Chapter 1

Meera Khater

first order linear

1. Classification
2. separable
3. first order linear
4. Bernoulli
5. Homogeneous
6. Exact and Integrating factors

Second order DE

1. Wronskian
2. second order homogeneous DE
3. Cauchy - Euler
4. Non-Homogeneous DE

DE/ODE : differential equation is an equation that contains derivative

[فيها مشتقة y', y'' أو dy, dx]

مثلاً

1) $y'' - 2xy' = e^x \longrightarrow$ DE لأن فيها مشتقة

2) $(x+1)\,dx = 2y^2\,dy \longrightarrow$ اقسم الطرفين على dx إذاً DE

1 Classification

$\longrightarrow$ order : the highest derivative in the DE (أعلى مشتقة)

مثلاً

1) $y''' - 2x(y'')^2 = e^x$ — order = 3

2) $y^{(4)} - 2xy = 0$ — order = 4

3) $x\,dx - (y+1)\,dy = 0$ — order = 1

ملاحظة فرق بين $y^{(4)}$ و y^4

$y^{(4)}$: مشتقة رابعة

y^4: قوة

$\rightarrow$ linearity

نراقب الـ y

linear form بال لازم يكون كله y' / y'' / y'''

يعني ممنوع
يكون عليهم
جذر أو قوة
أو ln أو sin
وممنوع نضرب
linear x linear
لأنه بعطي non-linear

كل الآتي ممنوع

$\sqrt{y}$ / $\sin y'$ / e^{y} / $\ln y$ / $\frac{1}{y}$ / y^2 / $y \cdot y'$

مثلًا

براقب الـ y

1) $y''' - 2x^2 y'' = \sin x$ — 3rd-order linear

2) $y''' - 2y' = (\sin x) y$ — 3rd-order linear

3) $y^{(4)} - 2x^2 y = \sin(\underline{\underline{xy}})$ — 4th-order non-linear

4) $y'' - 2x\underline{\underline{y y'}} = e^x$ — 2nd-order non-linear

Example

is $y(x) = e^{3x} - 2$
a solution of $y'' - 3y' = 0$?

$$y'' - 3y' = 0$$
$$9e^{3x} - 3(3e^{3x}) = 0$$
$$0 = 0$$

yes it is a solutio

$$\boxed{\begin{aligned} y &= e^{3x} - 2 \\ y' &= 3e^{3x} \\ y'' &= 9e^{3x} \end{aligned}}$$

DE

الفرق بينهم

ODE ⟵ ⟶ PDE

ordinary | partially

$y = f(x)$ | $U = f(x, y)$

* الاشتقاق يكون عادي طبيعي

* وتكون المعادلة مكونة من متغيرين فقط

$$y'' - 2xy = e^x$$

(هذا ما سندرس)

* هنا يكون الاشتقاق جزئي

* وتكون المعادلة مكونة من 3 متغيرات فأكثر

$$U_{xy} - 2U_x = x^2$$

الأولوية اله

$\boxed{2}$ separable افصل وكامل

$$\int f(x)\,dx = \int g(x)\,dx$$

أولًا بدنا نتعلم كيف نفصل

1) $2xy^2\,dx - 2(y+1)\,dy = 0$

$$2xy^2\,dx = 2(y+1)\,dy$$

$$2x\,dx = \frac{2(y+1)}{y^2}\,dy \quad \checkmark$$

2) $(x^2 + yx^2)\,dx = (y^2+1)\,dy$

$$x^2(1+y)\,dx = (1+y^2)\,dy$$

$$x^2\,dx = \frac{(1+y^2)}{(1+y)}\,dy \quad \checkmark$$

3) $(x^2y + x^2 + y + 1)\,dx = 2xy\,dy$

$$(x^2(1+y) + (1+y))\,dx = 2xy\,dy$$

$$(1+y)(1+x^2)\,dx = 2xy\,dy$$

$$\frac{1+x^2}{x}\,dx = \frac{2y}{1+y}\,dy \quad \checkmark$$

4) $(xy+1)\,dx = 2xy\,dy$ ✗ not separable

اللي جنب dy و dx لازم تكون (separable

- Function of x بالحالة الأولى
- Function of y الحالة الثانية
- Function of y ✗ Function of x الحالة الثالثة

5) If $(x^3y + x^n + y+1)\,dx = (x+y^m)\,dy$ was separable then what's the values of m, n?

$$(x^3y + x^3 + y+1)\,dx = (x+y^0)\,dy$$

$n = 3$

عشان ناخذ عامل مشترك وتصير الحالة الثانية

$m = 0$

عشان تصير الحالة الأولى

$$\left(x^3(1+y) + (1+y)\right)dx = x\,dy$$

$$(1+y)(1+x^3)\,dx = x\,dy$$

$$\frac{1+x^3}{x}\,dx = \frac{dy}{1+y}$$

---

1) $x\,dy = \cos^2 y\,dx$

الآن بدنا نفصل ونكامل

$$\frac{dy}{\cos^2 y} = \frac{dx}{x}$$

بالنسبة لـ c فاحنا رح انحطها في طرف واحد فقط

$$\int \sec^2 y\,dy = \int \frac{1}{x}\,dx$$

$$\tan y + C_1 = \ln|x| + C_2$$

$$\tan y = \ln|x| + C$$

IVP : هي DE معها شرط يعني لازم توجد ال C

initial value problem

في المثال السابق اذا أعطاني شرط $y(1) = 0$

$x = 1$
$y = 0$

$$\tan 0 = \ln(1) + C$$
$$0 = 0 + C$$
$$C = 0$$
$$\tan y = \ln|x|$$

2) $(x^2 + x^2y^2 + 1 + y^2)\,dx = 2y\,dy$

$$\left(x^2(1+y^2) + (1+y^2)\right)dx = 2y\,dy$$
$$(1+x^2)(1+y^2)\,dx = 2y\,dy$$
$$\int (1+x^2)\,dx = \int \frac{2y}{1+y^2}\,dy$$
$$\frac{x^3}{3} + x = \ln|1+y^2| + C$$

3) $e^{2x-3y}\,dx = e^{y-x}\,dy$

$$e^{2x} \cdot e^{-3y}\,dx = e^{y} \cdot e^{-x}\,dy$$
$$\frac{e^{2x}}{e^{-x}}\,dx = \frac{e^{y}}{e^{-3y}}\,dy$$
$$e^{3x}\,dx = e^{4y}\,dy$$
$$\int e^{3x}\,dx = \int e^{4y}\,dy$$
$$\frac{e^{3x}}{3} = \frac{e^{4y}}{4} + C$$

يكافئ الseparable
بالأولويات

3 fist-order linear

$$\left[y' + p(x)\,y = f(x)\right] \text{ --- } *$$

الخطوات:

1) $\mu = e^{\int p(x)\,dx}$

integrating factor

تأكد انه معامل
y' واحد
عشانه تحل الخطوة

2) Multiply $*$ by μ

3) simplify and integrate

solve

1. $x \dfrac{dy}{dx} + 2y = \dfrac{\sin x}{x}$

$x y' + 2y = \sin x / x$

$\underbrace{y' + \frac{2}{x}}_{p(x)} y = \underbrace{\sin x / x^2}_{f(x)} \text{ --- } *$

1) $e^{\int \frac{2}{x} dx} = e^{2\ln|x|} = e^{\ln x^2} = x^2 = \mu$

2) $*$ by μ = $x^2 y' + 2xy = \sin x$

3) $\int (x^2 \cdot \underbrace{y}_{\text{دائمًا}})' = \int \sin x = x^2 y = -\cos x + C$

دائمًا
زمن

اذا أعطاني شرط
بكمل وبطلع ال C

2. $$y' + \underbrace{\frac{6x}{x^2+1}}_{P(x)} y = \underbrace{\frac{1}{(x^2+1)^4}}_{f(x)} \quad --- *$$

1) $$e^{\int \frac{6x}{x^2+1}dx} = e^{3\ln|x^2+1|} = e^{\ln(x^2+1)^3}$$
$$= (x^2+1)^3 = \mu$$

2) $$(x^2+1)^3 y' + 6x(x^2+1)^2 y =$$

3) $$\int \left((x^2+1)^3 \cdot y \right)' = \int \frac{1}{x^2+1}$$
$$(x^2+1)^3 y = \tan^{-1}x + C$$

3. $$y\,dx = [\sin^2 y - x]\,dy$$

أولاً كأول نظرة المعادلة ليست linear

لأنه فيها $\sin y$

ولكن رح نحلها بناءً على x وليس على y

يعني المختصر المفيد

اذا قالك linear بشكل عام تركيزي فقط على ال y

ولكن اذا حددلي بالسؤال linear in x

وقتها بركز مع x وبتكون المعادلة بهاد الشكل

$$x' + p(y)x = f(y)$$

$$y\,dx = [\sin^2 y - x]\,dy$$

$$\frac{dx}{dy} = \frac{[\sin^2 y - x]}{y}$$

$$x' = \frac{\sin^2 y}{y} - \frac{x}{y}$$

$$\underbrace{x' + \underbrace{\frac{1}{y}}_{P(y)} x = \underbrace{\frac{\sin^2 y}{y}}_{F(y)}}_{} \text{ ----- } *$$

1) $$e^{\int \frac{1}{y} dy} = e^{\ln|y|} = y = \mu$$

2) $$y x' + x = \sin^2 y$$

3) $$\int (y \cdot \underbrace{x}) ' = \int \sin^2 y$$

الرمز

$$y x = \frac{1}{2} \int 1 - \cos 2y$$

$$y x = \frac{1}{2}\left(y - \frac{\sin 2y}{2}\right) + C$$

4. If $y' + \frac{1}{x} y = x^2 \, y^{2a-3}$
is linear then the values of a:

$$y^{2a-3} = 1$$

$$y^{2a-3} = y^0$$

$$2a-3 = 0$$

$$\left(a = \frac{3}{2}\right)$$

$$y^{2a-3} = y$$

$$2a-3 = 1$$

$$(a = 2)$$

4 Bernoulli DE

$$y' + p(x)y = f(x)\underline{\underline{y^n}}$$

not linear in y

$n \neq 0, 1$

note before we continue

$$\frac{d}{dx}(y(x))^n = n(y(x))^{n-1} y'(x)$$

is the same

$$\frac{d}{dx}(y)^n = n(y)^{n-1} y'$$

بس اختصار

اشتقاق قاعدة bernoulli

$$y' + p(x)y = f(x)y^n$$

let $u = y^{1-n}$

$$u' = (1-n)y^{-n}y'$$

Multiply the DE by $(1-n)y^{-n}$

$$(1-n)y^{-n}y' + (1-n)y^{-n}p(x)y = (1-n)y^{-n}f(x)y^n$$

$$(1-n)y^{-n}y' + (1-n)y^{1-n}p(x) = (1-n)f(x)$$

$$u' + (1-n)p(x)u = (1-n)f(x)$$

$$u' + \underline{P}(x)u = F(x) \longrightarrow \text{linear in } u$$

bernoulli اختصار ما يجب حفظه في

$$y' + p(x)\,y = f(x)\,y^n$$

let $u = y^{1-n}$ then the DE is reduced to

الهدف ان نخليها linear

$$u' + (1-n)\,p(x)\,u = (1-n)\,f(x)$$

وبنحلها ...

