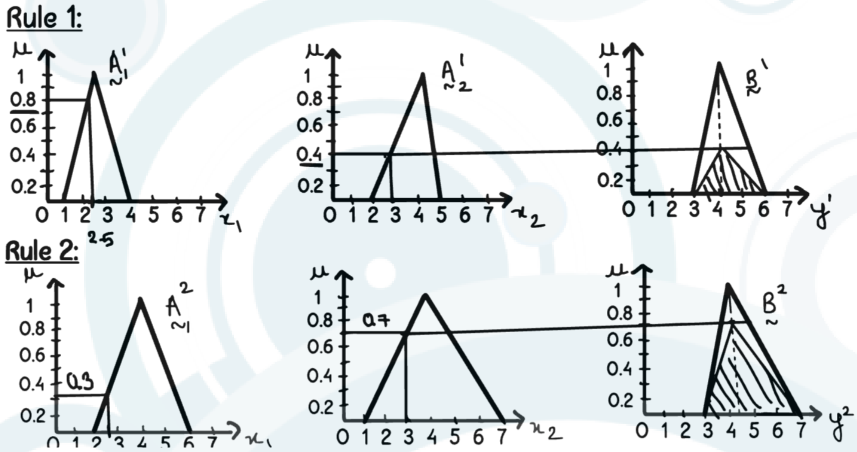
Steps:

1- Find corresponding values of x1 and x2 on the graph
And get the y coordinate from them

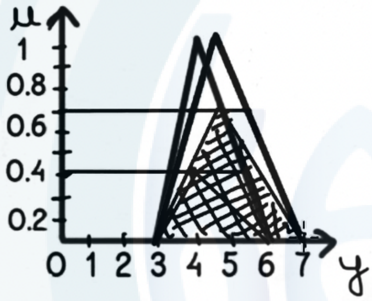


2- Extend a Line from the Rules to the Output B1 and B2
If the rule contains AND extend from the smaller value
If the rule contains OR extend from the bigger value

(The difference here that we don't truncate the function we just resize it)

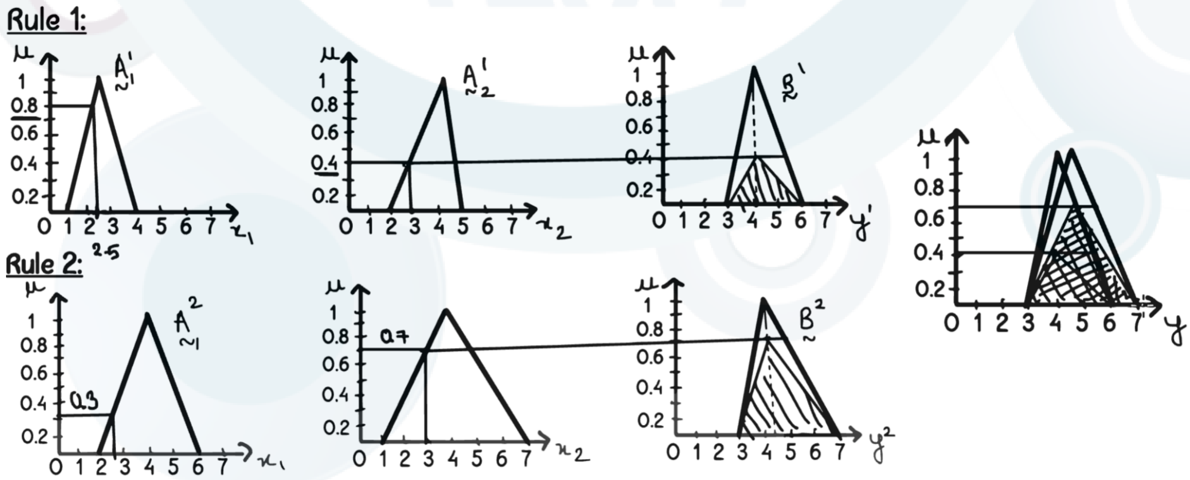


3- Draw the resized final output



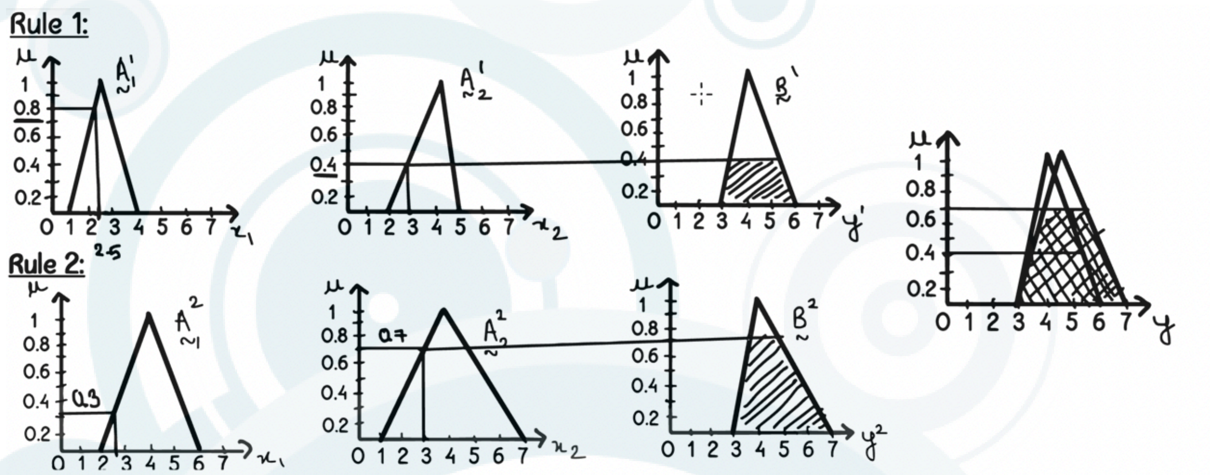
4- De-fuzzify the output using any of the defuzzification methods

Full overview:

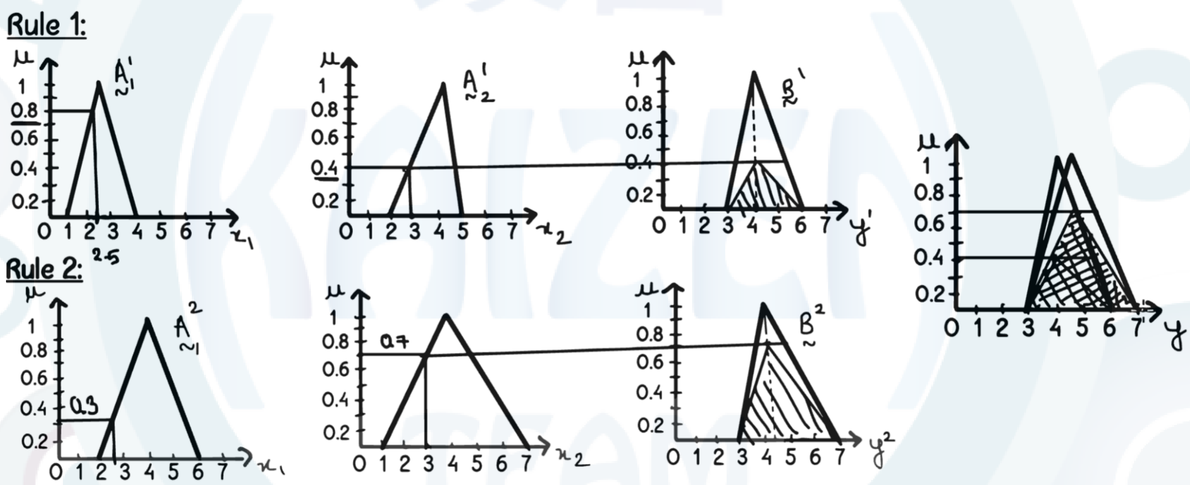


Mamdani Overview:

