A differential equation (D.E) is an equation with a function and one or more of its derivative

cH1

- The ordinary differential equation is an equation where the function depends on one indepindent variable.

* order ⇒ the highest derivative in the D.E (the Degree)

Ex: $y'' + 2xy' = x^2$ order = 2

$y''' + 2(y'')^2 = e^x$ order = 3

$x\,dx - y\,dy = 0$ order = 1

* Linearity ⇒ the ODE in linear if it is linear in y and its derivatives

Ex: $y''' - 2xy' = \sin x$ ⇒ 3^{rd}-order linear

$y'' - (y')^2 = x$ ⇒ non linear secound order

$y^{(4)} - \sin y = x$ ⇒ non linear ($\sin y$) 4^{th}-order

---

separable ODE is an first-order D.E which can be written as $f(x)\,dx = g(y)\,dy$

Ex: ① $x\cos^2 y\,dx + e^x\,dy = 0$

$$\frac{x\cos^2 y\,dx}{-e^x \cos^2 y} = \frac{-e^x\,dy}{-e^x \cos^2 y}$$

$-xe^{-x}\,dx = \sec^2 y\,dy$ ⇒ separable

②

$$\frac{dy}{dx} = \frac{x\,y\sin x}{y+1} \qquad y(0) = 1$$

$$\int \frac{(y+1)\,dy}{y} = \int x\sin x\,dx$$

x ⊕ sin x

1 ⊖ −cos x

0 −sin x

$y + \ln|y| = -x\cos x + \sin x + C$

$1 + 0 = 0 + 0 + C$

$\boxed{c = 1}$

$\boxed{y + \ln(y) = -x\cos x + \sin x + 1}$

First order linear D.E

if $P(x)$, $f(x)$ are continuous functions of x, then

$y' + P(x)y = f(x)$

in a first order linear D.E.

① The D.E should be written as:

$y' + P(x)y = f(x)$ ---- (*)

② find integrating factor $= e^{\int P(x)dx}$

③ Multiply (*) by $e^{\int P(x)dx}$

④ simplify and integrate.

Ex: $\frac{x\,dy}{x\,dx} + \frac{3y}{x} = \frac{\sin x}{x^2 \cdot x}$

$y' + \frac{3}{x}y = \frac{\sin x}{x^3}$ ---- (*)

$e^{\int \frac{3}{x}dx}$ ⇒ $\underline{\underline{x^3}}$

$\underbrace{x^3 y' + 3yx^2}$ = $\sin x$

$\int (yx^3)' = \int \sin x$ ⇒ $x^3 y = -\cos x + C$

Remark one can consider the following form → $x' + p(y)x = f(y)$

$x' = \frac{dx}{dy}$ ⇒ $e^{\int (P(y))\,dy}$ ①

Bernoulli D.E form: $y' + p(x)y = f(x)\, y^n \quad n \sim R - \{0,1\}$

where $p(x)$ and $f(x)$ are continuous

now let us consider the D.E $(*)$

$$y' + p(x)\, y = f(x)\, y^n \quad \ldots (*) \quad n \neq 0,1$$

this D.E. is <u>non linear</u>

to solve $\Rightarrow$ we assume

$$u = y^{1-n}$$

$$u' = (1-n)\, y^{-n}\, y'$$

multiply $(*)$ by $(1-n)\, y^{-n} \Rightarrow (1-n)\, y^{-n} y' + (1-n)\, y^{-n} p(x)\, y = (1-n)\, y^{-n} f(x)\, y^n$

which is linear !! $\quad u' + (1-n)\, p(x)\, u = (1-n)\, f(x) \ldots (**)$

Ex :- $\dfrac{x^2}{x^2} y' + \dfrac{2xy}{x^2} = \dfrac{y^3}{x^2}$

$$y' + \frac{2}{x} y = \frac{1}{x^2} y^3$$

Bernoulli

$$u = y^{1-3} \Rightarrow y^{-2}$$

$$u' + -2\, p(x)\, u = -2\, f(x)$$

$$u' + -\frac{2 \cdot 2}{x} u = -2\left(\frac{1}{x^2}\right) \ldots (**)$$

Linear

so

first order Linear

$$e^{\int p(x)\,dx} = e^{\int \frac{-4}{x} dx} \Rightarrow \boxed{x^{-4}}$$

$$u' x^{-4} + -4\, x^{-5} u = -2\, x^{-6}$$

$$\int (u \cdot x^{-4})'\,dx = \int -2\, x^{-6}\, dx$$

$$u\, x^{-4} = \frac{-2}{-5} x^{-5} + c$$

$$x^{-4} y^{-2} = \frac{2}{5} x^{-5} + c \quad \#$$

similary one can solve :

$$x' + p(y)\, x = f(y)\, x^n \quad n \neq (0,1)$$

let $u = x^{1-n}$ then the D.E can be reduced to

$$u' + (1-n)\, p(y)\, u = (1-n)\, f(y)$$

$$\frac{dy}{dx} = \frac{y}{x^2 y^3 - x} \Rightarrow \frac{dx}{dy} = \frac{x^2 y^3 - x}{y}$$

$$\frac{dx}{dy} = x^2 y^2 - \frac{x}{y}$$

$$x' + \left(\frac{1}{y}\right) x = x^2 y^2$$

$$u = x^{1-n} \Rightarrow \boxed{x^{-1}}$$

$$u' + (1-n)\, p(y)\, u = (1-n)\, f(y)$$

$$u' + (-1)\frac{1}{y} u = (-1)\, y^2$$

$$e^{\int -\frac{1}{y}} \Rightarrow e^{-\ln y} = y^{-1} = \boxed{\frac{1}{y}}$$

$$\frac{1}{y} u' - \frac{1}{y^2} u = -y$$

$$\int \left(\left(\frac{1}{y}\right)(u)\right)' = \int -y$$

$$\frac{1}{y} \cdot \frac{1}{x} = \frac{-1}{2} y^2 + c \quad \#$$

Homogeneous D.E : The D.E $\Rightarrow y' = f(x,y)$ is called homogeneous if it can be written as $y' = g\left(\frac{y}{x}\right)$

$\rightarrow$ can be solved by reducing it to a separable D.E as follows :-

let $u = \frac{y}{x} \Rightarrow y = ux$

$y' = u'x + u$

the D.E ⊛ becomes :-

$xu' + u = g(u)$

$x\frac{du}{dx} = g(u) - u \Rightarrow \frac{du}{g(u)-u} = \frac{dx}{x}$

(separable)

Ex: $\frac{dy}{dx} = \frac{2x^2 - y^2}{xy + x^2} \begin{matrix} \div x^2 \\ \div x^2 \end{matrix}$. (A)

$\frac{2 - \left(\frac{y^2}{x^2}\right)}{\left(\frac{y}{x}\right) + 1}$ #

Ex:- solve this ODE,

$\frac{dy}{dx} = \frac{xy + y^2 + x^2}{x^2} \begin{matrix} \div x^2 \\ \div x^2 \end{matrix}$