Solve

ملاحظة

$u' = \frac{du}{dx}$ اذا كانت المعادلة ترتبط بين x, u

و $u' = \frac{du}{dy}$ اذا كانت ترتبط بين y, u

① $x^2 y' + 2xy = y^3$

$$y' + \underbrace{\frac{2}{x}}_{p(x)} y = \underbrace{\frac{1}{x^2}}_{f(x)} \underbrace{y^3}_{y^n}$$

let $u = y^{-2}$ then the DE is reduced to

$$u' + (1-n)\,p(x)\,u = (1-n)\,f(x)$$

$$u' + (-2)\left(\frac{2}{x}\right)u = (-2)\left(\frac{1}{x^2}\right)$$

$$u' - \frac{4}{x}u = \frac{-2}{x^2} \quad \text{-----} \; (*)$$

تكملة الحل

1) $e^{\int -\frac{1}{x}x} = e^{-1\ln x} = x^{-1} = \mu$

2) $x^{-1}u' - 1\,x^{-5}u = -2\,x^{-6}$

3) $\int (x^{-1}\cdot u)' = \int -2x^{-6}$

$= x^{-1}\,u = \frac{2}{5}x^{-5} + C$

لاتنسى ترجع u لأصلها

4) $x^{-1}\,y^{-2} = \frac{2}{5}x^{-5} + C$

$= y^{-2} = \frac{2}{5}x^{-1} + x^{1}C$

تبسيط

$\boxed{2}$ $y' + x^2 y = \frac{e^{-x^3}\sinh x}{3y^2}$

$$y' + \underbrace{x^2}_{p(x)}y = \underbrace{\frac{e^{-x^3}\sinh x}{3}}_{f(x)}\underbrace{y^{-2}}_{y^n}$$

let $u = y^3$ then the DE is reduced

to $u' + (1-n)\,p(x)\,u = (1-n)\,f(x)$

$u' + (3)(x^2)\,u = \cancel{(3)}\left(\frac{e^{-x^3}\sinh x}{\cancel{3}}\right)$

$u' + 3x^2 u = e^{-x^3}\sinh x$ ---- (*)

تكملة الحل

1) $e^{\int 3x^2\,dx} = e^{x^3} = \mu$

2) $e^{x^3}u' + 3x^2 e^{x^3}u = \sinh x$

3) $\int \left(e^{x^3}u\right)' = \int \sinh x$

$e^{x^3}u = \cosh x + C$

$e^{x^3}(y^3) = \cosh x + c$

$\boxed{3}$ $\frac{dy}{dx} = \frac{x}{y - yx^2}$

Bernoulli in x

السؤال رح يذكر انها bernoulli in x ولا تيكون in y

$\frac{dx}{dy} = \frac{y}{x} - \frac{yx^2}{x}$

$x' = yx^{-1} - yx$

$x' + \underbrace{y}_{P(y)}x = \underbrace{y}_{F(y)}\underbrace{x^{-1}}_{x^n}$

let $u = x^2$ then the DE is reduced to

$u' + (1-n)P(y)u = (1-n)F(y)$

$u' + (2)(y)u = (2)(y)$

$u' + 2yu = 2y$ --- (*)

1) $e^{\int 2y\,dy} = e^{y^2}$

2) $e^{y^2}u' + 2y\,e^{y^2} = 2y\,e^{y^2}$

3) $\int (e^{y^2} \cdot u)' = \int 2y\,e^{y^2}$ تكامل بالتعويض

$e^{y^2}u = e^{y^2} + C$

$u = 1 + e^{-y^2}C$

4) $x^2 = 1 + e^{-y^2}C$

اذا اجا السؤال نفسه solve بدون ما يحددلك انه bernoulli in x بحله separable

$$\frac{dy}{dx} = \frac{x}{y - yx^2}$$

$$y(1-x^2)\,dy = x\,dx$$

$$\int y\,dy = \int \frac{x}{1-x^2}\,dx$$

$$\frac{1}{2}y^2 = \frac{-\ln|1-x^2|}{2} + C$$

$$y^2 = C - \ln|1-x^2|$$

$\boxed{5}$ Homogeneous DE

$$\left[y' = f\left(\frac{y}{x}\right) \right] \text{---} (*)$$

let $u = \frac{y}{x}$

$y = xu$

$y' = xu' + u$

$$xu' + u = f(u)$$

الهدف منها نوصل الـ DE

تحل separable

solve

① $y' = \dfrac{x^2 + y^2 + xy}{x^2}$

$$y' = 1 + \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right) \text{---} (*)$$

let $u = \frac{y}{x} \rightarrow y = xu \rightarrow y' = xu' + u$

$$xu' + u = 1 + u^2 + u$$

$$xu' = 1 + u^2$$

$$x\frac{du}{dx} = 1 + u^2$$

$$\int \frac{du}{1+u^2} = \int \frac{dx}{x}$$

$$\tan^{-1} u = \ln|x| + C$$

$$\tan^{-1}\left(\frac{y}{x}\right) = \ln|x| + c$$

$$\frac{y}{x} = \tan(\ln(x| + c)$$

$$y = x \cdot \tan(\ln|x| + C)$$

② $(x+y)\,dy = (x-y)\,dx$

$$\frac{dy}{dx} = \frac{x-y}{x+y}$$

let $u = \frac{y}{x}$

$y = xu$

$y' = xu' + u$

$$y' = \frac{1-\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)}$$

$$xu' + u = \frac{1-u}{1+u}$$

$$x\frac{du}{dx} = \frac{1-u}{1+u} - u$$

$$x\frac{du}{dx} = \frac{1-u-u-u^2}{1+u}$$

$$\int \frac{(1+u)}{1-2u-u^2}\,du = \int \frac{dx}{x}$$

$$-\frac{1}{2}\ln|1-2u-u^2| = \ln|x| + C$$

$$-\frac{1}{2}\ln\left|1-\frac{2y}{x}-\left(\frac{y}{x}\right)^2\right| = \ln|x| + C$$

past papers on homogeneous

note

كيف نعرف اذا بنحل homo ؟ لازم الأوس تكون متساوية

① $y' = \dfrac{x^3 + 2xy^n}{x^m + 2y^3}$

$x^3 \;,\; 2x^1y^2 \;,\; x^3 ,\; 2y^3$

$n = 2 \qquad m = 3$

② $y' = 2\ln x - \ln y^n + 1$

let $n=2$

$y' = 2\ln x - \ln y^2 + 1$

$= 2\ln x - 2\ln y + 1$

$= 2(\ln x - \ln y) + 1$

$= 2\ln\frac{x}{y} + 1$

$= 2\ln\frac{1}{u} + 1$

$xu' + u = -2\ln u + 1$

let $u = \frac{y}{x}$

then $\frac{1}{u} = \frac{x}{y}$

$y' = xu' + u$

then we solve it separable

③ $y' = \dfrac{y + x}{\sqrt{xy}}$

we can solve it as homo because $y^1, x^1, (y^1x^1)^{\frac{1}{2}}$

$y' = \dfrac{y}{\sqrt{x}\sqrt{y}} + \dfrac{x}{\sqrt{x}\sqrt{y}}$

$= \left(\frac{y}{x}\right)^{\frac{1}{2}} + \left(\frac{x}{y}\right)^{\frac{1}{2}}$

$= \sqrt{u} + \dfrac{1}{\sqrt{u}}$

let $u = \frac{y}{x}$

$y' = xu' + u$

we continue (separable)

④ $\dfrac{dy}{dx} = \dfrac{\sqrt{xy} + y}{x}$

let $u = \dfrac{y}{x}$

$y' = xu' + u$

$y' = \sqrt{\dfrac{y}{x}} + \dfrac{y}{x}$

$xu' + \cancel{u} = \sqrt{u} + \cancel{u}$

$x \dfrac{du}{dx} = \sqrt{u}$

$\int u^{-\frac{1}{2}} du = \int \dfrac{dx}{x}$

$2u^{\frac{1}{2}} = \ln|x| + C$

$\sqrt{\dfrac{y}{x}} = \dfrac{\ln|x|}{2} + C$

$\boxed{6}$

دراس

ما الها اسم

:)

الهدف

تتحول

separable

$y' = f(ax + by + c)$ --- (*)

$u = ax + by + C$

$u' = a + by'$

examples

1) $y' = (2x + 3y - 1)^3 - 1$ $\qquad u = 2x + 3y - 1$

2) $y' = 2(x+y)^3 - 1$ $\qquad u = x + y$

3) $y' = \dfrac{x+y}{2x+2y+1} = \dfrac{(x+y)}{2(x+y)+1}$ $\quad u = x+y$

4) $y' = \sin(2x-3y)+5$ $\quad u = 2x-3y$

5) $y' = 2x+2y-5$ $\quad u = 2x+2y$, $u = x+y$

6) $y' = \dfrac{2x-y+1}{y-2x+5} = \dfrac{2x-y+1}{5-(2x-y)}$ $\quad u = 2x-y$

7) $y' = \dfrac{2x-y+5}{x+2y+1}$ لاتحل على هذه الطريقة $\times$

كيف نعرف اذا تحل على هذه الطريقة أم لا

معامل x بالبسط $\times$ معامل y بالمقام $=$ معامل y بالبسط $\times$ معامل x بالمقام

مثلاً لو طبقناها على المثال ⑥

$(-2)\times(-1) = (1)\times(2)$

$2 = 2$ ✓

لو طبقناها على مثال ⑦

$(-1)(1) \neq (2)(2)$

Solve

$$\frac{dy}{dx} = (x+y+2)^2$$

$u = x+y+2$

$u' = 1+y'$

$y' = u'-1$

$y' = (x+y+2)^2$

$u'-1 = u^2$

$u' = u^2+1$

separable

$$\frac{du}{dx} = u^2+1$$

$$\int \frac{du}{u^2+1} = \int dx$$

$$\tan^{-1} u = x + C$$

$$u = \tan(x+C)$$

$$x+y+2 = \tan(x+C)$$

$\boxed{7}$ Exact and integrating Factor

Revision

* $f(x,y) = x^2 + y^2 + 2x^2y$

$\frac{df}{dx} = 2x + 4xy \qquad \frac{df}{dy} = 2y + 2x^2$

* $f(x,y) = x^2y^3 + e^{xy}$

$\frac{df}{dx} = 2xy^3 + ye^{xy} \qquad \frac{df}{dy} = 3x^2y^2 + xe^{xy}$

* $\int (2xy + x^2y^3)\, dx$

this is called parial Integral

$yx^2 + \frac{x^3y^3}{3} + \underbrace{f(y)}_{\text{constant}}$

* $\int (4xy^2 + \cos(xy))\, dy$

$\frac{4}{3}xy^3 + \frac{\sin(xy)}{x} + f(x)$

Exact DE

$$\boxed{M(x,y)\,dx + N(x,y)\,dy = 0}$$

لازم نوصل لهاي الصيغة قبل ما نبلش

$M \longrightarrow$ معاملها dx

$N \longrightarrow$ معاملها dy

if $\boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}}$ then the DE is Exact