Max-Min



Max-Product

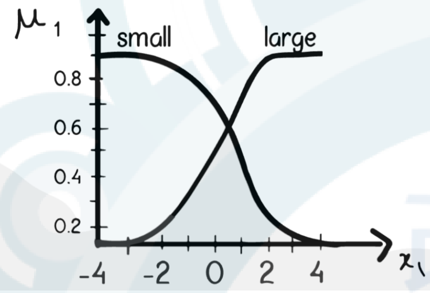


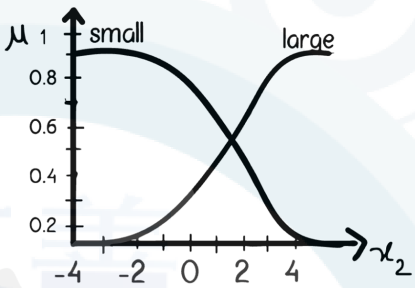
Sugeno Systems (Multi Input & Crisp Output Sugeno Systems.)

Example with steps:
Consider a two input single output Sugeno model with 4 rules as follows:

Rule 1: IF x_1 is small and x_2 is small, THEN $y_1 = -x_1 + x_2 + 1$
Rule 2: IF x_1 is small and x_2 is large, THEN $y_2 = -x_2 + 3$
Rule 3: IF x_1 is large and x_2 is small, THEN $y_3 = -x_1 + 3$
Rule 4: IF x_1 is large and x_2 is large, THEN $y_4 = -x_1 + x_2 + 2$

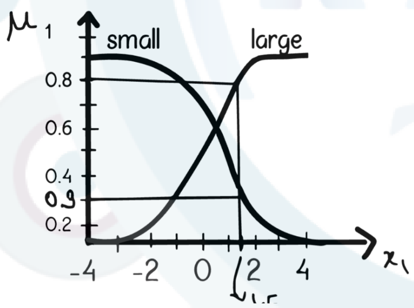
Find the output when $x_1 = 1.5$ and $x_2 = 2.5$

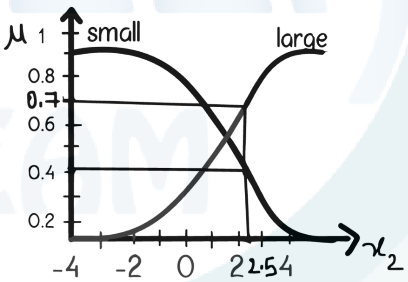




Steps:

1- We extend a line from the corresponding x1,x2 on the x1,x2 axis to the corresponding graph (small or large depending on the rule)





For X1:

Small: 0.3
Large: 0.8

For X2:

Small: 0.4
Large: 0.7

2- Find the corresponding values of each rule according to the operator used in the rule (max for "or" and min for "and)

For Rule 1:min(0.3,0.4) = 0.3
For Rule2: min(0.3,0.7) = 0.3
For Rule3: min(0.8,0.4) = 0.4
For Rule4: min(0.8,0.7) = 0.7

Note: we used "min" because all of the rules are using "AND" operator

3- Substitute the x1,x2 values in the equations of the output "y" for each rule

For Rule1: substitute x1=1.5 and x2=2.5 in y1 equation

$$y_1 = (-x_1) + x_2 + 1 = (-1.5) + 2.5 + 1 = 2$$

Therefore: y1=2
Repeat the same for all other rules

For all rules

y1=2
y2=0.5
y3=1.5
y4=6

4- apply the weighted average method

$$y^* = \frac{w_{y_1} \cdot y_1 + w_{y_2} \cdot y_2 + w_{y_3} \cdot y_3 + w_{y_4} \cdot y_4}{w_{y_1} + w_{y_2} + w_{y_3} + w_{y_4}}$$

$$= \frac{(0.3 \times 2) + (0.3 \times 0.5) + (0.4 \times 1.5) + (0.7 \times 6)}{(0.3) + (0.3) + (0.4) + (0.7)}$$

y*= 3.26

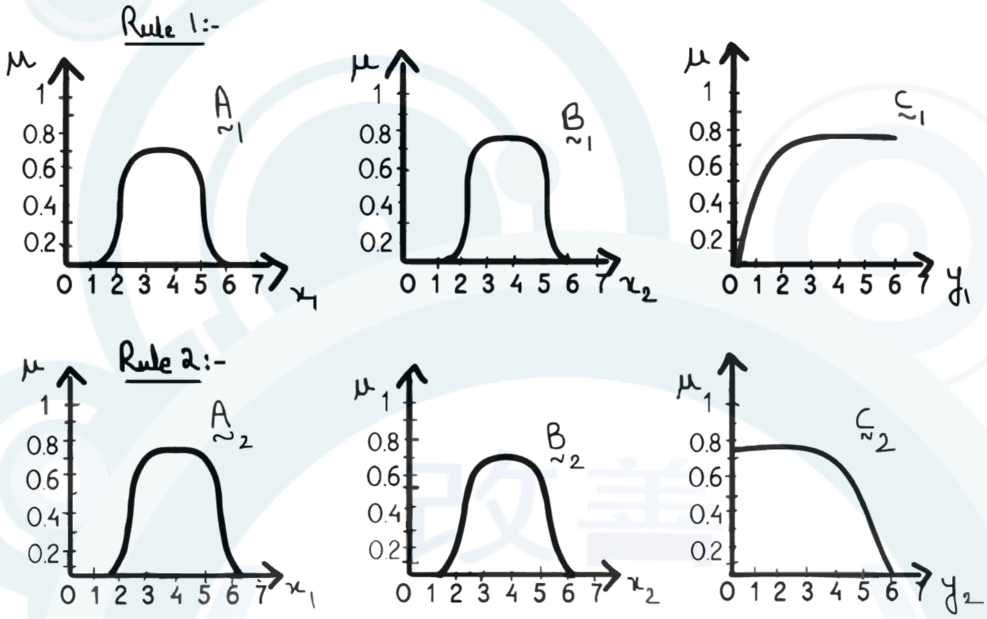
Sugeno Method is better than Mamdani because we can directly apply the weighted average method to find the defuzzified value.