$\frac{dy}{dx} = \frac{y}{x} + \left(\frac{y}{x}\right)^2 + 1$

$u = \frac{y}{x}$ $\quad ux = y \rightarrow y' = u'x + u$

$y' = u + u^2 + 1$

$\downarrow$

$u'x + u = u + u^2 + 1$

$\frac{du}{dx}x = u^2 + 1$

$\frac{du}{u^2+1} = \frac{1}{x}dx$

$\tan^{-1} u = \ln x + c$

$u = \tan(\ln x + c)$ #

$\frac{y}{x} = \tan(\ln x + c)$

$y = x\tan(\ln x + c)$ #

(B) Example of $(f(ax + by + c))$

$\frac{dy}{dx} = f(ax + by + c)$

Ex:- $\frac{dy}{dx} = (x + y + 2)^2$

let $u = x + y + 2$

$u' = 1 + y'$

$y' = 1 + u'$

substitute is ⊛

$\underset{②\ \text{بدل}}{-1 + u'} = \underset{①\ \text{بدل}}{u^2}$

$\frac{du}{dx} = u^2 + 1$

$\frac{du}{u^2+1} = dx \Rightarrow \tan^{-1} u = x + c$ #

Integrating factor

suppose the D.E

$M(x,y)dx + N(x,y)dy = 0$

$\frac{dM}{dy} = \frac{dN}{dx} \Rightarrow$ exact ✓

or

$\frac{dM}{dy} \neq \frac{dN}{dx}$

not exact ✗

$\Leftarrow$ So

$\mu = e^{\int R(y)\,dy}$ $\qquad R(y) = \frac{\frac{dM}{dy} - \frac{dN}{dx}}{-M}$

$\mu = e^{\int R(x)\,dx}$ $\qquad R(x) = \frac{\frac{dM}{dy} - \frac{dN}{dx}}{N}$

$(\mu(M(x,y)))\,dx + \mu(N(x,y))\,dy = 0 \Rightarrow$ exact ✓ $\Rightarrow$

How to solve $\Rightarrow$ Ex: solve (long way)

(تقليدي)

$\underbrace{2xy+y^3+\cos x}_{M}\,dx + \underbrace{x^2+3y^2x}_{N}\,dy = 0$

$\frac{dM}{dy} = 2x+3y^2$

$\frac{dN}{dx} = 2x+3y^2$

the solution is

$u(x,y) = c$

① $\left[\frac{du}{dx} = M\right] \Rightarrow \int \frac{du}{dx}dx = \int M\,dx$

$u = \int M\,dx$

$u = \int 2xy+y^3+\cos x\,dx$

$u(x,y) = x^2y + \sin x + f(y) + xy^3 \Rightarrow \frac{du}{dy} = x^2+3y^2x+f'(y)$

$\frac{du}{dy} = N$

$\underbrace{x^2+3y^2x+f'(y)}_{\frac{du}{dy}} = \underbrace{x^2+3y^2x}_{N}$

$f'(y) = 0$

the solution is :- $\Leftarrow$ $f(y) = 0 + c$

$u(x,y) = c$

$x^2y+y^3x+\sin x + c = c$

$x^2y+y^3x+\sin x = C$

Ex :- $[2x\cos y + 3x^2y]dx + [x^3 - x^2\sin y - y]dy = 0$ (short way)

(دوائر)

$\frac{dM}{dy} = -2x\sin y + 3x^2$

$=$ exact

$\frac{dN}{dx} = 3x^2 - 2x\sin y$

$\int (2x\cos y + 3x^2y)dx$

$x^2\cos y + x^3y$

$\int (x^3 - x^2\sin y - y)dy$

$x^3y + x^2\cos y - \frac{y^2}{2}$

نأخذ واحدة من المتشابه

$x^3y + x^2\cos y - \frac{y^2}{2} = C$

$(2y-6x)dx + (3x-4x^2y^{-1})dy = 0$ if $M(x,y) = xy^n$ is integrating factor find n

$\underbrace{[2xy^{n+1} - 6x^2y^n]}_{M}dx + \underbrace{[3x^2y^n - 4x^3y^{n-1}]}_{N}dy = 0$

$\frac{dM}{dy} = \frac{dN}{dx} \Rightarrow 2(n+1)xy^n - 6nx^2y^{n-1} = 6xy^n - 12x^2y^{n-1}$

$2(n+1) = 6$

$n+1 = 3$

$\boxed{n=2}$

$-6n = -12 = \boxed{n=2}$

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Secound order ODEs

if
$\underline{\underline{I}} \Rightarrow y'' + P(x)y' + q(x)y = f(x)$ --- ① { $q(x), f(x)$ continuous

$f(x) = 0$ the D.E ① is called homogeneous otherwaise it's called nonhomogeneous.

consider

① $y'' + p(x)y' + q(x)y = 0$ --- ② (if y_1, y_2 are two solutions of ② on I

let y_1, y_2 be two solutions of ② then $y = c_1 y_1 + c_2 y_2$ (c_1, c_2 constant) is also a solution on I. called the general solution

$y_1 = c y_2 \quad \left(\frac{y_1}{y_2} = c\right) \Rightarrow y_1, y_2$ are dependent (one solution)

$y_1 \neq c y_2 \quad \left(\frac{y_1}{y_2} \neq c\right) \Rightarrow y_1, y_2$ are independent (two solution)

$y_1 = c y_2 \rightarrow w(y_1, y_2) = 0$	dependent
$y_1 \neq c y_2 \rightarrow w(y_1, y_2) \neq 0$	independent

wranskian $\Rightarrow W(y_1, y_2)$
رانسكين

$$w(y_1, y_2) = y_1 \cdot y_2' - y_2 \cdot y_1'$$

Ex:- $y_1 = x^2$
$y_2 = 3x^2 \qquad \frac{y_1}{y_2} = \frac{x^2}{3x^2} = \frac{1}{3} = \text{constant}$

$$w(x^2, 3x^2) = x^2 \cdot 6x - 3x^2 \cdot 2x = 0$$

abel's theorem $\underline{\underline{1}}$ $y'' + p(x)y' + f(x)y = 0$

$$w(y_1, y_2)(x) = c e^{-\int p(x)\,dx}$$ ex $\Rightarrow$

Ex:- $y_1, y_2 \Rightarrow$ linearly independent solution of:-

$$t y'' + 2y' + t e^t y = 0 \quad \underline{\text{and}} \quad W(y_1, y_2)(2) = 3$$

find $W(y_1, y_2)(5)$

solution:-

1. $$y'' + \underbrace{\frac{2}{t}}_{p(x)} y' + e^t y = 0$$

$$e^{-\int p(x)\,dx} = e^{-\int \frac{2}{t}\,dt} \Rightarrow t^{-2} = \boxed{\frac{C}{t^2}} = W(y_1, y_2)(x)$$

then

$W(y_1, y_2)(2) = 3$

$$\frac{C}{4} = 3 \Rightarrow \boxed{C = 12}$$

so

$W(y_1, y_2)(5) = ??$

$$\frac{12}{25} = W(y_1, y_2)(5)$$

$W(f, g) = 2$

find $W(f, f + 2g)$

$W(f, f) + W(f, 2g)$

$W(f, f) + 2W(f, g)$

$0 + 2(2) = \boxed{4}$

* ملاحظة :- 1. $-W(y_2, y_1) = W(y_1, y_2)$

2. $W(y, y_1) = 0 \rightarrow$ Dependent $\rightarrow$ so $W(y_1, y_2) = 0$

$\frac{y_1}{y_2} =$ constant.