1) If $[2x+4xy^3]dx + [6x^2 f(y) + 3y^2]dy = 0$ is exact then $f(y) =$ ___

$$\frac{\partial M}{\partial y} = 12xy^2 \qquad \frac{\partial N}{\partial x} = 12x f(y)$$

$$12xy^2 = 12x f(y)$$

$$f(y) = y^2$$

$\boxed{2}$ If $[3x^2y+4xy^n]dx+[x^3+6x^2y^2+3y^2]dy=0$ is exact then $n =$ ____

$$\frac{\partial M}{\partial y} = 3x^2+4x(n\,y^{n-1}) \qquad \frac{\partial N}{\partial x} = 3x^2+12xy^2$$

$$\cancel{3x^2}+4xny^{n-1} = \cancel{3x^2}+12xy^2$$

$$4xny^{n-1} = 12xy^2$$

$$n\,y^{n-1} = 3y^2$$

$$(n=3)$$

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\left(\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\right) \Longrightarrow \text{exact}$$

the solution is $U(x,y)=c$

خصائص ل $U(x,y)$

$$\frac{\partial u}{\partial x} = M$$

$$\frac{\partial u}{\partial y} = N$$

Solve

$\boxed{3}$ $[3x^2+4xy^3+y^2]dx+[6x^2y^2+2xy]dy=0$

$$\frac{\partial M}{\partial y}=12xy^2+2y \qquad \frac{\partial N}{\partial x}=12xy^2+2y$$

$\therefore$ it is exact

The solution is $U(x,y)=c$

① $\frac{du}{dx}=M=3x^2+4xy^3+y^2$

$$\int du=\int(3x^2+4xy^3+y^2)\,dx$$

$$U(x,y)=x^3+2x^2y^3+y^2x+f(y)$$

② ثم اشتق بالنسبة لـ y

$$\underbrace{\frac{du}{dy}}_{N}=6x^2y^2+2xy+f'(y)$$

③ $$6x^2y^2+2xy=6x^2y^2+2xy+f'(y)$$

$$f'(y)=0$$

$$f(y)=0+C_1 \longrightarrow \text{مافي داعي نحطها}$$

④ $U(x,y)=c$

$$x^3+2x^2y^3+y^2x+C_1=C$$

$$x^3+2x^2y^3+y^2x=C$$

لو كان السؤال السابق دوائر ففي حل أسرع

$$[3x^2+4xy^3+y^2]dx+[6x^2y^2+2xy]dy=0$$

$$\int M\,dx = x^3+2x^2y^3+y^2x$$

$$\int N\,dy = 2x^2y^3+xy^2$$

نضع التكامل الأول كاملًا ثم نضيف عليه الحدود الجديدة

$$x^3+2x^2y^3+y^2x = C$$

نفس الجواب

$$\boxed{4}\ [2x\cos y+3x^2y]dx+[x^3-x^2\sin y-y]dy=0$$

طبعًا بالأول متأكد اذا exact

$$\frac{dM}{dy} \overset{?}{=} \frac{dN}{dx}$$

$$\int M\,dx = x^2\cos y+x^3y$$

$$\int N\,dy = x^3y+x^2\cos y-\frac{y^2}{2}$$

$$x^2\cos y+x^3y-\frac{y^2}{2}=C$$

$$\boxed{5}\ [xy^2-3x^2y-2x^3]dx+[x^2y-x^3]dy=0$$

$$\int M\,dx = \frac{x^2}{2}y^2-x^3y-\frac{x^4}{2}$$

$$\int N\,dy = \frac{x^2y^2}{2}-x^3y$$

$$\frac{x^2}{2}y^2-x^3y-\frac{x^4}{2}=C$$

$$x^2y^2-2x^3y-x^4=C$$

تبسيط $\times 2$

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\frac{dM}{dy} \neq \frac{dN}{dx} \qquad \text{(Not Exact)}$$

If μ is a function such that

$$\mu M\,dx + \mu N\,dy = 0$$

then μ is called an integrating factor

$\mu(x)$	$\mu(y)$
$\dfrac{\frac{dM}{dy} - \frac{dN}{dx}}{N} = R(x)$	$\dfrac{\frac{dM}{dy} - \frac{dN}{dx}}{-M} = R(y)$
$\mu = e^{\int R(x)\,dx}$	$\mu = e^{\int R(y)\,dy}$

1. solve $(y^2-3xy-2x^2)dx+(xy-x^2)dy=0$

1) $\dfrac{dM}{dy} = 2y-3x \qquad \dfrac{dN}{dx} = y-2x$

$\therefore$ not exact

2) $\dfrac{dM}{dy} - \dfrac{dN}{dx} = 2y-3x-y+2x = y-x$

3) $\dfrac{\dfrac{dM}{dy} - \dfrac{dN}{dx}}{N} = \dfrac{(y-x)}{x(y-x)} = \dfrac{1}{x}$

4) $e^{\int \frac{1}{x}dx} = e^{\ln|x|} = x$

5) $(y^2x-3x^2y-2x^3)\,dx+(x^2y-x^3)\,dy=0$

this is exact

$\int M\,dx = \dfrac{y^2x^2}{2} - x^3y - \dfrac{x^4}{2}$

$\int N\,dy = \dfrac{y^2x^2}{2} - x^3y$

$\frac{1}{2}y^2x^2 - x^3y - \frac{1}{2}x^4 = C$

$\boxed{2}$ solve $(3x^2y + y^2)dx + (2x^3 + 3xy)\,dy = 0$

1) $\frac{dM}{dy} = 3x^2 + 2y \quad , \quad \frac{dN}{dx} = 6x^2 + 3y$

$\therefore$ not exact

2) $\frac{dM}{dy} - \frac{dN}{dx} = 3x^2 + 2y - 6x^2 - 3y = -3x^2 - y$

3) $\dfrac{\frac{dM}{dy} - \frac{dN}{dx}}{-M} = \dfrac{-3x^2 - y}{y(-3x^2 - y)} = \dfrac{1}{y}$

4) $e^{\int \frac{1}{y} dy} = e^{\ln|y|} = y$

5) $(3x^2y^2 + y^3)\,dx + (2x^3y + 3xy^2)\,dy = 0$

This is exact

$$\int M\,dx = x^3y^2 + y^3x$$

$$\int N\,dy = x^3y^2 + xy^3$$

$$x^3y^2 + y^3x = C$$

$\boxed{3}$ Find the integrating factor

$$y\,dx - (e^{-2y} - 2xy)\,dy = 0$$

$$y\,dx + (2xy - e^{-2y})\,dy = 0$$

لازم يكون بالنص موجب

1) $\dfrac{dM}{dy} = 1$, $\dfrac{dN}{dx} = 2y$

2) $$\frac{\dfrac{dM}{dy} - \dfrac{dN}{dx}}{-M} = \frac{1-2y}{-y} = \frac{-1}{y} + 2$$

3) $$e^{\int (2 - \frac{1}{y})\,dy} = e^{2y - \ln|y|} = e^{2y} \cdot e^{\ln y^{-1}}$$

$$\therefore \mu = \frac{e^{2y}}{y}$$

4) $$e^{2y}\,dx + \left(2x e^{2y} - \frac{1}{y}\right) dy = 0$$

Note $$M(x,y)\,dx + N(x,y)\,dy = 0$$

if $$\frac{\dfrac{dM}{dy} - \dfrac{dN}{dx}}{N} = 2 \quad R(x)$$

Then $\mu = e^{\int 2\,dx}$
$= e^{2x}$

if $$\frac{\dfrac{dM}{dy} - \dfrac{dN}{dx}}{-M} = 2 \quad R(y)$$

then $\mu = e^{\int 2\,dy}$
$= e^{2y}$

$\boxed{4}$ If $\mu = y^n$ is an integrating factor of $2xy\,dx + (y^2 - x^2)\,dx = 0$ then $n = \ldots\ldots$

First Solution : $\frac{\partial M}{\partial y} = 2x$

$\frac{\partial N}{\partial x} = -2x$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M} = \frac{2x + 2x}{-2xy} = \frac{4x}{-2xy} = \frac{-2}{y}$$

$$\mu = e^{\int \frac{-2}{y} dy} = e^{-2\ln|y|} = y^{-2}$$

$y^n = y^{-2}$

$(n = -2)$

second solution :

$$2xy \cdot y^n\,dx + (y^2 y^n - x^2 y^n)\,dy = 0$$

$$2x\,y^{1+n}\,dx + (y^{2+n} - x^2 y^n)\,dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$2x(1+n)\,y^n = -2x\,y^n$$

$1+n = -1$

$(n = -2)$

Second order DE

$$y'' + p(x)\,y' + q(x)\,y = f(x)$$

If $(f(x) = 0)$ then the DE is homogeneous otherwise it is called non-homogeneous

$y'' - 2xy' + e^{x}y = 0 \longrightarrow$ homogeneous

$y'' - 2y' = x \longrightarrow$ non-homogeneous

$y'' - 2xy' - 3 = 0 \longrightarrow$ non-homogeneous

أول اشي بنقلها

$f(x) = 3 \neq 0$

$y'' - 2x^2y' - 3x^2 = 1 \longrightarrow$ non-homogeneous

$$y'' - 2x^2y' = \underbrace{1 + 3x^2}_{f(x)}$$

بهمنا حاليًا

Consider

homogeneous ال

$$y'' + p(x)y' + q(x)y = 0 \quad --- \; (*)$$

let y_1, y_2 be two solutions

$$\underbrace{y}_{\text{solution}} = C_1 y_1 + C_2 y_2$$

بشرط تكون
homo المعادلة

$$(y_2 = c\, y_1) \Longrightarrow \left(\frac{y_2}{y_1} = c\right)$$

y_1, y_2 are dependent

بعتبرهم حل واحد وبحتاج ألاقي حل ثاني

otherwise $\frac{y_2}{y_1} \neq c \Longrightarrow y_1, y_2$ are independent

بعتبرهم حلين

ex

$y_1 = x^4$

$y_2 = 2x^4$

$$\frac{y_1}{y_2} = \frac{1}{2}$$

dependent

$y_1 = e^{x}$

$y_2 = e^{3x}$

$$\frac{y_1}{y_2} = \frac{e^{x}}{e^{3x}} = e^{-2x}$$

independent

Wronskian

let y_1, y_2 be two solutions of ⊛

The wroskian of y_1, y_2 is denoted by

$$w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = (y_1 y_2' - y_2 y_1')$$

الأول في مشتقة الثاني ناقص الثاني في مشتقة الأول

$$W(y_1, y_2) = -W(y_2, y_1)$$

If $W(y_1, y_2) = 0$ then y_1, y_2 are dependent

ex:

$$w(x^4, 2x^4) = x^4(8x^3) - (2x^4)(4x^3) = 0$$

dependent

$$w(e^x, e^{3x}) = e^x(3e^{3x}) - (e^{3x})(e^x)$$

$$= 2e^{4x}$$

independent

اذاً بالمختصر اذا بدي أعرف الحلول اللي عندي dependent

بقسمهم على بعض

$$\frac{y_2}{y_1} = c$$

ثابت

بطلع ال W

$$W(y_2, y_1) = 0$$

ملاحظة (1)

