ch 18

Orders: أكبر مشتقة

Linear: (y) وكل مشتقاتها خطّيات

note that → $y \cdot y'$ → non Linear

if $y(x)$ is a solution for an equation → جِد المشتقات المطلوبة → عوّضهم → لازم تحققها

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separable: $f(x)\,dx = f(y)\,dy$

$f(x+y)\,dx$ or $f(x+y)\,dy$ → not separable

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first order Linear D.E: $y' + P(x)\,y = f(x)$ --- (*)

① $\mu = e^{\int P(x)dx}$ ② $(*) \cdot \mu$ ③ $(\mu \cdot y)' = R.H.S$ زبطوهم

④ كامل الطرفين

$x' + P(y)\,x = f(y)$ نفس الكلام على

note: $y' = \frac{dy}{dx}$, $x' = \frac{dx}{dy}$

Bernoulli: $y' + P(x)\,y = f(x)\,y^n$ --- (*)

↳ non Linear
must have (y^n)
$n \neq 0, 1$

① $u = y^{1-n}$, $u' = (1-n)\,y^{-n}\,y'$

② $(*) \cdot (1-n)\,y^{-n}$ → $u' + (1-n)\,P(x)\,u = (1-n)\,f(x)$

③ صارت Linear بكمّل حلها

$x' + P(y)\,x = f(y)\,x^n$ نفس الكلام على

Homogeneous: $y' = g\left(\frac{y}{x}\right)$

$\text{rational} = \dfrac{\text{poly of } (x \text{ and } y)}{\text{poly of } (x \text{ and } y)}$ $\to$ اقسم على $x^{\square}$ $\to$ homogeneous

نفس الدرجة $\to$

① $u = \frac{y}{x}$, $y = ux$, $y' = xu' + u$

② $xu' + u = g\left(\frac{y}{x}\right)$

$\frac{dy}{dx} = f(ax+by+c)$

① $u = ax+by+c$, $u' = a + by'$, $y' = \frac{u'-a}{b}$

② $y' = f(ax+by+c) = f(u)$

Exact: $M(x,y)\,dx + N(x,y)\,dy = 0$

$\to M\,dx + N\,dy = 0$

if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \to$ Exact

$\to u = \int M\,dx$, $u = \int N\,dy$ $\to$ الأول + الثاني $= C$

بدون تكرار

Integrating factors: $M(x,y)\,dx + N(x,y)\,dy = 0$ and $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = R(x) \to \mu(x) = e^{\int R(x)dx}$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M} = R(y) \to \mu(x) = e^{\int R(y)dy}$$

$\to$ the D.E. $\cdot\ \mu \to$ Exact

ch2

2nd-order homogeneous: $y'' + p(x)y' + q(x)y = 0$

if $y_1 = cy_2 \Longrightarrow y_1$ and y_2 are dependent solutions

$w(f,g) = fg' - gf'$ if $w(y_1, y_2) = 0 \rightarrow y_1$ and y_2 are dependent solutions

$w(y_1, y_2) = -w(y_2, y_1)$ $\quad w(y_1, y_1) = 0$

Abel's theorem: $y'' + p(x)y' + q(x)y = 0$

$\rightarrow w(y_1, y_2)_{(x)} = ce^{-\int p(x)dx}$

$w(f, f + bg) = \underbrace{w(f,f)}_{0} + w(f, bg) = bw(f,g)$

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y-missing

$\rightarrow u = y', u' = y'' \rightarrow$ ch1

x-missing

$\rightarrow u = y', u' = y'', u' = u\dfrac{du}{dy} \rightarrow$ ch1

Reduction of order: $y'' + p(x)y' + q(x)y = 0$

and y_1 is given $\rightarrow y_2 = y_1 \displaystyle\int \frac{e^{-\int p(x)dx}}{y_1^2} dx$

independent solution

Homogeneous Linear constant coefficients: $y'' + ay' + by = 0$

$\rightarrow r^2 + ar + b = 0 \rightarrow r_1 \neq r_2 \rightarrow y_1 = e^{r_1 x}, y_2 = e^{r_2 x}$

$\rightarrow r_1 = r_2 \rightarrow y_1 = e^{rx}, y_2 = xe^{rx}$

$\rightarrow r = \lambda \pm \mu i \rightarrow y_1 = e^{\lambda} \cos(\mu x), y_2 = e^{\lambda} \sin(\mu x)$