Reduction of order :-

$$y'' + p(x) y' + q(x) y = 0 \quad \text{--- (2)}$$

Given y_1 is a solution of (2)

then we can find a 2nd linearly independent solution y_2 using :-

$$y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{(y_1)^2}\,dx$$

Ex:- $x y'' + 2y' + xy = 0$

1. $\Rightarrow y'' + \frac{2}{x} y' + y = 0$

$y_1 = \frac{\cos x}{x}$ find the 2nd solution

2. $e^{-\int p(x)\,dx} \Rightarrow x^{-2}$

3. $$y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx$$

$$\Rightarrow \boxed{\frac{\sin x}{x}}$$

$y'' + ay' + by = 0$ --- (*)

بالبداية نحلل المعادلة التربيعية ونوجد r
ثم نوجد y

$y = e^{rx} \Rightarrow$ solution

عندما يكون هناك constant coefficiant

Ex: solve $y'' + y' - 2y = 0 \Rightarrow r^2 + r - 2 = 0$

$(r+2)(r-1)$

$r = -2 \Rightarrow y = e^{-2x}$

$r = 1 \Rightarrow y = e^{x}$

$y = e^{-2x}c_1 + e^{x}c_2$

Ex: solve $y'' + 4y' + 2y = 0 \Rightarrow r^2 + 4r + 2 = 0$

$r = -2 \pm \sqrt{2}$

$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$y_1 = e^{(-2-\sqrt{2})x}$

$y_2 = e^{(-2+\sqrt{2})x} \Rightarrow y = c_1 e^{(-2-\sqrt{2})x} + c_2 e^{(-2+\sqrt{2})x}$ *

* ملاحظة إذا كانت $r_2 = r_1$ إذن نضرب r_2 ب x لانه ممنوع تكرار الحل

$y_1 = e^{rx}$

$y_2 = xe^{rx}$

Ex: solve $\boxed{y'' + 4y + 4 = 0}$

$r^2 + 4r + 4$

$(r+2)(r+2) = 0$

$r_1 = -2$, $r_2 = -2$

$y = (c_1 + c_2 x)e^{-2x}$

y-missing and x-missing

non linear

$y'' = f(x, y')$ (y-missing)

$u = y' \rightarrow u' = y''$

$u' = f(x, u)$

$\frac{du}{dx} = f(x, u)$

or

$y'' = f(y, y')$ (x-missing)

$u = y' \rightarrow u' = y''$

$u' = f(y, u)$

$\frac{du}{dx} = f(y, u)$

X

so

$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}$ chain rule

$u' = u\frac{du}{dy}$

$u\frac{du}{dy} = f(y, u)$

Ex :- $y'' + \frac{2}{x} y' = \frac{1}{x^2} \Rightarrow$ y - missed

$u = y'$

$u' = y''$

$$u' + \frac{2}{x} u = \frac{1}{x^2}$$

$$\frac{du}{dx} + \frac{2}{x} u = \frac{1}{x^2} \quad ---- (*) \quad \text{non homo (so)}$$

$$e^{\int p(x) dx} = e^{\int \frac{2}{x} dx} = x^2$$

$$x^2 u' + 2x u = 1$$

$$\int (x^2 u)' dx = \int 1 \, dx$$

$$\frac{\cancel{x^2} u}{\cancel{x^2}} = \frac{x + c_1}{x^2} \Rightarrow \frac{dy}{dx} = \frac{x + c_1}{x^2}$$

$$dy = \frac{x + c_1}{x^2} dx$$

$$y = \ln|x| - \frac{c_1}{x} + c_2$$

Complex Numbers

Ex : solve $r^2 + 2r + 2 = 0$

$(a = 1 \quad B = 2 \quad c = 2)$

مراجعة

$\sqrt{-1} = i$

$\sqrt{-7} = 7i$

$i^2 = -1$

$$b^2 - 4ac = 4 - 4 \cdot 1 \cdot 2 = -4$$

$$\frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = \boxed{-1 \pm i}$$

$a + bi$

اشارة α مسموحة

اشارة B غير مهمة

complex roots :-

$$\boxed{r = \lambda \pm \mu i}$$

$$y_1 = e^{(\lambda + \mu i)x} \qquad y_2 = e^{(\lambda - \mu i)x}$$

الحل المستخدم

$$y_1 = e^{\lambda x} \cos \mu x \qquad y_2 = e^{\lambda x} \sin \mu x$$

solving

E solve Ex :- $y'' + 2y' + 5y = 0 \Rightarrow r^2 + 2r + 5 = 0$

$a=1 \quad B=2 \quad c=5$

$r = -1 \pm 2i$

$\frac{-2 \pm \sqrt{4 - 4 \cdot 5}}{2} \Rightarrow \boxed{-1 \pm 2i}$

$y_1 = e^{-x} \cos 2x$

$y_2 = e^{-x} \sin 2x$

General solution $\Rightarrow$

$y = c_1 e^{-x} \cos 2x + c_2 e^{-x} \sin 2x$

$(r - r_1)(r - r_2) = 0$

* ملاحظة لنجعل r_1 , r_2 حلول المعادلة

then

$r^2 - (r_1 + r_2) r + r_1 \cdot r_2 = 0$

Ex :- find the 2^{ed}-order linear homogeneous D.E whose solution is :-

$y = e^{2x} \cos 3x + c_2 e^{2x} \sin 3x$

complex

$r^2 - (r_1 + r_2) r + r_1 \cdot r_2 = 0$

$r = 2 \pm 3i$

$r^2 - (4) r + 13 \Rightarrow y'' - 4y' + 13y = 0$

$r_1 = 2 + 3i$

$r_2 = 2 - 3i$

$(2+3i)(2-3i)$

$4 - (3i)^2$

$4 + 9$

Non homogeneous 2^{nd}-order linear ODEs :-

$y'' + p(x) y' + q(x) y = r(x)$ --- ①

$r(x) \neq 0$

$y'' + p(x) y' + q(x) y = 0$ --- ②

General solution of ① is given by

$y = y_h + y_p$

r(x) could by → poly → e^x → sinx or cosx

*ملاحظة نمط السؤال التالي هو solve لكن بالعادة يأتي find the suitable form

Ex :- $y'' + 4y = x^2 + 12$

$r^2 + 4 = 0$

$r = \pm 2i$

$y_1 = \cos 2x$

$y_2 = \sin 2x$

$y_h = c_1 \cos 2x + c_2 \sin 2x$

$y_p = Ax^2 + Bx + C$

$y'_p = 2Ax + B$

$y''_p = 2A \xrightarrow{\text{نعوض}}$

$2A + 4(Ax^2 + Bx + C) = x^2 + 12$

نجرب تعويض ارقام بـ x نحتاج 3 معادلات لانه يوجد 3 متغيرات

$A = \frac{1}{4}$

$B = 0$

$C = \frac{23}{8}$

$y_p = \frac{1}{4} x^2 + \frac{23}{8} + c_1 \cos 2x + c_2 \sin 2x$

Write a suitable form → $y_{p(x)}$:-

① $y''-2y' = xe^{x}+2e^{2x}$

① ← نوجد y_h ⇐ $y''-2y'=0$

$r^2-2r=0$

$r(r-2)=0$

$r=0 \to e^{0x} \to 1 = y_1$

$r=2 \to e^{2x} \to e^{2x} = y_2$

$y_h = c_1 + e^{2x}c_2$

② نوجد y_p ⇐ $(Ax+B)e^{x}$ + Ce^{2x}

③ نقارن بين ال y_1, y_2 و صيغة ال y_p إذا وجد تشابه نضرب ب X

④ $y_p = (Ax+B)e^{x} + \underline{\underline{X}}Ce^{2x}$

⑤ نجمع $y = y_p + y_h$

$$\boxed{y = c_1 + e^{2x}c_2 + xce^{2x} + (Ax+B)e^{x}}$$

② $\cancel{y''+2y'=2x+6}$

$y''+2y'+2y = 2xe^{-x}\sin x+1$

اذا وجد في طرف $y_{p(x)}$

sin او cos ← نضيف الآخر الغير موجود

sin ← cos

cos ← sin

$y''+2y'+2y=0$

$r^2+2r+2=0$

$\frac{-b\pm\sqrt{b^2-4ac}}{2a} \Rightarrow \overset{\lambda}{-1} \pm \overset{\mu}{i}$

$y_1 = e^{-x}\cos x$

$y_2 = e^{-x}\sin x$

$y_{h(x)} = e^{-x}\cos x \cdot c_1 + c_2 e^{-x}\sin x$

$y_{p(x)} \to (Ax+B)e^{-x}\sin x\,\underline{\underline{X}} \cancel{+(Cx+D)e^{-x}\cos x}$