$\{y_1, y_2\}$ → Fundamental set of solution

→ Basis of solution

independent

ملاحظة (2)

$y = c_1 y_1 + c_2 y_2 \longrightarrow y_1, y_2$ independent

general solution يسمى

1) If $W(y_1, y_2) = 2e^{3x}$

Find $W(y_1 + y_2, 2y_1)$

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1' = 2e^{3x}$$

$$W(y_1 + y_2, 2y_1) = (y_1+y_2)(2y_1') - (2y_1)(y_1' + y_2')$$

$$= \cancel{2y_1y_1'} + 2y_2y_1' - \cancel{2y_1y_1'} - 2y_1y_2'$$

$$= 2(y_2y_1' - y_1y_2')$$

$$= 2\, W(y_2, y_1)$$

$$= 2(-2e^{3x}) = -4e^{3x}$$

حل آخر

$$W(y_1 + y_2, 2y_1)$$

$$W(y_1, 2y_1) + W(y_2, 2y_1)$$

$$(y_1)(2y_1') - (2y_1)(y_1') + y_2(2y_1') - (2y_1)(y_2')$$

$$0 + 2\, W(y_2, y_1)$$

$$-4e^{3x}$$

$W(y_1, y_2) = -W(y_2, y_1)$ إذاً

$W(y_1, y_1) = 0$

2) If $W(e^{2t}, f(t)) = 3e^{4t}$ Find $f(t)$

$$(e^{2t})(f'(t)) - (f(t))(2e^{2t}) = 3e^{4t}$$

$$e^{2t}\left(f'(t) - 2f(t)\right) = 3e^{4t}$$

$$f'(t) - 2f(t) = 3e^{2t}$$

$(f(t) = u)$

للتسهيل

$$\left[u' \underline{-2}\overset{p(t)}{u} = \overset{f(t)}{\underline{3e^{2t}}}\right]$$

First order linear

$$\mu = e^{\int -2\,dt} = e^{-2t}$$

$$e^{-2t}u' - e^{-2t}u = 3$$

$$\int (e^{-2t}u)' = \int 3\,dt$$

نعوض

$f(0)=1$

$$e^{-2t}u = 3t + C$$

$$f(t) = 3te^{2t} + Ce^{2t}$$

$$1 = 0 + C$$

$$C = 1$$

$$\left(f(t) = 3t\,e^{2t} + e^{2t}\right)$$

Abel's Theorem

ركز انه متعامل
مع homo

let y_1, y_2 be two solutions of

لازم معامل
y''
يكون واحد

$$y'' + p(t)y' + q(t)y = 0$$

then $W(y_1, y_2)(t) = c\, e^{-\int p(t)\,dt}$

1) $ty'' + 2y' + te^t y = 0$

$W(y_1, y_2)(2) = 3$, find $W(y_1, y_2)(5)$

$$y'' + \frac{2}{t}y' + e^t y = 0 \quad --(*)$$

DE
معروفة

وال W
عند نقطة
معروفة

$$W(y_1, y_2)(t) = c\, e^{-\int \frac{2}{t}\,dt}$$

$$= c\,t^{-2} = \frac{c}{t^2}$$

$$W(y_1, y_2)(2) = \frac{c}{(2)^2}$$

$$3 = \frac{c}{4}$$

$$(c = 12)$$

$$W(y_1, y_2)(5) = \frac{12}{(5)^2} = \frac{12}{25} = 0.48$$

y - missed

$$y'' = f(x, y')$$

ex
$$y'' = x^2 + (y')^2$$ → non-linear

بهاد الدرس عادي تكونه non-linear

$$y'' = e^x + (y')^2 + 10$$

$$y'' = f(x, y')$$

let $u = y'$

$$u' = y''$$

$$u' = f(x, u)$$ first order ODE

$u' = \frac{du}{dx}$ لأنه

solve : $y'' + \frac{2}{x}y' = \frac{1}{x^2}$

y - missed

$u = y'$

$u' = y''$ → first order DE

$$u' + \frac{2}{x}u = \frac{1}{x^2} \qquad \text{linear}$$

1) $e^{\int \frac{2}{x}dx} = e^{\ln x^2} = x^2$

2) $x^2u' + 2x\,u = 1$

3) $\int (x^2u)' = \int 1 \, dx$

$$x^2u = x + C$$

التكملة ↙

4) $y' = \frac{1}{x} + \frac{c}{x^2}$ again first order seperable

$$\frac{dy}{dx} = \frac{1}{x} + \frac{c}{x^2}$$

$$\int dy = \int \left(\frac{1}{x} + c x^{-2} \right) dx$$

$$y = \ln|x| + \frac{c_1 x^{-1}}{-1} + C_2$$

$$= \ln|x| + \frac{c_1}{x} + C_2$$

بنحتاج شرطين لايجاد c_1 , c_2

---

x-missed ⟷ y-missed

x-missed

$y'' = f(y, y')$

$u = y'$

$u' = y''$

$u' = f(y, u)$

$\underbrace{\frac{du}{dx}}$ $= f(y, u)$!

$\underbrace{u \frac{du}{dy}} = f(y, u)$

هذا الفرق

مع

y-missed

$$\boxed{\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = y' \frac{du}{dy} = u \frac{du}{dy}}$$

y-missed

$y'' = f(x, y')$

$u = y'$

$u' = y''$

$u' = f(x, u)$

Solve $\quad y y'' + (y')^2 = 0$

x-missed $\qquad y\left(u\dfrac{du}{dy}\right) + u^2 = 0 \qquad$ seperable

let $u = y'$

$u' = y''$

$u\dfrac{du}{dy} = y''$

$$y u \frac{du}{dy} = -u^2$$

$$\int \frac{du}{-u} = \int \frac{dy}{y}$$

$$-\ln|u| = \ln|y| + C$$

$$-\ln|y'| = \ln|y| + C$$

$$\ln|y'| + \ln|y| = C$$

$$e^{\ln|y'\cdot y|} = e^{C}$$

$$y y' = C \qquad \text{seperable}$$

$$\int y\,dy = \int C\,dx$$

$$\frac{1}{2}y^2 = C_1 x + C_2$$

solve $(y+1)y'' = (y')^2$

x-missed

$u = y'$

$u' = y''$

$u\frac{du}{dy} = y''$

$$(y+1)u\frac{du}{dy} = (u)^2$$

$$\int \frac{1}{u}\,du = \int \frac{dy}{y+1}$$

$$\ln|u| = \ln|y+1| + C$$

$$\ln|y'| = \ln|y+1| + c$$

$$e^{\ln|y'|} = e^{\ln|y+1|}e^{c}$$

$$y' = C(y+1)$$

$$\int \frac{dy}{y+1} = \int C\,dx$$

$$\ln|y+1| = C_1x + C_2$$

solve $y'' = \frac{-1}{2y^2}$ $\quad y(0) = 1$, $\quad y'(0) = -1$

x-missed

$u = y'$

$u' = y''$

$u\frac{du}{dy} = y''$

$$u\frac{du}{dy} = \frac{-1}{2y^2}$$

$$\int u\,du = \int \frac{-1}{2y^2}\,dy$$

$$\frac{u^2}{2} = \frac{1}{2y} + c$$

$$\frac{(y')^2}{2} = \frac{1}{2y} + c$$

↓ ต่อ

$$\boxed{\begin{array}{l} y' = -1 \\ y = 1 \\ x = 0 \end{array}}$$

$$(y')^2 = \frac{1}{y} + c$$

$$(-1)^2 = \frac{1}{1} + c$$

$$c = 0$$

$$(y')^2 = \frac{1}{y}$$

$$y' = \pm\frac{1}{\sqrt{y}}$$

$y'(0) = -1$ نأخذ الســالب لأنه

$$y' = \frac{-1}{\sqrt{y}}$$

$$\int \sqrt{y}\, dy = \int -\, dx$$

$$\frac{2}{3} y^{\frac{3}{2}} = -x + c$$

$$\frac{2}{3} = 0 + c$$

$$c = \frac{2}{3}$$

$$\frac{2}{3} y^{\frac{3}{2}} = -x + \frac{2}{3}$$

$$y = \sqrt[3]{\left(1 - \frac{3}{2}x\right)^2}$$

Reduction of order

$$y'' + p(x)\,y' + q(x)\,y = 0$$

homo

لازم معامل y'' يكون واحد

$$y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx$$

ex $y_1 = x$ is a solution of

$$(x^2 - x)\,y'' - x y' + y = 0$$

Find a 2^{nd} solution (Independent)

$$y'' - \frac{1}{x-1}\,y' + \frac{y}{x^2 - x} = 0$$

$$y_2 = y_1 \int \frac{e^{-\int p(x)}}{y_1^2}\,dx = x \int \frac{e^{-\int -\frac{1}{x-1}}}{x^2}\,dx$$