Tsukamoto Systems

Example with steps

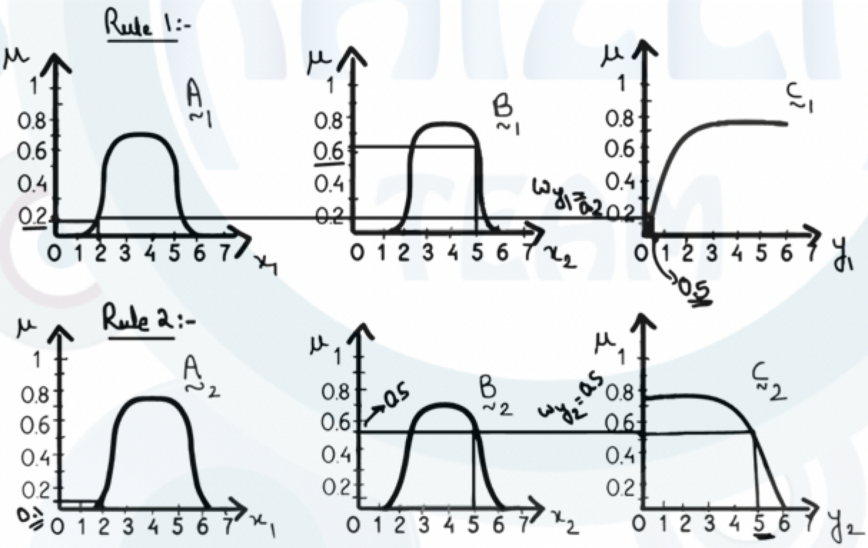
Rule 1: If x_1 is $\tilde{A}_1$ and x_2 is $\tilde{B}_1$, THEN y_1 is $\tilde{C}_1$ $\quad x_1 = 2 \quad x_2 = 5$

Rule 2: If x_1 is $\tilde{A}_2$ or x_2 is $\tilde{B}_2$, THEN y_2 is $\tilde{C}_2$



Steps:

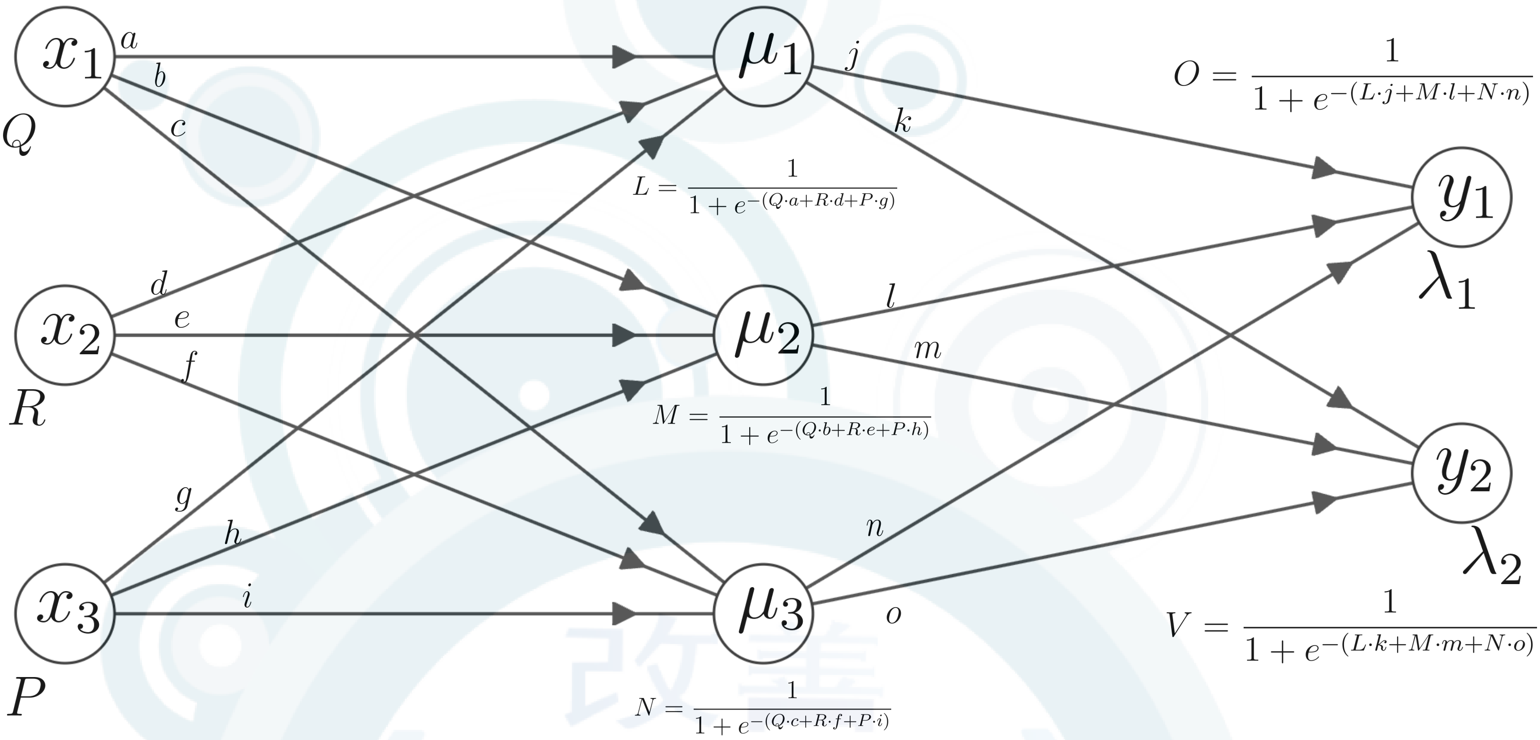
1- Just like Mamdani, find corresponding values of x1 and x2 on the graph
And get the y coordinate from them.Then, Extend a Line from the Rules to the Output
Find the corresponding Y1 and Y2 values from the output graph



2- Use the weighted average method to find the defuzzified value

$$y^* = \frac{w_{y_1} \cdot y_1 + w_{y_2} \cdot y_2}{w_{y_1} + w_{y_2}} = \frac{0.2 \cdot 0.5 + 0.5 \cdot 5}{0.2 + 0.5} = 3.714 \Rightarrow y^* \approx 3.71$$

Chapter6
Neural Networks



Given:
$Q,\ R,\ P$ for $X_1,\ X_2,\ X_3$ respectively
$\lambda_1,\ \lambda_2$ for $Y_1,\ Y_2$ respectively
$a,\ b,\ c,\ d,\ d,\ e,\ f,\ g,\ g,\ h,\ i$ are weights for $X_1,\ X_2,\ X_3$

Step 1: Follow the graph and do the calculations to get the answer of O and V (for y_1 and y_2)

Step 2: Calculate the errors for the output layers $Ey_1 = \lambda_1 - O, \quad Ey_2 = \lambda_2 - V$

Step 3: Calculate the errors for the hidden layers:

$$E\mu_1 = L\cdot(1-L)\cdot(j\cdot Ey_1 + k\cdot Ey_2)$$
$$E\mu_2 = M\cdot(1-M)\cdot(l\cdot Ey_1 + m\cdot Ey_2)$$
$$E\mu_3 = N\cdot(1-N)\cdot(n\cdot Ey_1 + o\cdot Ey_2)$$

Step 4: Calculate New Weights $W_{\text{new}} = W_{\text{old}} + \alpha\cdot\mu\cdot X$, α is the learning factor

$$Wx_{11}^{\text{new}} = a + \alpha\cdot E\mu_1\cdot Q \qquad Wx_{21}^{\text{new}} = d + \alpha\cdot E\mu_1\cdot R \qquad Wx_{31}^{\text{new}} = g + \alpha\cdot E\mu_1\cdot P$$
$$Wx_{12}^{\text{new}} = b + \alpha\cdot E\mu_2\cdot Q \qquad Wx_{22}^{\text{new}} = e + \alpha\cdot E\mu_2\cdot R \qquad Wx_{32}^{\text{new}} = h + \alpha\cdot E\mu_2\cdot P$$
$$Wx_{13}^{\text{new}} = c + \alpha\cdot E\mu_3\cdot Q \qquad Wx_{23}^{\text{new}} = f + \alpha\cdot E\mu_3\cdot R \qquad Wx_{33}^{\text{new}} = i + \alpha\cdot E\mu_3\cdot P$$