$+ Cx+De^{-x}\cos x\,\underline{\underline{X}}$

$+ E$

$y = xe^{-x}(Ax+B)\sin x + xe^{-x}(Cx+D)\cos x + e^{-x}\cos x(c_1) + (c_2)e^{-x}\sin x + E.$

Euler - Cauchy Equations:-

$$x^2 y'' + a x y' + b y = 0 \qquad x > 0 \ \cdots \ (*)$$

the solution here is $y = x^r$

- $r_1 \neq r_2$ — $y_1 = x^{r_1}$, $y_2 = x^{r_2}$
- $r_1 = r_2$ — $y_1 = x^r$, $y_2 = (\ln x)\, x^r$
- complex Roots $\lambda \pm \mu i$ → $y_1 = x^\lambda \cos(\mu \ln x)$, $y_2 = x^\lambda \sin(\mu \ln x)$

Ex:- $2x^2 y'' + 3 x y' - y = 0$

$$\downarrow$$

$$2 r(r-1) + 3r - 1 = 0$$

$$2r^2 + r - 1 = 0$$

$$(2r-1)(r+1)$$

$r = \frac{1}{2} \Rightarrow y = \sqrt{x}$

$r = -1 \Rightarrow y = \frac{1}{x}$

the General solution is :- $y = c_1 y_1 + c_2 y_2$

$$c_1 \sqrt{x} + c_2 \frac{1}{x}$$

$$x > 0$$

Ex:- $x^2 y'' - 5 x y' + 13 y = 0$

$$\downarrow$$

$$r(r-1) - 5r + 13 = 0$$

$$r^2 - 6r + 13 = 0$$

$a = 1$
$b = -6$
$c = 13$

$b^2 - 4ac$
$36 - 4(1)(13)$
$= -16$

$$\frac{-6 \pm \sqrt{-16}}{2}$$

$$\boxed{= 3 \pm 2i}$$

$\lambda = 3$ $\quad$ $\mu = 2$

$y_1 = x^3 \cos(2 \ln x)$

$y_2 = x^3 \sin(2 \ln x)$

the General solution is $\Rightarrow y = c_1 x^3 \cos(2\ln x) + c_2 x^3 \sin(2 \ln x)$

Ex:- if $\{ x^2 \cos(\ln x), x^2 \sin(\ln x) \}$ is a fundamental set of solution of $x^2 y'' + a x y' + b y = 0$ find (a, B)

$\lambda = 2 \rightarrow$
$\mu = 1 \rightarrow$

$$2 \pm i \xrightarrow[\text{ال } r]{\text{طالعين}} r^2 \mp 4r + 5 = 0$$

$r_1 = 2+i$
$r_2 = 2-i$

$4 - (+1) = 5$

$$r(r-1) - 3r + 5 = 0$$

$$x y'' - 3 x y' + 5 y = 0$$

$$\boxed{a = -3} \ \boxed{B = 5}$$

Variaten of Parameters :-

cH2

consider $\Rightarrow$ $y'' + p(x)\, y' + q(x)\, y = r(x)$ — (*) , $r(x) \neq 0$

when $p(x)$, $q(x)$ are continuous function of same open interval (I)

① let y_1, y_2 be independent solution of

$$y'' + p(x)\, y' + q(x)\, y = 0 \quad --- ②$$

$$y_{h(x)} = c_1 y_1 + c_2 y_2$$

نوجد الـ homo بالبداية

ثم

② we can find $y_{p(x)}$ for ① as follows :- حفظ

$$y_{p(x)} = -y_1 \int \frac{y_2}{W} r + y_2 \int \frac{y_1}{W} r$$

ملاحظة :- بالعادة تكون الإجابات موجودة دون حل التكامل.

Ex :- solve $y'' + y = \sec x$

① نوجد $y_{h(x)}$

② نوجد الرونسكين (W)

$y_{h(x)} = c_1 \sin x + c_2 \cos x$ } $r^2 + 1 = 0$

$r = \pm i$

$y_1 = \sin x$

$y_2 = \cos x$

* بالعادة عند اختيار y_1 و y_2 نعطي اقل قيمة لـ y_1 والقيمة الأكبر لـ y_2

$$W(y_1, y_2) = \sin x\,(-\sin x) - \cos x \cdot \cos x$$

$$-(\sin^2 x + \cos^2 x)$$

$$= \boxed{-1}$$

$r(x) = \sec x$

$$y_{p(x)} = -\sin \int \frac{\cos x}{-1} \cdot \sec x\, dx + \cos x \int \frac{\sin x}{-1} \sec x \, dx$$

$$y_{p(x)} = x \sin x + \cos x \ln(\cos x)$$

$$y = y_h + y_p \Rightarrow c_1 \sin x + c_2 \cos x + x \sin x + \cos x \ln|\cos x|$$

Revision بالقسمة التركيبية

نفس فكرة CH2 بس r^3

Ex:- r^3-2r^2-5r+6

الاختصار معامله مشترك

معاملات ال 6 ، $\pm 1 \quad \pm 2 \pm 3 \pm 6$

نبدأ ب الاصغر

نجرب $r=1$

$1-2-5+6 = 0$ ✓

	معاملات r			
$r=1$	1	-2	-5	6
اول عدد ينزل ×	↓	1	-1	-6
	1	-1	-6	0

لازم تكون 0

$= (r-1)(r^2-r-6)$

$= (r-1)(r-3)(r+2)$

$r = 1, 3, -2$

$y_1 = e^x \quad y_3 = e^{-2x}$

$y_2 = e^{3x}$

$y_{h(x)} = c_1e^x + c_2e^{-2x} + c_3e^{3x}$

solve → non homogeneous CH3

$y''' + y'' = 2x+1$

① نحل ال homo

$r^3 + r^2 = 0$

$r^2(r+1) = 0$

$r=0 \to 1$

$r=0 \to x$

$r=-1 \Rightarrow e^{-x}$

$y_{h(x)} = c_1 + c_2x + c_3e^{-x}$

② نحل ال nonhomo ونقارن

$2x+1$

طلع

$y_{p(x)} = (Ax+B)\,\underline{x^2}$

③ اذا طلب باقي الحل نعوض ← بالمعادلة ولذلك نوجد المشتقات الثلاث

$y'_{p(x)}$

$y''_{p(x)}$

$y'''_{p(x)}$

④ $y = y_p + y_h$

Variation of parameters

$y''' + P_2(x)y'' + P_1(x)y' + P_0(x)y = r(x)$ ← not sinx/cosx, not e^x, not poly, then

$y = y_h + y_p$

$$y_{p(x)} = y_1\int\frac{w_1}{w}r + y_2\int\frac{w_2}{w}r + y_3\int\frac{w_3}{w}r$$

y_1, y_2, y_3 are independent solutions of the homogeneous part.