$(r - r_1)(r - r_2) = 0 \rightarrow r^2 - (r_1 + r_2)r + r_1 r_2 = 0$

Euler-Cauchy : $x^2 y'' + axy' + by = 0$

$\to r(r-1) + axr + b = 0 \to$
- $r_1 \neq r_2 \to y_1 = x^{r_1}, y_2 = x^{r_2}$
- $r_1 = r_2 \to y_1 = x^r, y_2 = x^r \ln x$
- $r = \lambda \pm \mu i \to y_1 = x^\lambda \cos(\mu \ln x)$
  $y_2 = x^\lambda \sin(\mu \ln x)$

Non-homogeneous 2nd o: $y = y_h + y_p$

$ay'' + by' + cy = r(x)$

y_h: the homogeneous solution

$y'' + p(x)y' + q(x)y = r(x)$

y_p:

undetermined

① شكله حسب ال $r(x)$

② عوّض في ① سطر سطر

③ اضرب لحد ما يبطل في تشابه

④ كل $(\sin, \cos)$ بيجو

التشابه المقصود مع الحل الـhomogeneous

variation of parameters

$$y_p = -y_1 \int \frac{y_2}{W} r + y_2 \int \frac{y_1}{W} r$$

$W \to W(y_1, y_2)$

$(y_1, y_2) \to$ حلول الـhomogeneous

$r \to r(x)$

ch 3

ex: $r^3 - 2r^2 - 5r + 6 = 0$

$\to \pm 1, \pm 6, \pm 2, \pm 3$

$+1 \to 1 - 2 - 5 + 6 = 0$ ✓ $\to r = +1 \to (r-1)$

$r^3 - 2r^2 - 5r + 6 = (r-1)(r^2 - r - 6)$

$$\begin{array}{c|cccc} 1 & 1 & -2 & -5 & 6 \\ & & 1 & -1 & -6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

$= (r-1)(r-3)(r+2)$

$r_1 = 1 \quad r_2 = 3 \quad r_3 = -2$

$y_1 = e^{x} \quad y_2 = e^{3x} \quad y_3 = e^{-2x}$

ex: $x^3 y''' - 3x^2 y'' + 6x y' - 6y = 0$

$r(r-1)(r-2) - 3r(r-1) + 6r - 6 = 0$

$(r-1)\left(r(r-2) - 3r + 6\right) = 0$

⋮

same as chapter 2

$W(x, x^2, x^3)$

$$W \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = (x)\begin{vmatrix} 2x & 3x^2 \\ 2 & 6x \end{vmatrix} - (1)\begin{vmatrix} x^2 & x^3 \\ 2 & 6x \end{vmatrix} + 0$$

نقي صف او عامود
و امشي فيه

رقم الصف + رقم العامود

~~even → ⊕~~
~~odd → ⊖~~

odd → (−)

even → (+)

$$W_1 = \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix} \qquad W_2 = \begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & 1 & 6x \end{vmatrix} \qquad W_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix}$$

Variation of parameters → $y_p = y_1 \int \frac{W_1}{W} r + y_2 \int \frac{W_2}{W} r + y_3 \int \frac{W_3 r}{W}$

ch4

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4-6 = -2$$

$$A \cdot A^{-1} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} ay_1 + by_2 &= g_1 \\ cy_1 + dy_2 &= g_2 \end{aligned} \rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \rightarrow Ay = g$$

if $g = 0 \rightarrow$ homogeneous system

Eigen value $(\lambda) \rightarrow$ salar

Eigen vector $(\underline{X_\lambda}) \rightarrow$ matrix

non zero

to find $(X_\lambda) \rightarrow X_\lambda [A - \lambda I] = 0$

to find $(\lambda) \rightarrow |A - \lambda I| = 0$

$$\lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \qquad X_\lambda = \begin{bmatrix} a \\ b \end{bmatrix}$$

notes

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{matrix} ax + by \\ cx + dy \end{matrix}$$

if $\lambda_1 = \alpha + \beta i$, $\lambda_2 = \alpha - \beta i$ and $X_{\lambda_1} = a + bi \rightarrow X_{\lambda_2} = a - bi$

if $\lambda_1 = \lambda_2 = \lambda \rightarrow [A - \lambda_1 I] X_{\lambda_1} = 0$

$$[A - \lambda_2 I] X_{\lambda_2} = X_{\lambda_1}$$

$$y_1' = a y_1 + b y_2 + g_1$$
$$y_2' = c y_1 + d y_2 + g_2$$
$$\rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

$$\underline{y}' = A\underline{y} + \underline{g}$$ if $g = 0 \rightarrow$ homogeneous