Chapter 7
Type-1 Fuzzy Logic System (T1FLS)

K-means clustering

Example:

# of exp	Depth of Cut	Speed	Surface Roughness
1.0	0.2	100.0	5.0
2.0	0.3	200.0	7.0
3.0	0.2	150.0	10.0
4.0	0.15	400.0	8.0
5.0	0.21	300.0	7.0
6.0	0.17	200.0	6.0
7.0	0.25	100.0	7.0
8.0	0.31	200.0	10.0
9.0	0.27	300.0	9.0

- Specify k, the number of clusters to be generated
- Choose k points at random as cluster centers
- Assign each instance to its closest cluster center using Euclidean distance
- Calculate the centroid (mean) for each cluster, use it as a new cluster center
- Reassign all instances to the closest cluster center
- Iterate until the cluster centers don't change anymore

Calculate DC1+DC2

Distance between two points

$$|DC| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

sample:

DC 1 for point 1:

$$\mathrm{DC}_1 = \sqrt{(0.2 - 0.2)^2 + (100 - 150)^2 + (5 - 10)^2} = 50.2$$

DC2 for point 1:

$$\mathrm{DC}_2 = \sqrt{(0.2 - 0.25)^2 + (100 - 100)^2 + (5 - 7)^2} = 2$$

#	DP	SP	SR	DC1	DC2	DC?
1.0	0.2	100.0	5.0	50	2	2
2.0	0.3	200.0	7.0	50.08	100	1
3.0	0.2	150.0	10.0	0	50.09	1
4.0	0.15	400.0	8.0	250	300	1
5.0	0.21	300.0	7.0	150.03	200	1
6.0	0.17	200.0	6.0	50.01	100	1
7.0	0.25	100.0	7.0	50.09	0	2
8.0	0.31	200.0	10.0	50	100	1
9.0	0.27	300.0	9.0	150	200	1

Find new centers For Q1

New Centers:

$C_1 = (0.3, 200, 7){+}(0.2, 150, 10){+}(0.15, 400, 8){+}(0.21, 300, 7){+}(0.17, 200, 6){+}(0.31, 200, 10){+}(0.27, 300, 9)$

C1 = (0.23,250,8.14)

C2 = (0.225,100,6)

K-Means Iteration Table

# of exp	DP	SP	SR	DC_1	DC_2	Q1	DC_1	DC_2	Q2	DC_1	DC_2	Q3
1.0	0.2	100.0	5.0	50.2	2.0	2.0	150.03	1.0	2.0	166.6	16.7	2.0
2.0	0.3	200.0	7.0	50.08	100.0	1.0	50.01	100.0	1.0	66.6	83.4	1.0
3.0	0.2	150.0	10.0	0.0	50.09	1.0	100.01	50.15	2.0	116.6	33.5	2.0
4.0	0.15	400.0	8.0	250.0	300.0	1.0	150.0	300.0	1.0	133.4	283.0	1.0
5.0	0.21	300.0	7.0	150.03	200.0	1.0	50.01	200.0	1.0	33.41	183.0	1.0
6.0	0.17	200.0	6.0	50.01	100.0	1.0	50.04	100.0	1.0	66.6	83.4	1.0
7.0	0.25	100.0	7.0	50.09	0.0	2.0	150.0	1.0	2.0	166.6	16.6	2.0
8.0	0.31	200.0	10.0	50.0	100.05	1.0	50.03	100.08	1.0	66.6	84.4	1.0
9.0	0.27	300.0	9.0	150.0	200.01	1.0	50.0	200.0	1.0	33.4	183.0	1.0

For Q0 initial centers (for #3 and #7)

$C_1 = (0.2,\ 150,\ 10)$

$C_2 = (0.25,\ 100,\ 7)$

For Q1

$C_1 = (0.23,\ 250,\ 8.14)$

$C_2 = (0.225,\ 100,\ 6)$

For Q2

$C_1 = (0.235,\ 266.6,\ 7.83)$

$C_2 = (0.22,\ 116.6,\ 7.33)$

Fpr Q3

$C_1 = (0.235,\ 266.6,\ 7.83)$

$C_2 = (0.22,\ 116.6,\ 7.33)$

Now we stop because centers are not changing anymore!

Now: Calculate the Standard Deviation for each center
Note: the values here are assumed (not correct)

$$C_1 = (0.24,\ 266.6,\ 7.8) \rightarrow \sigma = (0.1,\ 50,\ 2)$$
$$C_2 = (0.21,\ 116.6,\ 7.3) \rightarrow \sigma = (0.12,\ 30,\ 1.5)$$

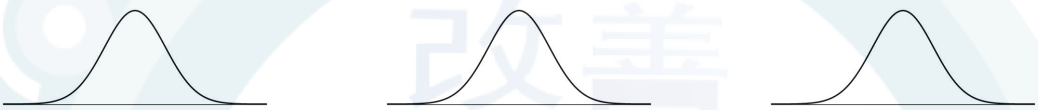
Now we do calculations on any point
For example: point #1 (0.2,100,5)

For Rule #1

For DP $\mu = e^{-0.5\left(\frac{0.2-0.24}{0.1}\right)^2} = 0.92$

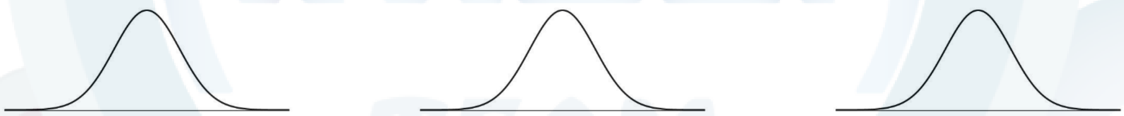
For SP $\mu = e^{-0.5\left(\frac{100-266.6}{50}\right)^2} = 3.88 \times 10^{-3}$

For SR $\Phi = \text{Firing Level} = \mu_y^{(\text{for output})} = (0.9) \times (3.88 \times 10^{-3}) = 3.5 \times 10^{-3}$



DP	SP	SR
0.92	3.88×10^{-3}	3.5×10^{-3}

For Rue #2



DP	SP	SR
0.99	0.86	0.85

$$\Pr_{DO} = \frac{\sum(\mu_y \times \text{center value for output})}{\mu_y^{(1)} + \mu_y^{(2)}} = \frac{[3.5 \times 10^{-3} \times 7.8] + [0.85 \times 7.3]}{3.5 \times 10^{-3} + 0.85} = 7.3$$