$$= (x)\int \frac{x-1}{x^2}\,dx = x\left(\ln|x| + \frac{1}{x}\right)$$

$$= x\ln|x| + 1$$

ex $x y'' + 2y' + xy = 0$

أولاً نقسم على x

$$y'' + \frac{2}{x}\,y' + y = 0$$

$$y_1 = \frac{\cos x}{x}, \quad y_2 = ?$$

$$y_2 = \frac{\cos x}{x} \int \frac{\frac{1}{x^2}}{\frac{\cos^2 x}{x^2}}\,dx$$

$$= \frac{\cos x}{x} \cdot \tan x = \frac{\sin x}{x}$$

homogeneous linear ODE with constant cofficients

hom ← متجانسة
linear ←
con coff

$$y'' + a y' + b y = 0$$

$$r^2 + a r + b = 0$$

$r_1 =$ $y_1 = e^{r_1 x}$

$r_2 =$ $y_2 = e^{r_2 x}$

case ①

$\Delta = +$

الها حلين مختلفين

case ②

$\Delta = 0$

الها حل مكرر مرتين

Case ③

$\Delta = -$

الها حلين وبكونو بالأعداد المركبة

solve $y'' + y' - 2y = 0$

$$r^2 + r - 2 = 0$$

$$(r+2)(r-1) = 0$$

$$r = -2, 1$$

$y_1 = e^{-2x}$ $y_2 = e^{x}$

general solution $y = c_1 e^{-2x} + c_2 e^{x}$

$\{e^{-2x}, e^{x}\}$ fundamental set

solve $2y'' + 3y' = 0$

$$2r^2 + 3r = 0$$

$$r(2r+3) = 0$$

$r = 0$ $r = -\frac{3}{2}$

$y_1 = e^{-\frac{3}{2}x}$

$y_2 = 1$

$y = c_1 e^{-\frac{3}{2}x} + c_2$

solve $y'' + 4y' + 2y = 0$

$r^2 + 4r + 2 = 0$

$\Delta = b^2 - 4ac = 16 - 8 = 8$

$r = \frac{-b \pm \Delta}{2a} = \frac{-4 \pm 2\sqrt{2}}{2} = -2 \pm \sqrt{2}$

$y = c_1 e^{(-2-\sqrt{2})x} + c_2 e^{(-2+\sqrt{2})x}$

find the 2^{nd}-order DE whose solution is $y = c_1 e^{2x} + c_2 e^{3x}$

$r = 2 \quad r = 3$

$(r-2)(r-3)$

$r^2 - 5r + 6$

$y'' - 5y' + 6y = 0$

case ② $\Delta = 0$

(coff are constant) بعاد الدرس

اذا طلع عنا قيمتين ال r متساويات

وطلعنا solution وبدنا الثاني

فش داعي نطبق القانون هناك فورًا اضرب بـ x

$r = 2, 2$

$y_1 = e^{2x} \quad y_2 = x e^{2x}$

Solve = $y'' - 6y' + 9y = 0$

$$r^2 - 6r + 9 = 0$$

$$(r-3)(r-3) = 0 \qquad r = 3, 3$$

$$y = c_1 e^{3x} + c_2 x e^{3x}$$

Find 2nd-order DE whose solution is $y = (c_1 + c_2 x) e^{-2x}$

$$r = -2, -2$$

$$(r+2)(r+2) = 0$$

$$r^2 + 4r + 4 = 0$$

$$y'' + 4y' + 4y = 0$$

Solve $y'' - 9y = 0 \qquad y(0) = -2, \; y'(0) = -12$

$$r^2 - 9 = 0 \qquad r = \pm 3$$

$$y_1 = e^{-3x} \qquad y_2 = e^{3x}$$

$$y = c_1 e^{-3x} + c_2 e^{3x} \longrightarrow -2 = c_1 + c_2$$

$$y' = c_1(-3e^{-3x}) + c_2(3e^{3x}) \longrightarrow -12 = -3c_1 + 3c_2$$

$$c_2 - c_1 = -4$$

$$\underline{c_2 + c_1 = -2}$$

$$2c_2 = -6$$

$$c_2 = -3$$

$$c_1 = 1$$

$$\left(y = e^{-3x} - 3e^{3x}\right)$$

This is called particular solution

Revision for complex number

$\sqrt{-1} = i \longrightarrow i^2 = -1$

$\sqrt{4} = 2 \qquad \sqrt{-4} = \sqrt{-1}\,\sqrt{4} = 2i$

$$z = \underbrace{a}_{\text{real}} + \underbrace{bi}_{\text{imaginary}}$$

ex of complex numbers: $2+3i$, $\underset{\text{real}}{2}$, $\underset{\text{pure imaginary}}{5i}$

Solve $r^2+2r+2=0$

$\Delta = b^2 - 4ac$
$= 4 - 4(2)$
$= -4$

$$r = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-2 \pm 2i}{2}$$

$$= \underbrace{-1}_{\lambda} \pm \underbrace{1}_{\mu}\, i$$

λ: مع الاشارة

μ: بدون الاشارة

Complex Roots

$r = \lambda \pm \mu i$

Complex solutions

$y_1 = e^{(\lambda + \mu i)x}$ $\qquad y_2 = e^{(\lambda - \mu i)x}$

Real Solutions

$y_1 = e^{\lambda x} \cos \mu x$ $\qquad y_2 = e^{\lambda x} \sin \mu x$

solve $\quad y'' + 2y' + 5y = 0$

$r^2 + 2r + 5 = 0$

$\Delta = b^2 - 4ac$
$= 4 - 20$
$= -16$

$\sqrt{\Delta} = 4i$

$r = \frac{-2}{2} \pm \frac{4i}{2}$
$= -1 \pm 2i$

$y_1 = e^{-x} \cos 2x \qquad y_2 = e^{-x} \sin 2x$

$y = c_1 e^{-x} \cos 2x + c_2 e^{-x} \sin 2x$

Solve $y'' + 9y = 0$

قاعدة اذا ما كان في جزء real فورًا الجواب $\cos \mu x$ $\sin \mu x$

$r^2 + 9 = 0$ $\lambda = 0$

$r = \pm 3i$ $\mu = 3$

$y = c_1 \cos 3x + c_2 \sin 3x$

Find 2^{nd}-order DE whose solution is

$$y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$$

$\lambda = 2$ $\mu = 3$

$r = 2 \pm 3i$

$r = 2 + 3i$ $r = 2 - 3i$

$(r - (2+3i))(r + (2-3i))$

$r^2 + 4r - 13 = 0$

$y'' + 4y' - 13y = 0$

قاعدة اللي بالوسط جمعهم والحد الأخير ضربهم

$r^2 - (r_1 + r_2)r + r_1 r_2$

Euler-Cauchy Equations

$$x^2 y'' + a x y' + b y = 0$$

$$(r)(r-1) + ar + b = 0$$

$\Delta = +$

حلين مختلفين

$$y_1 = x^{r_1}$$

$$y_2 = x^{r_2}$$

$\Delta = 0$

حلين متشابهين

$$y_1 = x^{r}$$

$$y_2 = \ln x \; x^{r}$$

$\Delta = -$

complex حلين

$$y_1 = x^{\lambda} \cos \mu \ln x$$

$$y_2 = x^{\lambda} \sin \mu \ln x$$

solve $\quad 2x^2 y'' + 3 x y' - y = 0$

$$2(r)(r-1) + 3(r) - 1 = 0$$

$$2r^2 + r - 1 = 0$$

$$(2r - 1)(r + 1)$$

$$r = \frac{1}{2} \qquad r = -1$$

$$y_1 = x^{-1}$$

$$y_2 = x^{\frac{1}{2}}$$

$$y = c_1 x^{-1} + c_2 x^{\frac{1}{2}}$$

solve $\quad x^2y'' - 5xy' + 9y = 0$

$$(r)(r-1) - 5r + 9 = 0$$

$$r^2 - 6r + 9 = 0$$

$$r = 3, 3$$

$$y_1 = x^3 \qquad y_2 = x^3 \ln x$$

$$x^2y'' - 5xy' + 13y = 0$$

$$(r)(r-1) - 5r + 13 = 0$$

$$r^2 - 6r + 13 = 0$$

$$\Delta = 36 - (4)(13)$$

$$= -16$$

$$\sqrt{\Delta} = 4i$$

$$r = \frac{6 \pm 4i}{2} = \underset{\lambda}{3} \pm \underset{\mu}{2i}$$

$$y_1 = x^3 \cos(2 \ln x)$$

$$y_2 = x^3 \sin(2 \ln x)$$

$$y = c_1 x^3 \cos(2 \ln x) + c_2 x^3 \sin(3 \ln x)$$

$$x y'' + 4 y' = 0$$

$$x^2 y'' + 4 x y' = 0$$

$$r(r-1) + 4r = 0$$

$$r(r+3) = 0$$

$$r = 0 \qquad r = -3$$

$$y_1 = x^{-3}$$

$$y_2 = 1$$

$$y = c_1 x^{-3} + c_2$$

given $y = c_1 x^2 + c_2 x^2 \ln x$

Find DE

$$r = 2, 2$$

$$(r-2)(r-2) = 0$$

$$r^2 - 4r + 4 = 0$$

$$r(r-1) - 3r + 4 = 0$$

$$x^2 y'' - 3x y' + 4y = 0$$

$\{ x^2 \cos \ln x , x^2 \sin \ln x \}$

find DF

$\lambda = 2 \qquad \mu = 1$

$r = 2 \pm i \qquad r^2 - (r_1 + r_2) r + r_1 r_2$

$r^2 - 4r + 5$

$r(r-1) - 3r + 5 = 0$

$x^2 y'' - 3x y' + 5y = 0$

coff are constant

$y'' + p(x)y' + q(x)y = r(x)$

non-homogeneous

→ poly
→ exp
→ sin/cos

$y = y_h + y_p$

$y_h = c_1 y_1 + c_2 y_2$

$y(0) = 0$
$y'(0) = 1$

solve $y'' + y = 2x^2 + 6$

$y'' + y = 0$

$r^2 + 1 = 0$

$r = \pm i \qquad \lambda = 0, \mu = 1$

$y_1 = \cos x$

$y_2 = \sin x$

$y_p = Ax^2 + Bx + C$

$y_p' = 2Ax + B$

$y_p'' = 2A$

$y_h = c_1 \cos x + c_2 \sin x$

$$2A + Ax^2 + Bx + C = 2x^2 + 6$$

$$Ax^2 + Bx + (2A + C) = 2x^2 + 6$$

$A = 2$ $\quad B = 0$ $\quad A + C = 6$

$C = 2$

$y_p = 2x^2 + 2$

$$y = y_h + y_p$$

$$y = c_1 \cos x + c_2 \sin x + 2x^2 + 2$$

من الشرطين نجد

c_1 , c_2

$$y'' - 2y' = e^{3x}$$

$$y'' - 2y' = 0$$

$$r^2 - 2r = 0$$

$r = 0 \qquad r = 2$

$y_1 = 1 \qquad y_2 = e^{2x}$

$$y_h = c_1 + c_2 e^{2x}$$

$y_p = Ae^{3x}$

$y_p' = 3Ae^{3x}$

$y_p'' = 9Ae^{3x}$

$$9Ae^{3x} - 6Ae^{3x} = e^{3x}$$

$3A = 1$

$A = \frac{1}{3}$

$$y = y_h + y_p = c_1 + c_2 e^{2x} + \frac{1}{3} e^{3x}$$

$$y'' + 2y' = 12\sin x \quad ---(*)$$

$$y'' + 2y' = 0$$

$$r^2 + 2r = 0$$

$$r = 0 \qquad r = -2$$

$$y_1 = 1 \qquad y_2 = e^{-2}$$

$$y_h = c_1 + c_2 e^{-2}$$

$$y = y_h + y_p$$

$$y_p = A\sin x + B\cos x$$

$$y_p' = A\cos x - B\sin x$$

$$y_p'' = -A\sin x - B\cos x$$

نعوضهم في (*)

ونجد A, B

$$y'' - 2y' = 2e^{2x}$$

$$r^2 - 2r = 0$$

$$r = 0 \qquad r = 2$$

$$y_1 = 1 \qquad y_2 = e^{2x}$$

$$y = c_1 + c_2 e^{2x}$$

$$y_p = x\left(Ae^{2x}\right)$$

$$= Axe^{2x}$$

Meera Khater

Chapter 3

1) we are dealing with : homogeneous (=0)
constant coefficient
higher order DEs

Revision $r^3 - 2r^2 - 5r + 6$

نأخذ عوامل ال 6
1 2 3 6
نجربهم موجب وسالب

عامل لأنه صفّر المعادلة $(r=1)$

	1	-2	-5	6
1		1	-1	-6
	1	-1	-6	0 → لازم صفر

$(r-1)(r^2-r-6) = 0$

$(r-1)(r-3)(r+2) = 0$

$r=-2 \quad r=1 \quad r=3$

Solve :

$\boxed{1}$ $y''' - 2y'' - 5y' + 6y = 0$

$r^3 - 2r^2 - 5r + 6 = 0$

تحليل هذه المعادلة

$r=-2 \quad r=1 \quad r=3$

تذكّر $y = e^{rx}$

$y_1 = e^{-2x} \qquad y_2 = e^{x} \qquad y_3 = e^{3x}$

معادلة تكعيبية
إذاً 3 حلول

$y = c_1 y_1 + c_2 y_2 + c_3 y_3$

$= c_1 e^{-2x} + c_2 e^{x} + c_3 e^{3x}$

$\boxed{2}$ $y^{(5)} - 3y^{(4)} + 3y''' - y'' = 0$

تذكّر الشروط

const coeff ، homo

$r^5 - 3r^4 + 3r^3 - r^2 = 0$

$r^2(r^3 - 3r^2 + 3r - 1) = 0$

معادلة من الدرجة الخامسة
إذاً 5 حلول

$(r = 1)$

	1	-3	3	-1
1		1	-2	1
	1	-2	1	0

تذكّر عند التكرار
نضرب ب x أول مرة
ثم x^2 ثاني مرة
ثم x^3 ...