How to find w ⇒ $w(y_1, y_2, y_3) \xrightarrow{\text{matrix}} \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$

$\Rightarrow y_1(y_2' * y_3'' - y_3' y_2'') - y_2(y_1' * y_3'' - y_3' * y_1'') + y_3(y_1' * y_2'' - y_2' * y_1'')$

$W_1\ (y_2, y_3)$

$W_2\ (y_3, y_1)$

$W_3\ (y_1, y_2)$

وانسكن عادي نحسب

أو

$w_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ 1 & y_2'' & y_3'' \end{vmatrix}$

$w_2 = \begin{vmatrix} " & 0 & " \\ " & 0 & " \\ " & 1 & " \end{vmatrix}$

$w_3 = \begin{vmatrix} " & " & 0 \\ " & " & 0 \\ " & " & 1 \end{vmatrix}$

Ex:- $x^3y''' - 3x^2y'' + 6xy' - 6y = 24x^4$, $x>0$

① $x^3y''' - 3x^2y'' + 6xy' - 6y = 0$

$y_1 = x$ $\quad y_3 = x^3$

$y_2 = x^2$ $\Rightarrow y_{h(x)} = c_1x + c_2x^2 + c_3x^3$

②
$W(x, x^2, x^3) = 2x^3$

③
$w_1 = x^4$, $w_2 = -2x^3$,
$w_3 = x^2$

④ $r(x) = \dfrac{24x^4}{x^3} = 24x$

y''' ‏لوحدها‎ ←

⑤
$$y_{p(x)} = x\int \frac{x^4}{2x^3} 24x\,dx + x^2\int \frac{-2x^3}{2x} 24x\,dx + x^3\int \frac{x^2}{2x^3} 24x\,dx$$

⑥
$$y = y_h + y_p$$

$c_1x + c_2x^2 + c_3x^3 + 4x^4$

CH 4 Homogeneous linear system with constant coefficients

$A^{-1} = \dfrac{1}{|A|}\begin{bmatrix} d & -B \\ -c & a \end{bmatrix}$ ‏معكوس‎

linear system

$ay_1 + By_2 = f_1$
$cy_1 + Dy_2 = f_2$ $\Rightarrow \begin{bmatrix} a & B \\ c & D \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$

$\begin{bmatrix} a & B \\ c & d \end{bmatrix} = A$ ‏كان لو‎

Consider the Matrix ($\underline{A}$)

($\underline{X}$ or ξ) called eigen vector

λ called eigen value $\rightarrow$ $\underbrace{\lambda^2 - |A|\lambda + |A|}$ (main diagonal)

$|A - \lambda I| = 0$ $\rightarrow$ to find eigen value

$$\begin{bmatrix} a-\lambda & B \\ c & D-\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Ex:- $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ find λ, $\underline{x}$

‏حالة (1) اختلاف‎ λ

$\begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} = 0$

$\lambda^2 - 2\lambda - 3$

$(\lambda+1)(\lambda-3)$

$(1-\lambda)^2 - 4 = 0$

$\sqrt{(1-\lambda)^2} = \sqrt{4}$

$1-\lambda = \pm 2$

$1-\lambda = 2$ $\quad\quad 1-\lambda = -2$

$\lambda = -1$ $\quad\quad$ ② $\quad \lambda = 3$

$\lambda = 3$ $\longrightarrow$ $\begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} \begin{bmatrix} a \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to find ξ or $\underline{X}$

$\lambda = -1$

$\downarrow$

$\begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} \begin{bmatrix} a \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} a \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$2a + B = 0 \Rightarrow \begin{bmatrix} a \\ -2a \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$4a + 2B = 0$

$\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} a \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-2a + B = 0 \longrightarrow B = 2a$ $\begin{bmatrix} a \\ 2a \end{bmatrix}$

$4a - 2B = 0$ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda = -1 \longleftrightarrow \xi^{(1)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\lambda = 3 \longleftrightarrow \xi^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\underline{y}_1^{(1)} = \xi^{(1)} e^{\lambda_1 t} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$

$\underline{y}^{(2)} = \xi^{(2)} e^{\lambda_2 t} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$

The general solution is $\underline{y} = c_1 \underline{y}^{(1)} + c_2 \underline{y}^{(2)}$

---

② حالة complex λ

Ex :- $y_1' = -y_1 + y_2$
$y_2' = -y_1 - y_2$
$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \xRightarrow[\lambda]{\text{نوجد}} \begin{bmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$(1+\lambda)^2 + 1 = 0 \Rightarrow \sqrt{(1+\lambda)^2} = \sqrt{-1} \Rightarrow 1 + \lambda = \pm i$

$\boxed{\lambda = -1 \pm i} \xrightarrow[\text{حل}]{\text{نأخذ}}$ مثلا $-1 + i$

واحد فقط

$\lambda = -1 + i$

$\begin{bmatrix} \cancel{-1+1}-i & 1 \\ -1 & \cancel{-1+1}-i \end{bmatrix} \begin{bmatrix} a \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$ $-ia + B = 0 \Rightarrow B = ia$ $\begin{bmatrix} a \\ ia \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix} = \xi^*$

$y^{(1)} = e^{(-1+i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} \Rightarrow e^{-t} \cdot e^{it} \begin{bmatrix} 1 \\ i \end{bmatrix} \Rightarrow e^{-t} (\cos t + i \sin t) \begin{bmatrix} 1 \\ i \end{bmatrix}$

$e^{it} \downarrow (\cos t + i \sin t)$

$e^{-t} \begin{bmatrix} \cos t + i \sin t \\ -\sin t + i \cos t \end{bmatrix} \Rightarrow$

③

$$\rightarrow e^{-t}\left(\begin{bmatrix}\cos t\\ -\sin t\end{bmatrix}+i\begin{bmatrix}\sin t\\ \cos t\end{bmatrix}\right)$$ فصلنا القسم i عن الحقيقي

$$\underline{y}_{(1)} = \underbrace{c_1 e^{-t}\begin{bmatrix}\cos t\\ -\sin t\end{bmatrix}}_{\underline{y}^{(1)}} + \underbrace{c_2 e^{-t}\begin{bmatrix}\sin t\\ \cos t\end{bmatrix}}_{\underline{y}^{(2)}}$$

---

③ حالة Equal (λ)

Ex:- $\underline{y}' = \begin{bmatrix}4 & 1\\ -1 & 2\end{bmatrix}\underline{y}$ $\xrightarrow{\text{find } \lambda}$ $\begin{vmatrix}4-\lambda & 1\\ -1 & 2-\lambda\end{vmatrix} = 0 \Rightarrow \lambda^2 - 6\lambda + 9$

$(\lambda-3)(\lambda-3)$

$\lambda = 3, 3$

$\lambda = 3 \Rightarrow \begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}\begin{Bmatrix}\alpha\\ \beta\end{Bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$

$\underline{\xi} = \begin{bmatrix}1\\ -1\end{bmatrix} \Rightarrow y^{(1)}_{(t)} = e^{3t}\begin{bmatrix}1\\ -1\end{bmatrix}$

to find 2^{nd} independent solution

$$y^{(2)}_{(t)} = \underbrace{\underline{\xi}\, t\, e^{3t}}_{\text{الحل الاول مضروب ب } t} + \underline{\eta}\, e^{3t}$$

$$\underline{y}^{(2)}_{(t)} = t e^{3t}\begin{bmatrix}1\\ -1\end{bmatrix} + e^{3t}\begin{bmatrix}0\\ 1\end{bmatrix}$$

$$= \left(t\begin{bmatrix}1\\ -1\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}\right)e^{3t}$$

how to find $\underline{\eta}$ (generalized eigenvector)

$$[A - \lambda I]\,\underline{\eta} = \underline{\xi}$$

$\lambda = 3$

$$\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}\begin{bmatrix}\alpha_2\\ \beta_2\end{bmatrix} = \begin{bmatrix}1\\ -1\end{bmatrix}$$

$\alpha + \beta = 1$

$\alpha = 0$, $\beta = 1$ $\alpha = 1$, $\beta = 0$

$\begin{bmatrix}0\\ 1\end{bmatrix}$ or $\begin{bmatrix}1\\ 0\end{bmatrix}$

$\underline{\eta} = \begin{bmatrix}0\\ 1\end{bmatrix}$ or $\begin{bmatrix}1\\ 0\end{bmatrix}$

Undetermined Coefficients

* نفس التطبيق القديم لكن هنا عند وجود تشابه بين ~~...~~ احد الحلول ال y_h نرفع الثابتة درجة واحدة اما اذا كان y_h مكرر نرفع درجتين

انماط السؤال

→ solve $\xrightarrow{\text{نوجد قيم الثابتة}}$

→ find the suitable form

Ex: $\underline{y}' = \begin{bmatrix}-3 & 1\\ 1 & -3\end{bmatrix}\underline{y} + \begin{bmatrix}-6\\ 2\end{bmatrix}e^{-2t}$

① find the solution y_h

$$y^{(1)} = \begin{bmatrix}1\\ 1\end{bmatrix}e^{-2t}, \quad y^{(2)} = \begin{bmatrix}1\\ -1\end{bmatrix}e^{-4t}$$

$$\underline{y}^{(P)}_{(t)} = (\underline{a}t + \underline{b})\,e^{-2t}$$

$$= \left(\begin{bmatrix}a_1\\ a_2\end{bmatrix}t + \begin{bmatrix}b_1\\ b_2\end{bmatrix}\right)e^{-2t}$$

* لا تنسى المقارنة

④

Nono Jenuws !!