Now, Given Alpha = 0.3 (learning factor)

$$\mu_{\text{new}} = \mu_{\text{old}} - \alpha \left[(Pr - y)(\bar{y} - Pr)\left(X_{\text{input}} - \mu_{\text{old}}\right)\right] \cdot \Phi$$

$$m_{\text{new}} = (0.24 - 0.3)\left(\frac{7.3-5}{7.8-7.3}\right)\left(\frac{0.2-0.24}{0.1^2}\right) \times 3.5 \times 10^{-3} = 0.244$$

$$m_{\text{new}} = 266.7 - 0.3\left(\frac{7.3-5}{7.8-7.3}\right)\left(\frac{100-266.7}{50^2}\right) \times 3.5 \times 10^{-3} = 266.7$$

$$m_{\text{new}} = 116.6 - 0.3\left(\frac{7.3-5}{7.3-7.3}\right)\left(\frac{100-116.6}{30^2}\right) \times 0.85 = 116.6$$

$$\bar{y}_{\text{new}} = \bar{y}_{\text{old}} - \alpha\left(\text{Pr} - y_{\text{actual}}\right) \cdot \Phi$$

$$\bar{y}_{\text{new}}^{R_1} = 7.8 - 0.3\left(\frac{7.3-5}{7.3-5}\right) \cdot 3 \times 10^{-3} = 7.79$$

$$\bar{y}_{\text{new}}^{R_2} = 7.3 - 0.3\left(\frac{7.3-5}{7.3-5}\right) \cdot 0.85 = 6.71$$

$$\sigma_{\text{new}} = \sigma_{\text{old}} - \alpha\left(\text{Pr} - y_{\text{actual}}\right)\left(\bar{y} - \text{Pr}\right)\left(\frac{(x-m)^2}{\sigma^3}\right) \cdot \Phi$$

$$\sigma_{\text{new}}^{R_1,\text{DOC}} = 0.1 - 0.3\left(\frac{7.3-5}{7.8-7.3}\right)\left(\frac{0.2-0.24}{0.1^2}\right)^2 \cdot 3.5 \times 10^{-3} = 0.098$$

$$\sigma_{\text{new}}^{R_1,\text{SP}} = 50 - 0.3\left(\frac{7.3-5}{7.8-7.3}\right)\left(\frac{100-266.7}{50^2}\right)^2 \cdot 3.5 \times 10^{-3} = 49.9$$

Chapter 9
Decision Making with Fuzzy

The information affecting an issue is likely incomplete or uncertain; hence, the outcomes are uncertain, irrespective of the decision made or the alternative chosen.

Engineers are primarily concerned with two types of decisions:
1. Operational decisions: an optimal action is sought to avoid a specific set of hazards
2. Strategic decisions: preparation for or anticipation of events in the future.

Fuzzy Synthetic Evaluation

Example with steps:

Suppose we want to measure the value of a microprocessor to a potential client. In conducting this evaluation, the client suggests that certain criteria are important. They can include performance (MIPS), cost ($), availability (AV), and software (SW). Performance is measured by millions of instructions per second (MIPS); a minimum requirement is 10 MIPS. Cost is the cost of the microprocessor, and a cost requirement of "not to exceed" 500 has been set. Availability relates to how much time after the placement of an order the microprocessor vendor can deliver the part; a maximum of eight weeks has been set. Software represents the availability of operating systems, languages, compilers, and tools to be used with this microprocessor. Suppose further that the client is only able to specify a subjective criterion of having "sufficient" software. A particular microprocessor (CPU) has been introduced into the market. It is measured against these criteria and given ratings categorized as excellent (e), superior (s), adequate (a), and inferior (i).

$$R = \begin{array}{c|cccc} & e & s & a & i \\ \hline \text{MIPS} & 0.1 & 0.3 & 0.4 & 0.2 \\ \$ & 0 & 0.1 & 0.8 & 0.1 \\ \text{AV} & 0.1 & 0.6 & 0.2 & 0.1 \\ \text{SW} & 0.1 & 0.4 & 0.3 & 0.2 \end{array}$$

$$w = \left[\frac{0.4}{P}, \frac{0.3}{C}, \frac{0.2}{A}, \frac{0.1}{S}\right]$$

Sol:

Using Max-min Composition نقارن الصف مع كل عمود

1) Max [min(0.1,0.4), min(0,0.3), min(0.1,0.2), min (0.1,0.1)]
2) Max [0.1,0,0,1,0.1] = 0.1
3) Repeat the steps

$$\tilde{e} = \tilde{w} \circ \tilde{R} = \{0.1, 0.3, 0.4, 0.2\}$$

Fuzzy Ordering

Decisions are sometimes made on the basis of rank, or ordinal ranking: which issue is best, which is second best, and so forth.

Truth Value

Example: Order the following fuzzy sets according to Truth Value Method:

$$\tilde{I}_1 = \left\{ \frac{1}{3} + \frac{0.8}{7} \right\}, \quad \tilde{I}_2 = \left\{ \frac{0.7}{4} + \frac{1.0}{6} \right\}, \quad \text{and} \quad \tilde{I}_3 = \left\{ \frac{0.8}{2} + \frac{1}{4} + \frac{0.5}{8} \right\}$$

We take the possibilities:
I1≥I2, I1≥I3, I2≥I1, I2≥I3, I3≥I1, I3≥I2

For each one. We check if the denominator follows the rule:

<table>
<tr><th colspan="5">I1≥I2= 0.8</th></tr>
<tr><td>3≥4</td><td>❌</td><td></td><td rowspan="4">max(0.7,0.8)</td><td rowspan="4">0.8</td></tr>
<tr><td>3≥6</td><td>❌</td><td></td></tr>
<tr><td>7≥4</td><td>✅</td><td>min(0.8,0.7)= 0.7</td></tr>
<tr><td>7≥6</td><td>✅</td><td>min(0.8,1.0)= 0.8</td></tr>
</table>

<table>
<tr><th colspan="5">I1≥I3= 0.8</th></tr>
<tr><td>3≥2</td><td>✅</td><td>min(1.0,0.8)= 0.8</td><td rowspan="6">max(0.8,0.8,0.8)</td><td rowspan="6">0.8</td></tr>
<tr><td>3≥4</td><td>❌</td><td></td></tr>
<tr><td>3≥8</td><td>❌</td><td></td></tr>
<tr><td>7≥2</td><td>✅</td><td>min(0.8,0.8)= 0.8</td></tr>
<tr><td>7≥4</td><td>✅</td><td>min(0.8,1.0)= 0.8</td></tr>
<tr><td>7≥8</td><td>❌</td><td></td></tr>
</table>

<table>
<tr><th colspan="5">I2≥I1= 1.0</th></tr>
<tr><td>4≥3</td><td>✅</td><td>min(0.7,1.0)= 0.7</td><td rowspan="4">max(1.0,1.0)</td><td rowspan="4">1.0</td></tr>
<tr><td>4≥7</td><td>❌</td><td></td></tr>
<tr><td>6≥3</td><td>✅</td><td>min(1.0,1.0)= 1.0</td></tr>
<tr><td>6≥7</td><td>❌</td><td></td></tr>
</table>

I2≥I3= 1.0				
4≥2	✅	min(0.7,0.8)= 0.7	max(0.7,0.7,0.7,1.0)	1.0
4≥4	✅	min(1,0.7)= 0.7		
4≥8	❌			
6≥2	✅	min(1.0,0.8)= 0.8		
6≥4	✅	min(1.0,1.0)= 0.8		
6≥8	❌			