$(r^2)(r-1)(r^2 - 2r + 1) = 0$

$r=0 \quad r=0 \quad r=1 \quad r=1 \quad r=1$

$y_1 = 1 \quad y_2 = x \quad y_3 = e^x \quad y_4 = xe^x \quad y_5 = x^2e^x$

$$y = c_1 + c_2x + c_3e^x + c_4xe^x + c_5x^2e^x$$

$\boxed{3}$ $y^{(4)} + 2y'' + y = 0$

$r^4 + 2r^2 + 1 = 0$

$(r^2)^2 + 2r^2 + 1 = 0$ assume $u = r^2$ بس للتسهيل

$u^2 + 2u + 1 = 0$

$(u+1)(u+1) = 0$

$u = -1 \qquad u = -1$

$r^2 = -1 \qquad r^2 = -1$

$r = \pm i \qquad r = \pm i$

تذكّر $y = e^{\lambda x}\cos \mu x$

$= e^{\lambda x}\sin \mu x$

$y_1 = \cos x \qquad y_3 = x\cos x$

$y_2 = \sin x \qquad y_4 = x\sin x$

$$y = c_1\cos x + c_2\sin x + c_3x\cos x + c_4x\sin x$$

ملخص سريع للي قبل

* شروط الحل بهاي الطريقة const coeff / homo

* عدد الحلول لازم يكونوا مساويه للدرجة أو (order)

* اذا الحلول كانت طبيعية ... $r=1/r=-6$ وقتها $y_1 = e^{rx}$

* اذا الحلول كانت complex $r=\lambda \pm \mu x$ وقتها $y_1 = e^{\lambda x}\cos \mu x$
$y_2 = e^{\lambda x}\sin \mu x$

* بجميع الحالات اذا تكرر الحل بضرب بـ x

* $y = c_1 y_1 + c_2 y_2 + c_3 y_3 + \dots$

Find a 3^{rd}-order linear homogeneous DE whose solution is $y = c_1 + c_2 x + c_3 e^{2x}$

خلطي e معناها الحالة الأولى اللي هي homo

$y_1 = 1$ $\quad$ $y_2 = x$ $\quad$ $y_3 = e^{2x}$

$r = 0$ $\quad$ $r = 0$ $\quad$ $r = 2$

$r^2(r-2) = 0$

$r^3 - 2r^2 = 0$

$y''' - 2y'' = 0$

2) We are dealing with: homogeneous variable coefficiant higher order DEs "Euler - Cauchy"

$x^3y''' + x^2y'' + xy' + y = 0$

solve $x^3y''' - 3x^2y'' + 6xy' - 6y = 0$

$r(r-1)(r-2) - 3r(r-1) + 6r - 6 = 0$

$r(r-1)(r-2) - 3r(r-1) + 6(r-1) = 0$

$(r-1)\left[r^2 - 5r + 6\right] = 0$

$(r-1)(r-2)(r-3) = 0$

$r=1 \quad r=2 \quad r=3$

$y_1 = x \quad y_2 = x^2 \quad y_3 = x^3$

$y = c_1 x + c_2 x^2 + c_3 x^3$

تذكّر بحالة cauchy euler

$y_1 = x^r$

في حالة ال complex

$y_1 = x^{\lambda} \cos(\mu \ln x)$

$y_2 = x^{\lambda} \sin(\mu \ln x)$

في حالة التكرار نضرب ب $\ln x$

	الحالة الأولى homo	الحالة الثانية cauchy
الشرط (غيرانها تكون homo)	constant coefficiant $y''' + ay'' + by' + cy = 0$	variable coefficient $x^3y''' + ax^2y'' + bxy' + cy = 0$
طريقة الحل	$r^3 + ar^2 + br + c = 0$	$r(r-1)(r-2) + ar(r-1) + br + c = 0$
في حالة الحلول طبيعية (رقم عادي)	$y_1 = e^{rx}$	$y_1 = x^r$
في حالة الحلول complex	$y_1 = e^{\lambda x} \cos \mu x$ $y_2 = e^{\lambda x} \sin \mu x$	$y_1 = x^{\lambda} \cos(\mu \ln x)$ $y_2 = x^{\lambda} \sin(\mu \ln x)$
عند تكرار الحلول	نضرب ب x	نضرب ب $\ln x$

3) we are dealing with: non-homogeneous higher order DEs

y_p حل

undetermined ⟷ Variation of parameters

لها شروط → homo with const coeff

exp
poly
sin/cos

ليس لها شروط

الخطوات

$\boxed{1}$ $y_h \rightarrow$ general

$\boxed{2}$ $y_p \rightarrow$ particular

$\boxed{3}$ $y = y_h + y_p$

Solve

$$y''' + y'' = 2x+1$$

يعني non-homo
DE لا تساوي صفر

$\boxed{1}$ y_h

$$y''' + y'' = 0$$

على الحالة الأولى

$$r^3 + r^2 = 0$$

$$r^2(r+1) = 0$$

$$r=0 \quad r=0 \quad r=-1$$

$$y_1 = 1 \quad y_2 = x \quad y_3 = e^{-x}$$

$$y_h = c_1 + c_2 x + c_3 e^{-x}$$

$\boxed{2}$ y_p

$$y_p = x^2(Ax+B)$$

$$[\, y_p = Ax^3 + Bx^2 \,]$$

this is the suitable form
بالتواثر هوه بنوقف

نضع مكانه
B , A
واحد
ونقارنه كل حد
اذا موجود بالحلول
اذا كان موجود احدهما
أو كلاهما نضرب بـ x
ثم نعيد الآلية حتى نضمن
حتى نضمنه انه ولا حد
موجود نفسه بالحلول

$$y' = 3Ax^2 + 2Bx$$

$$y'' = 6Ax + 2B$$

$$y''' = 6A$$

لو بدنا
نطلع A, B
بنشتق 3 مرات
وبنعوض بال DE

$$y''' + y'' = 2x+1$$

معامل x = معامل x ، الحد الثابت = الحد الثابت

$$6A + 6Ax + 2B = 2x+1$$
$$6Ax + 6A + 2B = 2x+1$$

$6A = 2$, $A = \frac{1}{3}$

$6A + 2B = 1$, $2 + 2B = 1$, $B = -\frac{1}{2}$

$$y_p = \frac{x^3}{3} - \frac{x^2}{2}$$

this is the particular solution

$$y = y_h + y_p$$
$$= \underbrace{c_1 + c_2 x + c_3 e^{-x}}_{\text{general}} + \underbrace{\frac{x^3}{3} - \frac{x^2}{2}}_{\text{particular}}$$

solve

$$y^{(4)} + 4y'' = 2x - 12$$

$$r^4 + 4r^2 = 0$$
$$r^2(r^2+4) = 0$$

$r = 0$, $r = 0$, $r = \pm 2i$

$y_1 = 1$, $y_2 = x$, $y_3 = \cos 2x$, $y_4 = \sin 2x$

$$y_h = c_1 + c_2 x + c_3 \cos 3x + c_4 \sin 2x$$

$$y_p = x^2(Ax+B) = Ax^3 + Bx^2$$

$y' = 3Ax^2 + 2Bx$

$y'' = 6Ax + 2B$

$y''' = 6A$

$y^{(4)} = 0$

$$0 + 4(6Ax + 2B) = 2x - 12$$
$$24Ax + 8B = 2x - 12$$

$24A = 2$, $A = \frac{1}{12}$

$8B = -12$, $B = -\frac{3}{2}$

$$y = y_h + y_p = c_1 + c_2 x + c_3 \cos 3x + c_4 \sin 2x + \frac{x^3}{12} - \frac{3x^2}{2}$$

Write a suitable form for y_p

① $y^{(4)} + y'' = 2e^{3-x} + e^{-x}$

$\rightarrow 2e^{3}e^{-x} + e^{-x}$

$e^{-x}\underbrace{(2e^{3}+1)}_{\text{const}}$

$r^4 + r^3 = 0$

$r^3(r+1) = 0$

$r=0 \quad r=0 \quad r=0 \quad r=-1$

$y_1 = 1 \quad y_2 = x \quad y_3 = x^2 \quad y_4 = e^{-x}$

$y_p = x(Ae^{-x})$

$y_p = Axe^{-x}$

② $y^{(4)} + 2y''' + 2y'' = x^2+3+2e^{-x}+xe^{x}\sin x$

$r^4 + 2r^3 + 2r^2 = 0$

$r^2(r^2+2r+2) = 0$

$\Delta = -4$

$r = \dfrac{-b \pm \sqrt{\Delta}}{2a}$

$= \dfrac{-2 \pm 2i}{2}$

$= -1 \pm i$

$r=0 \quad r=0 \quad r=-1\pm i$

$y_1 = 1 \quad y_2 = x \quad y_3 = e^{-x}\cos x \quad y_4 = e^{-x}\sin x$

$y_p = x^2(Ax^2+Bx+C)$

$+ De^{-x}$

$+ (Ex+F)e^{x}\sin x + (Gx+H)e^{x}\cos x$

نقارن كل
سطر على حدا

$y_p = (Ax^2+Bx+C)x^2 + De^{-x} + (Ex+F)e^{x}\sin x + (Gx+H)e^{x}\cos x$

③ $y^{(5)} - 2y^{(4)} + y''' = \sin x + 2\cos x + x$

$r^5 - 2r^4 + r^3 = 0$

$r^3(r^2 - 2r + 1) = 0$

$r^3(r-1)(r-1) = 0$

$r=0 \quad r=0 \quad r=0 \quad r=1 \quad r=1$

$y_1 = 1 \quad y_2 = x \quad y_3 = x^2 \quad y_4 = e^x \quad y_5 = xe^x$

فقط للمقارنة

$y_p = A\sin x + B\cos x$
$x^3(Cx+D)$

$y_p = A\sin x + B\cos x + (Cx+D)x^3$

ملاحظة لو كانت

$= 2x\sin x + 3\cos x + e^x\cos 2x$

$2x\sin x + 3\sin x + e^x\cos 2x$

$(2x+3)\sin x + e^x\cos 2x$

نعتبر $\cos x$ نفس $\sin x$ لتسهيل الحل (للاختصار)