∴ find the suitable form $y^{(p)}$ of the system

$$y_1' = 5y_1 + 3y_2 - 2e^{-t} + 1$$

$$y_2' = -y_1 + y_2 + e^{-t} - 5t + 7$$

solution:- $\underline{y}' = \begin{bmatrix} 5 & 3 \\ -1 & 1 \end{bmatrix} \underline{y} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ -5 \end{bmatrix} t + \begin{bmatrix} 1 \\ 7 \end{bmatrix}$

the eigenvalue $= \boxed{\lambda = 2}$ e^{2t}, $\boxed{\lambda = 4}$ e^{4t} $\Rightarrow y^{(p)} = \underline{a} e^{-t} + \underline{b} t + \underline{c}$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{-t} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} t + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Variation of parameters !- ← نوجد $y^{(p)}$ عن طريق هذه العلاقة

$$\underline{y}^{(p)} = \underline{\Phi}(t) \int \underline{\Phi}^{-1} \underline{F}(t)\, dt$$

Φ : fundamental Matrix

Ex:- $\underline{y}' = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} \underline{y} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t}$

$\begin{bmatrix} a & B \\ C & D \end{bmatrix} = \Phi(t) = \begin{bmatrix} y^{(1)} & y^{(2)} \end{bmatrix}$

$\Phi^{-1}(t) = \dfrac{1}{|\Phi|} \begin{bmatrix} D & -B \\ -C & a \end{bmatrix}$

$F(t) =$ موجودة في السؤال (nonhomogeneous term)

① we solve

$\underline{y}' = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} \underline{y}$

$y^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{t}$ $\quad y^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$

$\Phi(t) = \begin{bmatrix} 2e^{t} & e^{2t} \\ e^{t} & e^{2t} \end{bmatrix}$

$\Phi^{-1} = \dfrac{1}{e^{3t}} \begin{bmatrix} e^{2t} & -e^{2t} \\ -e^{t} & 2e^{t} \end{bmatrix}$

$|\Phi| = 2e^{3t} - e^{3t} = \boxed{e^{3t}}$

$F = \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix}$

$$y^{(p)} = \begin{bmatrix} 2e^{t} & e^{2t} \\ e^{t} & e^{2t} \end{bmatrix} \int \frac{1}{e^{3t}} \begin{bmatrix} e^{2t} & -e^{2t} \\ -e^{t} & 2e^{t} \end{bmatrix} \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix}$$

$$\therefore \underline{y} = \underline{y}^{(h)} + \underline{y}^{(p)}$$

laplace Transform :- CH6

let f be a function of t , $0 \le t < \infty$

The function (if it exits)

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t) e^{-st} dt$$

Ex:- $\mathcal{L}\{5\} = \frac{5}{s}$

$\mathcal{L}\{t^2\} = \frac{2!}{s^3} = \frac{2}{s^3}$

$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$

قواعد ال laplace المهمة

$\mathcal{L}\{c\} = \frac{c}{s}$ $s>0$

$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ $s>a$

$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ $s>0$

$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$ $s>0$

$\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}$ $s>0$

$\mathcal{L}\{\sinh(at)\} = \frac{a}{s^2-a^2}$ $s>|a|$

$\mathcal{L}\{\cosh(at)\} = \frac{s}{s^2-a^2}$ $s>|a|$

= * **Theorem** : $\mathcal{L}\{e^{at} f(t)\} = F_{(s-a)}$

نعوض لاي قيمة ل s

*ملاحظة :- يمكن توزيع ال laplace ($\mathcal{L}$) في الجمع والطرح

$\mathcal{L}\{a f(t) + B g(t)\} = a\mathcal{L}\{f(t)\} + B\mathcal{L}\{g(t)\}$

Ex:- $\mathcal{L}\{t^3 e^{2t}\}$

$= \mathcal{L}\{t^3\}_{s-2} \Rightarrow \left.\frac{3!}{s^4}\right|_{s-2} \Rightarrow \boxed{\frac{6}{(s-2)^4}}$

---

* The Inverse laplace Transform

if $\mathcal{L}\{f(t)\} = F(s)$

* $\mathcal{L}^{-1}\{F_{(s-a)}\} = e^{at} f(t)$

Ex:- $\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+4}\right\}$

$e^{t}\mathcal{L}\left\{\frac{1}{s^2+4}\right\} \Rightarrow \frac{e^t}{2}\sin 2t$

Ex $\mathcal{L}^{-1}(\ )$

① $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$

② $\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2}t^2$

③ $\mathcal{L}^{-1}\left\{\frac{3}{s^3} + \frac{s}{s^2+9}\right\}$

$\frac{3}{2}t^2 + \cos 3t$

---

unit step function "Heaviside function"

$$u(t-a) = \begin{cases} 0 & 0 \le t < a \\ 1 & t \ge a \end{cases} \quad \boxed{a \ge 0}$$

$u(t) = 1$

$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$

$\mathcal{L}\{u(t-a) f(t)\} = e^{-as}\mathcal{L}\{f(t+a)\}$

Ex:- $f(t) = t^2 + u(t-1)$ find $f(2)$

$f(2) = (2)^2 + 1 = 5$

Remark :-

$\sin(\pi + t) = -\sin t$
$\sin(\pi - t) = \sin t$
$\cos(\pi + t) = -\cos t$
$\cos(\pi - t) = -\cos t$

$\sin(\frac{\pi}{2} + t) = \cos t$
$\sin(\frac{\pi}{2} - t) = \cos t$
$\cos(\frac{\pi}{2} + t) = -\sin t$
$\cos(\frac{\pi}{2} - t) = \sin t$

Ex :-

① $\mathcal{L}\{u(t-2)\} = \frac{e^{-2s}}{s}$

② $\mathcal{L}\{t\,u(t-2)\} = e^{-2s}\mathcal{L}\{t+2\} = e^{-2s}\left[\frac{1}{s^2} + \frac{2}{s}\right]$

③ $\mathcal{L}\{\sin t\, u(t-\frac{\pi}{2})\} = e^{-\frac{\pi}{2}}\mathcal{L}\{\sin(t+\frac{\pi}{2})\}$
$e^{-\frac{\pi}{2}}\mathcal{L}\{\cos t\} = e^{-\frac{\pi}{2}s} \cdot \frac{s}{s^2+1}$ #

④ $\mathcal{L}\{e^{5t} t\, u(t-2)\}$

@ → $\mathcal{L}\{t\, u(t-2)\} = e^{-2s}\mathcal{L}\{t+2\}$
$e^{-2s}\left\{\frac{1}{s^2} + \frac{2}{s}\right\}$