I3≥I1= 1.0				
2≥3	❌		max(1.0,0.5,0.5)	1.0
2≥7	❌			
4≥3	✅	min(1.0,1.0)= 1.0		
4≥7	❌			
8≥3	✅	min(0.5,1.0)= 0.5		
8≥7	✅	min(0.5,0.8)= 0.5		

I3≥I2= 0.7				
2≥4	❌		max(1.0,0.5,0.5)	0.7
2≥6	❌			
4≥4	✅	min(0.7,1.0)= 0.7		
4≥6	❌			
8≥4	✅	min(0.5,0.7)= 0.5		
8≥6	✅	min(0.5,1.0)= 0.5		

We get the following:

I1≥I2= 0.8
I1≥I3= 0.8
I2≥I1= 1.0
I2≥I3= 1.0
I3≥I1= 1.0
I3≥I2= 0.7

Now we need to calculate the following:

$$T(\tilde{I}_1 \geq \tilde{I}_2, \tilde{I}_3), \quad T(\tilde{I}_2 \geq \tilde{I}_1, \tilde{I}_3), \quad T(\tilde{I}_3 \geq \tilde{I}_1, \tilde{I}_2)$$

$$\begin{aligned} T(I_1 \geq I_2, I_3) &= \min[\, (I_1 \geq I_2), (I_1 \geq I_3) \,] \\ &= \min[\, 0.8, 0.8 \,] \\ &= 0.8 \end{aligned}$$

$$\begin{aligned} T(I_2 \geq I_1, I_3) &= \min[\, (I_2 \geq I_1), (I_2 \geq I_3) \,] \\ &= \min[\, 1.0, 1.0 \,] \\ &= 1.0 \end{aligned}$$

$$\begin{aligned} T(I_3 \geq I_1, I_2) &= \min[\, (I_3 \geq I_1), (I_3 \geq I_2) \,] \\ &= \min[\, 1.0, 0.7 \,] \\ &= 0.7 \end{aligned}$$

Therefore,

$$T(\tilde{I}_1 \geq \tilde{I}_2, \tilde{I}_3) = 0.8, \quad T(\tilde{I}_2 \geq \tilde{I}_1, \tilde{I}_3) = 1.0, \quad T(\tilde{I}_3 \geq \tilde{I}_1, \tilde{I}_2) = 0.7$$

Therefore,

$I_2 > I_1 > I_3$

Non-transitive Ranking

When we compare objects that are fuzzy, ambiguous, or vague, we may well encounter a situation where there is a contradiction in the classical notions of ordinal ranking and transitivity in the ranking.

• Example: When comparing red to blue, we prefer red; when comparing blue to yellow, we prefer blue; but when comparing red and yellow we might prefer yellow.

• For non-transitive ranking, the relativity function is introduced

$f_y(x)$ as the membership value of x with respect to y
$f_x(y)$ as the membership value of y with respect to x

$$f(x \mid y) = \frac{f_y(x)}{\max\left[f_y(x),\ f_x(y)\right]}$$

Example: Given

$$\begin{aligned}
&f_{x_1}(x_1) = 1,\ f_{x_1}(x_2) = 0.5,\ f_{x_1}(x_3) = 0.3,\ f_{x_1}(x_4) = 0.2\\
&f_{x_2}(x_1) = 0.7,\ f_{x_2}(x_2) = 1,\ f_{x_2}(x_3) = 0.8,\ f_{x_2}(x_4) = 0.9\\
&f_{x_3}(x_1) = 0.5,\ f_{x_3}(x_2) = 0.3,\ f_{x_3}(x_3) = 1,\ f_{x_3}(x_4) = 0.7\\
&f_{x_4}(x_1) = 0.3,\ f_{x_4}(x_2) = 0.1,\ f_{x_4}(x_3) = 0.3,\ f_{x_4}(x_4) = 1
\end{aligned}$$

Solution:
Sample of calculations:

$$f(x/y) = \frac{f_y(x)}{\max(f_y(x),\ f_x(y))}$$

$$f(x_1/x_1) = \frac{f_{x_2}(x_1)}{\max(f_{x_2}(x_1),\ f_{x_1}(x_2))} = \frac{0.7}{\max(0.7,\ 0.5)} = 1$$

$$f(x_2/x_3) = \frac{f_{x_3}(x_2)}{\max(f_{x_3}(x_2),\ f_{x_2}(x_3))} = \frac{0.3}{\max(0.3,\ 0.8)} = \frac{3}{8} = 0.375$$

We get the relativity matrix:

$$C = \begin{array}{c|cccc}
 & x_1 & x_2 & x_3 & x_4 \\
\hline
x_1 & 1 & 1 & 1 & 1 \\
x_2 & 0.71 & 1 & 0.38 & 0.11 \\
x_3 & 0.6 & 1 & 1 & 0.43 \\
x_4 & 0.67 & 1 & 1 & 1
\end{array}$$

To determine the overall ranking, we need to find the smallest value in each of the rows of the C matrix.

X1: 1 X2: 0.11 X3:0.43 X4: 0.67 → The order will be: X1 > X4 > X3 > X2

Preference and Consensus

Preference

First, we need to define a reciprocal relation as a fuzzy relation:

$r_{ii} = 0,\ \text{for } 1 \leq i \leq n.$ I can't prefer something over itself (لا يمكن تفضيل الشيء على نفسه)

$r_{ij} + r_{ji} = 1,\ \text{for } i \neq j.$ I prefer (x) over (y) by 30%. Then I prefer (y) over (x) by 70%

Two common measures of preference are defined here as average fuzziness and average certainty:

Average Fuzziness:

$$F(\tilde{R}) = \frac{\mathrm{tr}(\tilde{R}^2)}{n(n-1)/2}.$$

Average Certainty:

$$C(\tilde{R}) = \frac{\mathrm{tr}(\tilde{R}\tilde{R}^{\mathrm{T}})}{n(n-1)/2}.$$

$\mathrm{tr}()$ and $()^{\mathrm{T}}$ denote the trace and transpose, respectively

$$\text{Trace} = \mathrm{tr}\left(\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix}\right) = 1+5+9 = 15 = \text{ Sum of diagonal}$$

$$A = \begin{bmatrix}a & b & c\\ d & e & f\end{bmatrix}_{2\times 3}, \quad A^{\mathrm{T}} = \begin{bmatrix}a & d\\ b & e\\ c & f\end{bmatrix}_{3\times 2}$$

OR C(R) =1-FR

Because:

$$F(\underset{\sim}{R}) + C(\underset{\sim}{R}) = 1.$$

Consensus

Three types of consensus:

Type 1: One clear choice || The remaining alternatives have equal secondary preference

$$M_1^* = \begin{array}{c c} & \begin{array}{cccc} A_1, & A_2, & A_3, & A_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} 0 & 0 & 0.5 & 0.5 \\ 1 & 0 & 1 & 1 \\ 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \end{bmatrix} \end{array}$$

I prefer Alternative 2 (A2) over A1,A3 and A4. The other alternatives are equally preferred over each other (50%) for each.
A2 > (A1,A3,A4) & (A1=A3=A4)

Type 2: One clear choice || The remaining alternatives have definite secondary preference.

$$M_2^* = \begin{array}{c c} & \begin{array}{cccc} A_1, & A_2, & A_3, & A_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