$y_p = (Ax+B)\sin x + (Cx+D)\cos x$
$+ Ee^x\cos 2x + Fe^x\sin 2x$

هيك أنهينا ال undetermined

شروط جهة ال homo انه يكون const coeff

جهة ال non-homo انه يكون exp / sin/cos / poly وبزبط خليط منهم

آلية الحل : نحل ال homo ونوجد y_h

نجد y_p عن طريق كتابة الصيغة العامة لل non-homo

ثم نكمل الحل بالمقارنة أولًا ثم بإيجاد الرموز A,B,... اذا كان بالامتحان solve

عند للمقارنة كل ما وجد تكرار نضرب بـ x

Variation of parameters

Revision

$$\begin{vmatrix} 2 & 4 \\ 6 & 8 \end{vmatrix} = 4\begin{vmatrix} 2 & 1 \\ 6 & 2 \end{vmatrix}$$

هون بنختار
عامل مشترك
من صف/عمود

الفرق بين المحددة
والمصفوفة

$$\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = 2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

هون لازم
من الكل

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

مثلاً

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 4 & 1 \end{vmatrix} = (1)\begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} - (1)\begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} + (1)\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}$$

$$= 6 + 3 - 3 = 6$$

$$W(e^{x}, e^{2x}, e^{-x}) = \begin{vmatrix} e^{x} & e^{2x} & e^{-x} \\ e^{x} & 2e^{2x} & -e^{-x} \\ e^{x} & 4e^{2x} & e^{-x} \end{vmatrix} = (e^{x})(e^{2x})(e^{-x})\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 4 & 1 \end{vmatrix}$$

$$= 6e^{2x}$$

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

ملاحظة الاشارات جوّا المحددة هيك

بهاد الدرس احنا
مهتمين بال 3rd-order

Find the following:

① $W(x, x^2, x^3) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$

$= x \begin{vmatrix} 2x & 3x^2 \\ 2 & 6x \end{vmatrix} - (1) \begin{vmatrix} x^2 & x^3 \\ 2 & 6x \end{vmatrix} + 0$

$= 6x^3 - 4x^3$

$= 2x^3$

② $W_1(x, x^2, x^3) = \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix} = (1) \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^4$

③ $W_2(x, x^2, x^3) = \begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & 1 & 6x \end{vmatrix} = (-1) \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = -2x^3$

④ $W_3(x, x^2, x^3) = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = (1) \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2$

$$y''' + p(x)y'' + q(x)y' + g(x)y = r(x)$$

① $y_h = c_1y_1 + c_2y_2 + c_3y_3$

② $y_p = y_1\int \frac{W_1}{W} r(x) + y_2\int \frac{W_2}{W} r(x) + y_3\int \frac{W_3}{W} r(x)$

③ $y = y_h + y_p$

$r(x)$
بناخدها
بعد ما نتأكد
انه معامل y''' واحد

solve

[1] $x^3y''' - 3x^2y'' + 6xy' - 6y = 24x^4$

undef... ليس
coeff لأن
aren't
constant

1) $r(r-1)(r-2) - 3r(r-1) + 6r - 6 = 0$

$(r-1)[r(r-2) - 3r + 6] = 0$

$(r-1)(r^2 - 5r + 6) = 0$ cauchy-euler

$(r-1)(r-2)(r-3) = 0$

$r = 1$ $\quad r = 2$ $\quad r = 3$

$y_1 = x$ $\quad y_2 = x^2$ $\quad y_3 = x^3$

انتبه

$r(x) = \frac{24x^4}{x^3}$

$r = 24x$

2) $W(x, x^2, x^3) = 2x^3$

$W_1(x, x^2, x^3) = x^4$

$W_2(x, x^2, x^3) = -2x^3$

$W_3(x, x^2, x^3) = x^2$

محلولين
بالصفحة
اللي قبل

$$y_p = y_1\int \frac{W_1}{W} r + y_2 \int \frac{W_2}{W} r + y_3 \int \frac{W_3}{W} r$$

$$= x\int \frac{x^4}{2x^3}(24x) + x^2\int \frac{-2x^3}{2x^3}(24x) + x^3 \int \frac{x^2}{2x^3}(24x)$$

$$= 12x\int x^2\,dx - 24x^2\int x\,dx + 12x^3 \int 1\,dx$$

$$= 12x\left(\frac{x^3}{3}\right) - 24x^2\left(\frac{x^2}{2}\right) + 12x^3(x)$$

$$= 4x^4 - 12x^4 + 12x^4 = 4x^4$$

$$y = y_h + y_p$$

$$= \underbrace{c_1 x + c_2 x^2 + c_3 x^3}_{\text{general}} + \underbrace{4x^4}_{\text{particular}}$$

قوانين لاختصار الوقت

$$W_1 = W(y_2, y_3)$$

$$W_2 = -W(y_1, y_3) = W(y_3, y_1)$$

$$W_3 = W(y_1, y_2)$$

سؤال سنوات

$$W(x, e^x, xe^x)$$

الفكرة هو انه مشتقة xe^x هي مشتقة ضرب يعني مثلاً

$$W(e^x, xe^x)$$

$$= (e^x)(xe^x + e^x) - (xe^x)(e^x)$$

$$= e^{2x}$$

W بالنص سالب

مشتقة الضرب بالنص موجب

$\boxed{2}$ $y''' + y' = \sec x$

$r^3 + r = 0$

$r(r^2+1) = 0$

$y_h = c_1 + c_2 \cos x + c_3 \sin x$

$r = 0 \qquad r = \pm i$

$y_1 = 1 \qquad y_2 = \cos x \qquad y_3 = \sin x$

$$W(1, \cos x, \sin x) = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix}$$

$$= \begin{vmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{vmatrix} = 1$$

$W_1 = W(y_2, y_3) = \cos^2 x + \sin^2 x = 1$

$W_2 = -W(y_1, y_3) = -\cos x$

$W_3 = W(y_1, y_2) = -\sin x$

$$y_p = y_1 \int \frac{W_1}{W} r + y_2 \int \frac{W_2}{W} r + y_3 \int \frac{W_3}{W} r$$

$$= \int 1 (\sec x) dx + \cos x \int \frac{-\cos x}{\cos x} dx + \sin x \int \frac{-\sin x}{\cos x} dx$$

$$y_p = \ln|\sec x + \tan x| - x \cos x + \sin x \ln|\cos x|$$

Chapter 4

Revision

أشياء لازم تكون عارفها بخصوص ال matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 2a+b & 3a-2b \\ 2c+d & 3c-2d \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2a+3b \\ 2c+3d \end{bmatrix}$$

الضرب بكون صف × عمود

$$\begin{bmatrix} 2a \\ 3b \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$$

$2a = 6$ $3b = 12$

$a = 3$ $b = 4$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

تعتبر زي كأنها واحد

$IA = A$

$AI = A$

System of ODE

بهاد ال Chapter مهتمين بال 2x2

$$y_1' = ay_1 + by_2$$

$$y_2' = cy_1 + dy_2$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Matrix form

$$\underline{y}' = A\underline{y}$$

$\underline{y}' = A\underline{y}$ → homo system

$\underline{y}' = A\underline{y} + \underline{F}$ → non-homo system

if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ matrix

المحددة

$|A| \neq 0$ then A^{-1} exists

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

كيف بنحسب ال inverse ؟

$\frac{1}{|A|}$ المحددة

مضروب بسالب

معكوسه

ex: if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$|A| = 4 - 6 = -2$

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

$$A \cdot A^{-1} = A^{-1} \cdot A = I$$

We are dealing with homogeneous system

multiple of $\underline{x}$

$$A\underline{x} = \lambda \underline{x}$$

eiginvalue can be zero

eiginvector can't be zero

اذا أردنا ايجاد λ eiginvalues

$$|A - \lambda I| = 0$$

this is a determinant

اذا أردنا ايجاد $\underline{x}$ eiginvector

$$[A - \lambda I]\underline{x} = \underline{0}$$

this is a matrix

كيف بنطلع ال $A - \lambda I$ ؟

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$$

اختصاراً للوقت فقط اطرح من ال main diagonal λ

eigin values $|A - \lambda I| = 0$

حفظ

eiginvector $[A - I\lambda]\underline{x} = 0$

example Find all eigenvalues and eigenvectors

for $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

$\boxed{1}$ $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(1-\lambda) - 4 = 0$$

$$\lambda^2 - 2\lambda + 1 - 4 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda+1)(\lambda-3) = 0$$

$$\lambda_1 = -1 \qquad \lambda_2 = 3$$

$\boxed{2}$ $[A - \lambda I]\underline{x} = \underline{0}$

مناخذ معادلة وحدة منهم

$$\begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(\lambda_1 = -1)$ $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$2a + b = 0$$

$$b = -2a$$

$$\underline{x}^{(1)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$(\lambda_2 = 3)$ $\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$-2a + b = 0$$

$$b = 2a$$

$$\underline{x}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

eigin values $\lambda_1 = -1$ $\lambda_2 = 3$

eigin vectors $\underline{x}^{(1)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ $\underline{x}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$y_1' = a y_1 + b y_2$$

$$y_2' = c y_1 + d y_2$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\underline{y}' = A \underline{y}$$

$$\underline{y^{(1)}} = e^{\lambda t} \underline{x^{(1)}}$$

$$\underline{y^{(2)}} = e^{\lambda t} \underline{x^{(2)}}$$

Solve $y_1' = y_1 + y_2$

$y_2' = 4y_1 + y_2$

$$\underline{y}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \underline{y}$$

first thing we should find eiginvalues and eiginVectors for $\begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix}$

$\lambda_1 = -1 \longrightarrow \underline{x^{(1)}} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

حليناهم قبل

$\lambda_2 = 3 \longrightarrow \underline{x^{(2)}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

So $\underline{y^{(1)}} = e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\underline{y^{(2)}} = e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$y = c_1 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Solve $\underline{y}' = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \underline{y}$

eigin values: $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-4-\lambda) + 6 = 0$$
$$-4 - \lambda + 4\lambda + \lambda^2 + 6 = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$
$$(\lambda+1)(\lambda+2) = 0$$

$$\lambda_1 = -1 \qquad \lambda_2 = -2$$

eigin vectors: $[A - \lambda I]\underline{x} = \underline{0}$

$$\begin{bmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\lambda_1 = -1$

$$\begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2a - 2b = 0$$
$$a = b$$

$$\underline{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\underline{y}^{(1)} = e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_2 = -2$

$$\begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3a - 2b = 0$$
$$b = \frac{3a}{2}$$

$$\underline{x}^{(2)} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\underline{y}^{(2)} = e^{-2t} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$y = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

So we finished the first part

were $\lambda_1 \neq \lambda_2$

to get λ $\quad |A-\lambda I|=0$ or $\lambda^2-(\operatorname{tr}A)\lambda+|A|=0$

مجموع main diagonal $\quad$ المحددة

to get $\underline{x}$ $\quad [A-\lambda I]\underline{x}=0$

to get $\underline{y}$ $\quad \underline{y}=\underline{x}\,e^{\lambda t}$

Now What if we got a complex eiginValues?