$\mathcal{L}\{e^{5t} t\, u(t-2)\} = \left(e^{-2t}\left[\frac{1}{s^2} + \frac{2}{s}\right]\right)_{s-5}$

$e^{-2(s-5)}\left[\frac{1}{(s-5)^2} + \frac{2}{s-5}\right]$

قواعد مهمة

$\mathcal{L}\{u(t-a) f(t)\} = e^{-as}\mathcal{L}\{f(t+a)\}$

$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)\mathcal{L}^{-1}\{F(s)\}_{t-a}$

$u(t-a) f(t-a)$

Ex: find ①

$\mathcal{L}^{-1}\left\{\frac{s\, e^{-s}}{s^2-1}\right\}$

$u(t-1)\,\mathcal{L}^{-1}\left\{\frac{s}{s^2-1}\right\}_{t-1}$

$u(t-1)\cosh(t-1)$

② $\mathcal{L}^{-1}\left\{\frac{4e^{-2s}}{s-7}\right\}$

$4u(t-2)\,\mathcal{L}^{-1}\left\{\frac{1}{s-7}\right\}_{t-2}$

$4u(t-2)\left(e^{7t}\right)_{t-2} =$

$= 4u(t-2)\, e^{7(t-2)}$

* عند التحويل ل $\mathcal{L}$ ← نحل الشيفت ثم ال لابلاس

* عند التحويل من $\mathcal{L}^{-1}$ ← $\mathcal{L}$ نحل اللابلاس ثم الشيفت

let $f(t) = \begin{cases} f_1(t) & 0 \le t \le a \\ f_2(t) & a \le t \le B \\ f_3(t) & t \ge B \end{cases}$

$f(t) = f_1(t)[u(t) - u(t-a)] + f_2(t)[u(t-a) - u(t-B)]$
$+ f_3(t)\, u(t-B)$

Ex:- $f(t) = \begin{cases} t & 0 \le t < 2 \\ 0 & t \ge 2 \end{cases}$

find $F(s)$.

$\boxed{2}$

$f(t) = t[u(t) - u(t-2)] + 0\,u(t-2)$

$t - t\,u(t-2)$

$\mathcal{L}\{f(t)\} = \frac{1}{s^2} - e^{-2s}\left[\frac{1}{s^2} + \frac{2}{s}\right]$

$$* \quad \frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \Rightarrow \frac{A(s+1)+Bs}{s(s+1)}$$

$$1 = A(s+1) + Bs$$

$$\boxed{s=0} \rightarrow \boxed{A=1} \qquad \boxed{s=-1} \rightarrow \boxed{B=-1} \quad \Rightarrow \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

Ex:- $\mathcal{L}^{-1}\left\{ \frac{1}{s(s+1)} \right\}$

$$\mathcal{L}^{-1}\left\{ \frac{1}{s} - \frac{1}{s+1} \right\} = 1 - e^{-t}$$

قواعد مهمة :-

[1] $\mathcal{L}\{y'(t)\} = sY(s) - y(0)$

[2] $\mathcal{L}\{y''(t)\} = s^2 Y(s) - sy(0) - y'(0)$

Ex:- solve $y' + y = u(t-1)$, $y(0) = 0$

$$\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{u(t-1)\}$$

$$sY(s) + 0 + Y(s) = \frac{e^{-s}}{s}$$

$$\frac{(s+1)Y(s)}{(s+1)} = \frac{e^{-s}}{s(s+1)} \Rightarrow \mathcal{L}^{-1}Y(s) \ \mathcal{L}^{-1}\frac{e^{-s}}{s(s+1)}$$

$$y(t) = u(t-1)\left(1 - e^{-t}_{(t-1)}\right)$$

$$y(t) = u(t-1)\left(1 - e^{1-t}\right)$$

Dirac Delta Function :-

① $\delta(t-a) = \begin{cases} \infty & t = a \\ 0 & t \neq a \end{cases}$

② $\int_{-\infty}^{\infty} \delta(t-a)\,dt = 1$

Theorem :- if $g(t)$ is a cts function, then

$$\int_{-\infty}^{\infty} \delta(t-a)\, g(t)\, dt = g(a)$$

Ex:-

1- $\int_{0}^{\infty} (1 + e^{-t})\,\delta(t-2)\,dt = 1 + e^{-2}$

2- $\int_{0}^{\infty} \delta(t - \frac{\pi}{2}) \sin t \, dt = \sin\frac{\pi}{2} = 1$

Theorem :- For $a \geq 0$ and $g(t) \in C^\infty$:-

① $\mathcal{L}\{\delta(t-a)\} = e^{-as}$

② $\mathcal{L}\{g(t)\delta(t-a)\} = e^{-as} g(a)$

Ex:- find $\mathcal{L} \Rightarrow$

① $\mathcal{L}\{\delta(t-3)\} = e^{-3s}$

② $\mathcal{L}\{(2t+1)\delta(t-1)\} = (2(1)+1)e^{-s} = 3e^{-s}$

3) solve:- $y'' + \pi^2 y = \delta(t-1)$, $y(0) = 1$, $y'(0) = 0$

$\mathcal{L}\{y''\} + \mathcal{L}\{\pi^2 y\} = \mathcal{L}\{\delta(t-1)\}$

$s^2 Y(s) - s y(0) - y'(0) + Y(s)\pi^2 = e^{-s}$

$s^2 Y(s) - s - 0 + Y(s)\pi^2 = e^{-s}$

$\therefore \mathcal{L}^{-1}\, Y(s)\ \mathcal{L}^{-1} = \dfrac{e^{-s} + s}{s^2 + \pi^2}$

$\therefore\ y(t) = \mathcal{L}\left(\dfrac{e^{-s}}{s^2+\pi^2}\right) + \mathcal{L}\left(\dfrac{s}{s^2+\pi^2}\right) \Rightarrow u(t-1)\,\mathcal{L}\left(\dfrac{1}{s^2+\pi^2}\right)_{t-1} + \cos(\pi t) \Rightarrow$

$\cos(\pi t) + u(t-1)\sin\pi(t-1)$

قواعد مهمة :-

[1] $\mathcal{L}\{t f(t)\} = -F'(s) \Rightarrow -\dfrac{d}{ds}(F(s))$

[2] $\mathcal{L}^{-1}\{F'(s)\} = -t f(t)$

Ex:- find $\mathcal{L} \Rightarrow$

a- $\mathcal{L}\{t \sin 2t\} \Rightarrow \dfrac{-d}{ds}\left[\dfrac{2}{s^2+4}\right] \Rightarrow -\dfrac{-2(2s)}{(s^2+4)^2} \Rightarrow \boxed{\dfrac{4s}{(s^2+4)^2}}$

B- $\mathcal{L}^{-1}\left\{\underbrace{\ln\left(\dfrac{s^2+4}{s^2}\right)}_{F(s)}\right\}$

let $F(s) = \ln\left(\dfrac{s^2+4}{s^2}\right) \rightarrow F(s) = \ln(s^2+4) - \ln s^2 \rightarrow F(s) = \ln(s^2+4) - 2\ln s$

$\mathcal{L}^{-1} F'(s) = \mathcal{L}^{-1}\left(\dfrac{2s}{(s^2+4)} - \dfrac{2}{s}\right) \Rightarrow -t f(t) = 2\cos 2t - 2$

$f(t) = \dfrac{2\cos 2t - 2}{-t}$

(III) $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(u)\,du$

قواعد مهمة :-

Ex:- Find $\mathcal{L}\left\{\frac{\sin t}{\cancel{t}}\right\} \Rightarrow \int_s^{\infty} \frac{1}{u^2+1}\,du \Rightarrow \tan^{-1} u \Big|_s^{\infty} = \tan^{-1}\infty - \tan^{-1} s$