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$\underline{y}' = A\underline{y}$

homogeneous

- $\lambda_1 \neq \lambda_2 \rightarrow y = c_1 x_{\lambda_1} e^{\lambda_1 t} + c_2 x_{\lambda_2} e^{\lambda_2 t}$
- $\lambda_1 = \lambda_2 \rightarrow y = c_1 x_{\lambda_1} e^{\lambda_1 t} + c_2 \left(t x_{\lambda_1} e^{\lambda_1 t} + x_{\lambda_2} e^{\lambda_2 t} \right)$
- $\lambda = \alpha \pm \beta i \rightarrow x_{\lambda_1} = (\vec{a}) + i(\vec{b})$

$x_{\lambda_2} = (\vec{a}) - i(\vec{b})$

$\rightarrow y = c_1 y_1 + c_2 y_2$

$\cancel{y_1 = e^{\alpha t} \begin{bmatrix} \vec{a}\cos t \\ -\vec{b}\sin t \end{bmatrix}}$

$\cancel{y_2 = e^{\alpha t} \begin{bmatrix} \vec{b}\cos t \\ +\vec{a}\sin t \end{bmatrix}}$

$$y_1 = e^{\alpha t} \left[\vec{a}\cos t - \vec{b}\sin t \right]$$
$$y_2 = e^{\alpha t} \left[\vec{b}\cos t + \vec{a}\sin t \right]$$

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non homogeneous

$\underline{y}' = Ay + g \quad , g \neq 0$

$y = y_h + y_p$

y_p

- Undetermined

اعطيها شكل حسب g

$\cancel{y_p \rightarrow (\underline{a}t + \underline{b}) e^{-2t}}$

مكرر $y_p = a e^{-2t} \rightarrow y_p = (\underline{a}t + \underline{b}) e^{-2t}$

- Variation of parameter

$\phi = \left[y^{(1)} \; y^{(2)} \right]$

$y' = A\underline{y} + F(t)$

$y_p = \phi(t) \int \phi^{-1}(t) F(t) \, dt$

ch 6

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)\, e^{-st}\, dt$$

$$\left.\begin{array}{lll} \mathcal{L}\{1\} = \dfrac{1}{s} & \mathcal{L}\{e^{at}\} = \dfrac{1}{s-a} & \mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}} \\ \mathcal{L}\{\sin(at)\} = \dfrac{a}{s^2+a^2} & \mathcal{L}\{\cos(at)\} = \dfrac{s}{s^2+a^2} & \end{array}\right] s>0$$

$$\left.\begin{array}{ll} \mathcal{L}\{\sinh(at)\} = \dfrac{a}{s^2-a^2} & \mathcal{L}\{\cosh(at)\} = \dfrac{s}{s^2-a^2} \end{array}\right] s>|a|$$

$$\mathcal{L}\{a f(t) + b g(t)\} = a\,\mathcal{L}\{f(t)\} + b\,\mathcal{L}\{g(t)\}$$

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}\,\mathcal{L}^{-1}\{F(s)\}$$

إكمال مربع

① $\left(\frac{b}{2}\right)^2$

② $\pm\left(\frac{b}{2}\right)^2$

$$u(t-a) = \begin{cases} 0 & 0 \le t < a \\ 1 & t \ge a \end{cases} \quad , a \ge 0$$

$u(t) = 1$ $\qquad$ $\mathcal{L}\{u(t-a)\} = \dfrac{e^{-as}}{s}$

$$\mathcal{L}\{u(t-a) f(t)\} = e^{-as}\,\mathcal{L}\{f(t+a)\}$$

$\sin(\pi + t) = -\sin t$ $\qquad$ $\sin\left(\frac{\pi}{2} \pm t\right) = \cos t$

$\sin(\pi - t) = \sin t$ $\qquad$ $\cos\left(\frac{\pi}{2} + t\right) = -\sin t$

$\cos(\pi \pm t) = -\cos t$ $\qquad$ $\cos\left(\frac{\pi}{2} - t\right) = \sin t$

$$f(t) = \begin{cases} f_1(t) & , 0 \le t < a \\ f_2(t) & , a \le t < b \\ f_3(t) & , t \geq b \end{cases}$$

$$\rightarrow f(t) = f_1(t)\,[u(t-0) - u(t-a)] + f_2(t)\,[u(t-a) - u(t-b)]$$

$$+ f_3(t)\,[u(t-b)]$$

$$\mathcal{L}^{-1}\{e^{-as} F(s)\} = u(t-a)\, \mathcal{L}^{-1}\{F(s)\}_{t-a}$$

$$= u(t-a)\, f(t-a)$$

note: $\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} = \frac{A(s+1) + Bs}{s(s+1)}$