I prefer Alternative 2 (A2) over A1,A3 and A4. But I can't rank the other alternatives:
A2>(A1,A3,A,4)
But!
A3>A1
A3>A4
A4>A1 ??? That's why I can't rank other alternatives

Type 3: ***(Type Fuzzy)*** Unanimous decision for the most preferred choice, but the remaining alternatives have infinitely many fuzzy secondary preferences.

$$M_f^* = \begin{array}{c c} & \begin{array}{cccc} A_1, & A_2, & A_3, & A_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} 0 & 0 & 0.5 & 0.6 \\ 1 & 0 & 1 & 1 \\ 0.5 & 0 & 0 & 0.3 \\ 0.4 & 0 & 0.7 & 0 \end{bmatrix} \end{array}$$

A2>(A1,A3,A4)
But!
A3>A1 by 50% and A3>A4 by 30%
But!
A4>A3 by 70% باختصار: يختلف عن النوع الثاني بان التفضيل يكون بنسب متفاوتة للخيارات الأخرى

The cardinalities of these matrices are the following:

$$\left|\mathrm{M}_1^*\right| = n \qquad \text{(Type I)}$$
$$\left|\mathrm{M}_2^*\right| = \left(2^{(n^2-3n+2)/2}\right)(n) \qquad \text{(Type II)}$$
$$\left|\mathrm{M}_f^*\right| = \infty \qquad \text{(Type } \textit{fuzzy}\text{)}$$

Distance to consensus metric can be defined as follows:

$\mathrm{m}(\tilde{R}) = 1 - \sqrt{\frac{2}{n}}$ for a Type I (M_1^*) consensus relation

$\mathrm{m}(\tilde{R}) = 0$ for a Type II (M_2^*) consensus relation.

- Question: when does $\mathrm{m}(M_1^*)$ equal $\mathrm{m}(M_2^*)$?

Answer: when the cardinality (number of alternatives) = 2

Example:

	A_1	A_2
A_1	0	0
A_2	1	0

Example with steps:

The Environmental Protection Agency (EPA) is faced with the challenge of cleaning up contaminated groundwater at many sites around the country. In order to ensure an efficient cleanup process, it is crucial to select a firm that offers the best remediation technology at a reasonable cost.

The EPA is deciding among four environmental firms. The professional engineers at the EPA compared the four firms and created a consensus matrix, shown below:

$$\tilde{R} = \begin{bmatrix} 0 & 0.5 & 0.8 & 0.4 \\ 0.5 & 0 & 0.9 & 0.2 \\ 0.2 & 0.1 & 0 & 0.1 \\ 0.6 & 0.8 & 0.9 & 0 \end{bmatrix}$$

What is the distance to M_1* consensus?

Solution:

Step 1: Find RxR (R^2):

$$R^2 = \begin{bmatrix} 0.65 & 0.4 & 0.81 & 0.18 \\ 0.3 & 0.5 & 0.58 & 0.29 \\ 0.11 & 0.18 & 0.34 & 0.1 \\ 0.58 & 0.39 & 1.2 & 0.49 \end{bmatrix}$$

Step 2: Calculate trace of R^2 (sum of diagonal):

$$tr(R^2) = 0.65 + 0.5 + 0.34 + 0.49 = 1.98$$

Step3: Calculate F(R):

$$F(R) = \frac{tr(R^2)}{n(n-1)/2} = \frac{1.98}{4(4-1)/2} = 0.33$$

Step 4: Calculate C(R):

$$C(R) = 1 - F(R) = 1 - 0.33 = 0.67$$

Step 5: Calculate m(R):

$$\begin{aligned} m(R) &= 1 - (2C(R) - 1)^{\frac{1}{2}} \\ &= 1 - (2 \cdot 0.67 - 1)^{\frac{1}{2}} = 0.417 \end{aligned}$$

Step 6: Calculate M1*:

$$\text{For } M_1^* : \quad m = 1 - \left(\frac{2}{n}\right)^{\frac{1}{2}} = 1 - \left(\frac{2}{4}\right)^{\frac{1}{2}} = 0.293$$

Thus, Distance from M_1^* consensus is $0.417 - 0.293 = 0.124$

Multi-Objective Decision Making

Decisions are, more often than not, made in an environment where many objectives need to be considered.

Two primary issues in multi-objective decision making are to acquire meaningful information regarding the satisfaction of the objectives by the various choices and to rank the relative importance of each of the objectives.

To evaluate the alternatives, the objectives are usually combined. This process requires subjective information from the decision authority concerning the importance of each objective.

The grade of membership that the decision function, D, has for each alternative a is given as:

$$\mu_D(a) = \min[\mu_{O_1}(a),\ \mu_{O_2}(a),\ \ldots,\ \mu_{O_r}(a)]$$

The optimum decision will then be the alternative that satisfies:

$$\mu_D(a^*) = \max_{a \in A}(\mu_D(a))$$

When each objective is associated with a weight expressing its importance, the decision can be given as follows:

$$D = \mathrm{M}(O_1, b_1) \cap \mathrm{M}(O_2, b_2) \cap \cdots \cap \mathrm{M}(O_r, b_r)$$

Implication is the operation that relates the objective and its importance:

$$\mathrm{M}(O_i(a), b_i) = b_i \longrightarrow O_i(a) = \overline{b_i} \vee O_i(a)$$

$$C_i = \overline{b_i} \cup O_i$$

$$\mu_{C_i}(a) = \max[\mu_{\overline{b_i}}(a),\ \mu_{O_i}(a)] \qquad D = \bigcap_{i=1}^{r}(\overline{b_i} \cup O_i)$$

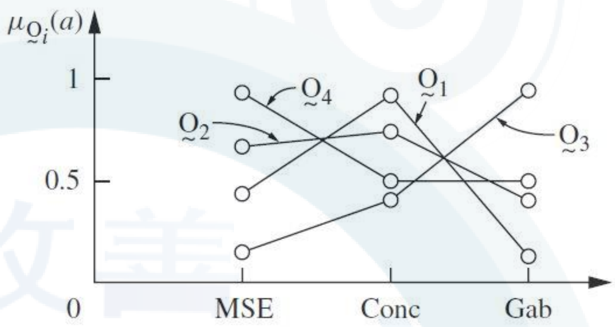
Example with Steps:

Alternatives:
$A = \{\text{MSE},\ \text{Conc},\ \text{Gab}\} = \{a_1, a_2, a_3\}.$

Objectives:
$O = \{\text{Cost},\ \text{Main},\ \text{SD},\ \text{Env}\} = \{O_1, O_2, O_3, O_4\}.$