$\frac{\pi}{2} - \tan^{-1} s$

(IV) $\mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \frac{F(s)}{s}$

$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(u)\,du$

Ex:- $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} \Rightarrow \mathcal{L}^{-1}\left\{\frac{\frac{1}{s^2+1}}{s}\right\} = \int_0^t \sin u\,du \Rightarrow -\cos u \Big|_0^t$

$\Rightarrow -[\cos t - 1] = 1 - \cos t$

Ex:- $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} \Rightarrow \mathcal{L}^{-1}\left\{\frac{\frac{1}{s(s^2+1)}}{s}\right\} = \int_0^t [1-\cos u]\,du = u - \sin u \Big|_0^t$

$\Rightarrow t - \sin t$

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* $\mathcal{L}\{\delta(t-a)\} = \int_0^{\infty} \delta(t-a)\, e^{-st}\,dt$

$\boxed{= e^{-as}}$

* $\mathcal{L}\{g(t)\,\delta(t-a)\} = \int_0^{\infty} \delta(t-a)\, g(t)\, e^{-st}\,dt = \boxed{g(a)\, e^{-as}}$

CH 5 Power Series Method

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

if $c = 0$ then the power series called Maclaurin series.

let $N(x) y'' + P(x) y' + r(x) y = 0$

we can assume G.s $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$

[1] Types of x_0

ordinary $a(x_0) \neq 0$

$\lim_{x \to x_0} P(x) = \text{exist}$

$\lim_{x \to x_0} r(x) = \text{exist}$

singular $a(x_0) = 0$

$\lim_{x \to x_0} P(x) = \text{doesn't exist}$

or

$\lim_{x \to x_0} r(x) = \text{doesn't exist}$

singular point

regular singular

if $\lim_{x \to x_0} (x - x_0) \frac{P(x)}{N(x)}$ exist

and

$\lim_{x \to x_0} (x - x_0)^2 \frac{r(x)}{N(x)}$ exist

irregular singular

اذا كانت احدى النهايتين

D.N.E

او كلاهما

Ex:- consider $xy'' - xy' + y = 0$

① show that $x_0 = 0$ is regular singular

② Determine the roots of the indicial equation at $x_0 = 0$

③ find the Frobenius series solution corresponding to the larger root

④ what would be the form of the second linearly independent solution.

[2]

Power series solution of

$y'' + x^2 y = 0$ about $x_0 = 0 \Rightarrow \boxed{\text{ordinary point}}$

let $y = \sum_{n=0} a_n x^n$ be a solution

$$y' = \sum_{n=1} a_n n (x)^{n-1}$$

$$y'' = \sum_{n=2} a_n n(n-1)(x)^{n-2}$$

substitute y, y'' in the D.E

$$\Rightarrow \sum_{n=2} n(n-1) a_n x^{n-2} + x^2 \sum_{n=0} a_n x^{n} = 0$$

[1] $$\sum_{n=2} n(n+1) a_n x^{n-2} + \sum_{n=0} a_n x^{n+2} = 0 \longrightarrow$$

$$\sum_{n=0} (n+2)(n+1)\, a_{n+2}\, x^n + \sum_{n=2} a_{n-2}\, x^n = 0$$

$$\Rightarrow 2a_2 + 6a_3 x + \sum_{n=2} (n+2)(n+1)\, a_{n+2}\, x^n + \sum_{n=2} a_{n-2}\, x^n = 0$$

نقطة البدء

$a_2 = 0 \qquad a_3 = 0 \qquad (n+2)(n+1)\, a_{n+2} + a_{n-2} = 0 \qquad \underline{n \geq 2}$

الاكبر $\Leftarrow$ $a_{n+2} = \dfrac{-a_{n-2}}{(n+2)(n+1)} \qquad n \geq 2$

$n=2 \quad a_4 = \dfrac{-a_0}{12}$

$n=3 \quad a_5 = \dfrac{-a_1}{20}$

$n=4 \quad a_6 = 0$

$n=5 \quad a_7 = 0$

$n=6 \quad a_8 = \dfrac{-a_4}{(8)(7)} = \dfrac{a_0}{(8)(7)(12)}$

$n=7 \quad a_9 = \dfrac{-a_5}{(9)(8)} = \dfrac{a_1}{(9)(8)(20)}$

$\Rightarrow$ the solution of the D.E :-

$$y(x) = \sum_{n=0} a_n x^n$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$a_0 + a_1 x - \frac{a_0}{12} x^4 - \frac{a_1}{20} x^5 \dots$$

$$= a_0 \left[1 - \frac{x^4}{12} \dots\right] + a_1 \left[x - \frac{x^5}{20}\right]$$

Reguler singular points

$$A(x)y'' + B(x)y' + C(x)y = 0$$

x_0 is said to be regular singular if

① $\lim\limits_{x \to x_0} \dfrac{B(x)}{A(x)} (x - x_0) = P_0 < \infty$

② $\lim\limits_{x \to x_0} \dfrac{C(x)}{A(x)} (x - x_0)^2 = q_0 < \infty$

other wise, x_0 is called an irreguler singuler point

if x_0 is regular singular, then we define the indicial equation as :-

$$r(r-1) + P_0 r + q_0 = 0$$

$\boxed{2}$

Ex :- $y''(x^2 - 4x + 3) + 4xy' + 2xy = 0$

$x^2 - 4x + 3 = 0 \Rightarrow (x = 1, 3)$ (singular points)

at $\boxed{x_0 = 1}$

$\lim\limits_{x \to 1} \dfrac{4x}{x^2 - 4x + 3} (x-1) \Rightarrow -2 \quad \underline{P_0}$

$\lim\limits_{x \to 1} \dfrac{2x}{x^2 - 4x + 3} (x-1)^2 \Rightarrow 0 \quad q_0$

$x_0 = 1$ is a regular singuler

$x_0 = 3$ " " "

in the previous example :- $r(r-1) + 2r + 0 = 0$

the roots are $(3, 0) \Leftarrow r \quad \Leftarrow r^2 - 3r = 0$

Frobenius' Theorem :-

if $x = x_0 \rightarrow$ regular singular point of $A(x)\,y'' + B(x)\,y' + C(x)\,y = 0$

then there exits at least one solution of the form :-

$$y = \sum_{n=0} a_n (x - x_0)^{n+r} \;,\quad a_0 \neq 0$$ (where (r) is constant to Be determined.)

$$y' = \sum_{n=0} (n+r)\, a_n\, (x - x_0)^{n+r-1}$$

$$y'' = \sum_{n=0} (n+r)(n+r-1)\, a_n\, (x - x_0)^{n+r-2}$$

---

the indicat roots r_1 and r_2 of the singularity are real .

we distingwish three cases :-

$\boxed{1}$ $r_1 > r_2$ and $r_1 - r_2 \notin \mathbb{Z}$ عدد كسري

$$y_1(x) = \sum_{n=0} a_n x^{n+r_1} \;,\quad a_0 \neq 0$$

$$y_2(x) = \sum_{n=0} a_n x^{n+r_2} \;,\quad a_0 \neq 0$$

$\boxed{2}$ $r_1 > r_2$ and $r_1 - r_2 \in \mathbb{Z}$ عدد صحيح

$$y_1(x) = \sum_{n=0} a_n x^{n+r_1} \quad a_0 \neq 0$$

$$y_2(x) = c\, y_1(x) \ln x + \sum_{n=0} a_n x^{n+r_2} \quad a_0 \neq 0$$

constant which may be zero

$\boxed{3}$ $r_1 = r_2 = r$

$$y_1(x) = \sum_{n=0} a_n x^{n+r} \quad a_0 \neq 0$$

$$y_2(x) = y_1(x) \ln x + \sum_{n=1} a_n x^{n+r}$$

$\boxed{3}$