$\rightarrow A(s+1) + Bs = 1$

$\boxed{s=0} \rightarrow A = 1$

$\boxed{s=-1} \rightarrow B = -1$

اصفار الأقواس

$\rightarrow \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$

$\mathcal{L}\{y'(t)\} = sY(s) - y(0)$

$\mathcal{L}^{-1}\{\frac{1}{s(s+1)}\} = 1 - e^{-t}$

$\mathcal{L}\{y''(t)\} = s^2 Y(s) - s\,y(0) - y'(0)$, for I.V.P. D.E.

solving steps: ① $\mathcal{L}\{$ لكلا الطرفين $\}$

② جد قيمة $Y(s)$

③ $\mathcal{L}^{-1}[$ قيمة $Y(s)] = y(t)$

Dirace Delta:

$$\delta(t-a)=\begin{cases}\infty & t=a\\ 0 & t\neq a\end{cases}$$

$$\int_{-\infty}^{\infty}\delta(t-a)\,dt=1$$

$$\int_{-\infty}^{\infty}\delta(t-a)\,g(t)\,dt=g(a)$$

for $a\geq 0$ and $g(t)$ cts: $\mathcal{L}[\delta(t-a)]=e^{-as}$

$$\mathcal{L}\{\delta(t-a)\,g(t)\}=e^{-as}g(a)$$

$$\mathcal{L}\{\delta(t-a)\}=\int_{0}^{\infty}\delta(t-a)\,e^{-st}\,dt=e^{-as}$$

$$\mathcal{L}[t\,f(t)]=-F'(s)$$

$$\mathcal{L}^{-1}\{F'(s)\}=-t\,f(t)$

[illegible]

$$\cot^{-1}(ax)=-\frac{\text{مشتقة الداخل}}{1+(ax)^2}$$

$$\tan^{-1}(ax)+\frac{\text{مشتقة الداخل}}{1+(ax)^2}$$

($\ln$, $\tan^{-1}$, $\cot^{-1}$ نفس الشي)

ex: $\mathcal{L}^{-1}\left\{\ln\left(\frac{s^2+4}{s^2}\right)\right\}=$

① $F(s)=\ln\left(\frac{s^2-4}{s^2}\right)=\ln(s^2-4)-2\ln(s)$

② $F'(s)=\frac{2s}{s^2+4}-\frac{2}{s}$

③ $\mathcal{L}^{-1}\{F'(s)\}=2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}-2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$

④ $-t\,f(t)=2\cos 2t-2\rightarrow f(t)=\frac{2-\cos 2t}{t}$

شرط (2)

ex: $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+16)^2}\right\}$

[1] $\mathcal{L}^{-1}$ المعروف   [2] مشتقته هي المطلوب

① $F(s) = \frac{1}{s^2+16} \xrightarrow{\text{of } f(t)} f(t) = \frac{1}{4}\sin 4t$ جوا نهار

② $\mathcal{L}^{-1}$ + اشتق .

③ $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+16)^2}\right\} = \boxed{}$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(u)\,du$$

notes:
$\sinh 0 = 0$
$\cosh 0 = 1$

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(u)\,du$$

$$\mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \frac{F(s)}{s}$$

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system using Laplace:

ex: $y_1' = 2y_1 + y_2 \quad , y_1(0) = 1$
$y_2' = y_1 + 2y_2 \quad , y_2(0) = 0$

$\mathcal{L}\{y_1'\} = 2\mathcal{L}\{y_1\} + \mathcal{L}\{y_2\}$

$sY_1(s) - y_1(0) = 2Y_1(s) + Y_2(s)$

$\vdots$

$sY_1(s) - 2Y_1(s) - Y_2(s) = 1$

repeat for $y_2' \rightarrow \quad -Y_1(s) + (s-2)Y_2(s) = 0$

next page →

$$Y_{1}(s) = \frac{\begin{vmatrix} 1 & -1 \\ 0 & s-2 \end{vmatrix}}{\begin{vmatrix} s-2 & -1 \\ -1 & s-2 \end{vmatrix}} = \frac{s-2}{(s-2)^2-1} \rightarrow y_1(t) = \mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2-1}\right\}$$

$$Y_{2}(s) \quad \frac{\begin{vmatrix} s-1 & 1 \\ -1 & 0 \end{vmatrix}}{\begin{vmatrix} s-2 & -1 \\ -1 & s-2 \end{vmatrix}} = \frac{1}{(s-2)^2-1} \rightarrow y_2(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2-1}\right\}$$