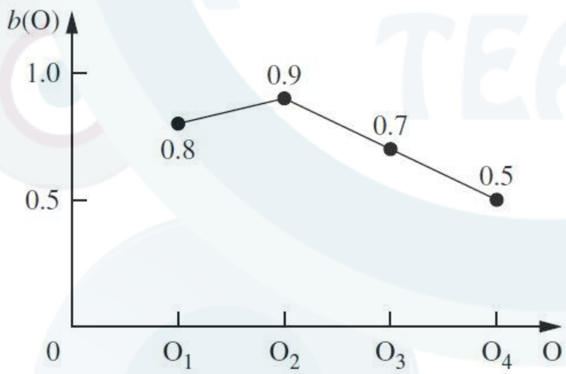
Preferences (the weight for each objective):
$P = \{b_1, b_2, b_3, b_4\} \rightarrow [0,\ 1].$

$$\tilde{O}_1 = \left\{\frac{0.4}{\text{MSE}} + \frac{1}{\text{Conc}} + \frac{0.1}{\text{Gab}}\right\}$$
$$\tilde{O}_2 = \left\{\frac{0.7}{\text{MSE}} + \frac{0.8}{\text{Conc}} + \frac{0.4}{\text{Gab}}\right\}$$
$$\tilde{O}_3 = \left\{\frac{0.2}{\text{MSE}} + \frac{0.4}{\text{Conc}} + \frac{1}{\text{Gab}}\right\}$$
$$\tilde{O}_4 = \left\{\frac{1}{\text{MSE}} + \frac{0.5}{\text{Conc}} + \frac{0.5}{\text{Gab}}\right\}$$



The engineer wishes to investigate two decision scenarios. Each scenario propagates a different set of preferences from the owner, who wishes to determine the sensitivity of the optimum solutions to the preference ratings. In the first scenario, the owner lists the preferences for each of the four objectives:

First Scenario:



Solution:

Step 1: Calculate the negation for all the "bs"

The negation equals (1-b), eg. for b1 → 1-0.8 = 0.2

$$\tilde{O}_1 = \left\{ \frac{0.4}{\text{MSE}} + \frac{1}{\text{Conc}} + \frac{0.1}{\text{Gab}} \right\}$$
$$\tilde{O}_2 = \left\{ \frac{0.7}{\text{MSE}} + \frac{0.8}{\text{Conc}} + \frac{0.4}{\text{Gab}} \right\}$$
$$\tilde{O}_3 = \left\{ \frac{0.2}{\text{MSE}} + \frac{0.4}{\text{Conc}} + \frac{1}{\text{Gab}} \right\}$$
$$\tilde{O}_4 = \left\{ \frac{1}{\text{MSE}} + \frac{0.5}{\text{Conc}} + \frac{0.5}{\text{Gab}} \right\}$$

$$\overline{b_1} = 0.2$$
$$\overline{b_2} = 0.1$$
$$\overline{b_3} = 0.3$$
$$\overline{b_4} = 0.5$$

Step 2: Calculate:

$$\begin{aligned} D(a_1) = D(\text{MSE}) &= (\overline{b_1} \cup O_1) \cap (\overline{b_2} \cup O_2) \cap (\overline{b_3} \cup O_3) \cap (\overline{b_4} \cup O_4) \\ &= (0.2 \vee 0.4) \wedge (0.1 \vee 0.7) \wedge (0.3 \vee 0.2) \wedge (0.5 \vee 1) \\ &= 0.4 \wedge 0.7 \wedge 0.3 \wedge 1 = 0.3 \end{aligned}$$

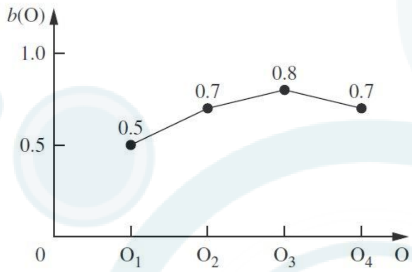
$$\begin{aligned} D(a_2) = D(\text{Conc}) &= (0.2 \vee 1) \wedge (0.1 \vee 0.8) \wedge (0.3 \vee 0.4) \wedge (0.5 \vee 0.5) \\ &= 1 \wedge 0.8 \wedge 0.4 \wedge 0.5 = 0.4 \end{aligned}$$

$$\begin{aligned} D(a_3) = D(\text{Gab}) &= (0.2 \vee 0.1) \wedge (0.1 \vee 0.4) \wedge (0.3 \vee 1) \wedge (0.5 \vee 0.5) \\ &= 0.2 \wedge 0.4 \wedge 1 \wedge 0.5 = 0.2 \end{aligned}$$

Step3: Calculate the Maximum:

$$D^* = \max\{D(a_1), D(a_2), D(a_3)\} = \max\{0.3, 0.4, 0.2\} = 0.4$$

Second Scenario



$$\tilde{O}_1 = \left\{ \frac{0.4}{\text{MSE}} + \frac{1}{\text{Conc}} + \frac{0.1}{\text{Gab}} \right\}$$
$$\tilde{O}_2 = \left\{ \frac{0.7}{\text{MSE}} + \frac{0.8}{\text{Conc}} + \frac{0.4}{\text{Gab}} \right\}$$
$$\tilde{O}_3 = \left\{ \frac{0.2}{\text{MSE}} + \frac{0.4}{\text{Conc}} + \frac{1}{\text{Gab}} \right\}$$
$$\tilde{O}_4 = \left\{ \frac{1}{\text{MSE}} + \frac{0.5}{\text{Conc}} + \frac{0.5}{\text{Gab}} \right\}$$

Step 1: Calculate the negation for each b

$\overline{b_1} = 0.5 \quad \overline{b_2} = 0.3 \quad \overline{b_3} = 0.2 \quad \overline{b_4} = 0.3$

Step 2: Calculate

$$\begin{aligned} D(a_1) = D(\text{MSE}) &= (\overline{b_1} \cup O_1) \cap (\overline{b_2} \cup O_2) \cap (\overline{b_3} \cup O_3) \cap (\overline{b_4} \cup O_4) \\ &= (0.5 \vee 0.4) \wedge (0.3 \vee 0.7) \wedge (0.2 \vee 0.2) \wedge (0.3 \vee 1) \\ &= 0.5 \wedge 0.7 \wedge 0.2 \wedge 1 = 0.2 \end{aligned}$$

$$\begin{aligned} D(a_2) = D(\text{Conc}) &= (0.5 \vee 1) \wedge (0.3 \vee 0.8) \wedge (0.2 \vee 0.4) \wedge (0.3 \vee 0.5) \\ &= 1 \wedge 0.8 \wedge 0.4 \wedge 0.5 = 0.4 \end{aligned}$$

$$\begin{aligned} D(a_3) = D(\text{Gab}) &= (0.5 \vee 0.1) \wedge (0.3 \vee 0.4) \wedge (0.2 \vee 1) \wedge (0.3 \vee 0.5) \\ &= 0.5 \wedge 0.4 \wedge 1 \wedge 0.5 = 0.4 \end{aligned}$$

There is a tie between D(a2) and D(a3)
To break the tie, remove the minimum value from both of them and take the second minimum. Repeat if the tie still exists

For a2:
$1 \wedge 0.8 \wedge \cancel{0.4} \wedge 0.5$
= 0.5

For a3:
$0.5 \wedge \cancel{0.4} \wedge 1 \wedge 0.5$
=0.5

The tie still exists. So we repeat

For a2:
$1 \wedge 0.8 \wedge \cancel{0.4} \wedge \cancel{0.5}$
= 0.8

For a3:
$\cancel{0.5} \wedge \cancel{0.4} \wedge 1 \wedge \cancel{0.5}$ = ,we removed both “0,5” because they have the same value
=1

Step3: Calculate the Maximum:

$$D^* = \max\{D(a_1), D(a_2), D(a_3)\} = \max\{0.2, 0.8, 1\} = 1$$