$\lambda=\alpha\pm\beta i$

solve
$$y_1'=-y_1+y_2$$
$$y_2'=-y_1-y_2$$

$$\underline{y}'=\begin{bmatrix}-1 & 1\\ -1 & -1\end{bmatrix}\underline{y}$$

$$|A-I\lambda|=\begin{vmatrix}-1-\lambda & 1\\ -1 & -1-\lambda\end{vmatrix}=0$$

$$(\lambda+1)^2+1=0$$

$$\lambda^2+2\lambda+2=0$$

$$\Delta=4-8=-4$$

$$\lambda=\frac{-2\pm 2i}{2}=-1\pm i$$

$\alpha=-1$

$\beta=1 \rightarrow$ بناخدها بدون اشارة

باخذ حبس λ وحدة (ما بتفرق أنو باخذ)

$$[A-\lambda I]\underline{x} = \begin{bmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-i\vec{a} + \vec{b} = 0$$

$$\vec{b} = i\vec{a}$$

$$\underline{x} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\underline{y} = e^{(-1+i)t}\begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\underline{y} = e^{-t}e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$= e^{-t}(\cos t + i\sin t)\begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$= e^{-t}\begin{bmatrix} \cos t + i\sin t \\ -\sin t + i\cos t \end{bmatrix}$$

$$= e^{-t}\left(\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + i\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}\right)$$

$$= \underbrace{e^{-t}\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}}_{\text{first solution}} + i\,\underbrace{e^{-t}\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}}_{\text{second solution}}$$

$$\underline{y}' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \underline{y}$$

$$|A - \lambda I| = 0 \qquad \begin{vmatrix} -1-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} = 0$$

$$(\lambda+1)^2 + 4 = 0$$

$$(\lambda+1) = \pm 2i$$

$$\lambda = -1 \pm 2i$$

for ex lets take $\lambda = -1+2i$

$$[A - \lambda I]\,\underline{x} = \underline{0}$$

$$\begin{bmatrix} -2i & -4 \\ 1 & -2i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2i\,a - 4b = 0$$

$$b = \frac{1}{2} i a$$

$$\underline{x} = \begin{bmatrix} 2 \\ i \end{bmatrix}$$

$$\underline{y} = \begin{bmatrix} 2 \\ i \end{bmatrix} e^{(-1+2i)t}$$

$$= e^{-t} e^{i2t} \begin{bmatrix} 2 \\ i \end{bmatrix}$$

$$= e^{-t} (\cos 2t + i \sin 2t) \begin{bmatrix} 2 \\ i \end{bmatrix}$$

$$= e^{-t} \begin{bmatrix} 2\cos 2t + i\, 2\sin 2t \\ -\sin 2t + i \cos 2t \end{bmatrix}$$

$$= e^{-t} \left(\begin{bmatrix} 2\cos 2t \\ -\sin 2t \end{bmatrix} + i \begin{bmatrix} 2\sin 2t \\ \cos 2t \end{bmatrix} \right)$$

$$= \underbrace{e^{-t} \begin{bmatrix} 2\cos 2t \\ -\sin 2t \end{bmatrix}}_{\text{first sol}} + i \underbrace{e^{-t} \begin{bmatrix} 2\sin 2t \\ \cos 2t \end{bmatrix}}_{\text{second sol}}$$

What if we got equal eiginvalues $\lambda_1 = \lambda_2$?

$\underline{y}^{(1)} = e^{\lambda t} \underline{x}^{(1)}$ اذا كان الجواب الأول

$\underline{y}^{(2)} = t e^{\lambda t} \underline{x}^{(1)} + \underbrace{\eta}_{\text{generalized eiginvector}} e^{\lambda t}$ الجواب الثاني صيكو

Solve $\underline{y}' = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} \underline{y}$

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$(\lambda-4)(\lambda-2) + 1 = 0$$

$$\lambda^2 - 2\lambda - 4\lambda + 8 + 1 = 0$$

$$\lambda^2 - 6\lambda + 9 = 0$$

$$\lambda_1 = 3 \quad \lambda_2 = 3$$

$$[A - \lambda I]\, \underline{x} = \underline{0}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \underline{x}^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$a + b = 0 \qquad b = -a \qquad \underline{y}^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t}$$

$$\underline{y}^{(2)} = t e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \underline{\eta}\, e^{3t}$$

to find η

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

بنحط $\underline{\eta}$ مكان $\underline{x}$

وبدل $\underline{0}$ بنحط $\underline{x}$

$$a + b = 1$$

$$b = 1 - a \qquad \underline{\eta} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$y = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t} + c_2 \left(t e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t} \right)$$

Solve $\underline{y}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \underline{y}$

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$(\lambda-3)(\lambda-1)+1 = 0$$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$\lambda_1 = 2 \quad \lambda_2 = 2$$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-a - b = 0$$

$$b = -a$$

$$\underline{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$-a - b = 1$$

$$b = -a - 1$$

$$\underline{\eta} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$y = c_1 e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \left(t e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{3t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right)$$

تلخيص سريع

$\lambda_1 \neq \lambda_2$

$|A - \lambda I| = 0$

$\lambda_1 \quad \lambda_2$

$[A - \lambda I]\underline{x} = \underline{0}$

$\underline{x}^{(1)} \quad \underline{x}^{(2)}$

then

$\underline{y}^{(1)} = \underline{x}^{(1)} e^{\lambda t}$

$\underline{y}^{(2)} = \underline{x}^{(2)} e^{\lambda t}$

$\lambda_1 = \lambda_2$

$|A - \lambda I| = 0$

$\lambda_1 = \lambda_2$

$[A - \lambda I]\underline{x} = \underline{0}$

$\underline{x}^{(1)} = \underline{x}^{(2)}$

then

$\underline{y}^{(1)} = \underline{x}^{(1)} e^{\lambda t}$

$\underline{y}^{(2)} = t\underline{x}^{(1)} e^{\lambda t} + \underline{\eta} e^{\lambda t}$

علمًا بأنه

$\underline{\eta}$

توجد عنه طريقة

$[A - \lambda I]\underline{\eta} = \underline{x}$

λ is complex

$|A - \lambda I| = 0$

$\lambda = \alpha \pm \beta i$

$[A - \lambda I]\underline{x} = \underline{0}$

علمًا بأنه

$e^{\pm\theta i} = \cos\theta \pm i\sin\theta$

$e^{2t} e^{-it} \begin{bmatrix} 1 \\ i \end{bmatrix}$

$e^{2t}(\cos t - i\sin t)\begin{bmatrix} 1 \\ i \end{bmatrix}$

$e^{2t}\begin{bmatrix} \cos t & -i\sin t \\ \sin t & i\cos t \end{bmatrix}$

$e^{2t}\left(\begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + i\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}\right)$

Non-Homo System

$$y = y^{(h)} + y^{(p)}$$

$y^{(p)}$ — Undetermined / Variation

$y^{(h)}$ — $\lambda_1 \neq \lambda_2$ / $\lambda_1 = \lambda_2$ / λ is complex

$$\underline{y}' = A\underline{y} + \underline{F}$$

(Solve) $y_1' = -3y_1 + y_2 - 6$

$y_2' = y_1 - 3y_2 + 2$

$$\underline{y}' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \underline{y} + \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

1) $y^{(h)}$ $\quad |A - \lambda I| = 0$; $\lambda^2 - (\text{tr}A)\lambda + |A|$

$$\lambda^2 + 6\lambda + 8 = 0$$

$$(\lambda + 2)(\lambda + 4) = 0$$

$$\lambda_1 = -2 \qquad \lambda_2 = -4$$

$$[A - \lambda I]\underline{x} = \underline{0}$$

$(\lambda_1 = -2)$ $\quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad -a + b = 0$

$b = a$

$$\underline{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$(\lambda_2 = -4)$ $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$a + b = 0 \qquad b = -a$

$\underline{x}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\underline{y}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} \qquad \underline{y}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$

$y^{(h)} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$

$y^{(p)} = \vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

المقارنة هنا تكون بين

eiginvalues_ال

$y' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} -6 \\ 2 \end{bmatrix}$

$-3a_1 + a_2 - 6 = 0$

$a_1 - 3a_2 + 2 = 0$

$-9a_1 + \cancel{3a_2} - 18 = 0$

$a_1 \cancel{- 3a_2} + 2 = 0$

$-8a_1 - 16 = 0$

$a_1 = -2$

$a_2 = 0$

$y = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} + \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

Find the suitable form for y^p

① $\underline{y}' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \underline{y} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$

$\lambda = -2 \qquad \lambda = -4$ حليناه قبل

$$y^{(P)} = \vec{a}\, e^{-2t}$$

$\lambda = -2, \lambda = -4$ ب $\lambda = -2$ نقارن

يوجد تكرار اذا نرفع درجة ال $\vec{a}$ درجة واحدة

$$y^{(P)} = (\vec{a}t + \vec{b})\, e^{-2t}$$

② $y_1' = -3y_1 + y_2 - 2t$

$y_2' = y_1 - 3y_2 + e^{-2t} + 1$

$$\underline{y}' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t}$$

$\lambda = -2, -4$

eiginValues نقارن

$$y^{(P)} = \vec{a}t + \vec{b} + \vec{c}\, e^{-2t}$$

$$y^{(P)} = \vec{a}t + \vec{b} + (\vec{c}t + \vec{d})\, e^{-2t}$$

لو كان مشترك مرتين
بنرفع درجته مرتين

$$(\vec{c}t^2 + \vec{d}t + \vec{e})$$

$$y_1' = 5y_1 + 3y_2 - 2e^{-t} + 1$$

$$y_2' = -y_1 + y_2 + e^{-t} - 5t + 7$$

$$\underline{y}' = \begin{bmatrix} 5 & 3 \\ -1 & 1 \end{bmatrix} \underline{y} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ -5 \end{bmatrix} t + \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$\lambda^2 - 6\lambda + 8$$

$$\lambda = 2 \qquad \lambda = 4$$

$$y(p) = \vec{a}\, e^{-t} \qquad \lambda = -1$$

$$+ \vec{b}\, t + \vec{c} \qquad \lambda = 0$$

Variation of Parameters

$$y^p = \phi \int \phi^{-1} \underline{F}\, dt$$

$\underline{F}$: non-homo part

ϕ : Fundamental matrix

$$\phi = \begin{bmatrix} | & | \\ y^1 & y^2 \\ | & | \end{bmatrix}$$

Solve for yp

$$\underline{y}' = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t}$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 1)(\lambda - 2) = 0$$

$$\lambda = 1 \qquad \lambda = 2$$

$$\begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-a + 2b = 0$$

$$a = 2b$$

$$\underline{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\underline{y}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{t}$$

$$\begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2a + 2b = 0$$

$$b = a$$

$$\underline{x}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\underline{y}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$$

$$\phi = \begin{bmatrix} 2e^{t} & e^{2t} \\ e^{t} & e^{2t} \end{bmatrix}$$

$$\underline{F} = \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix}$$

$$\phi^{-1} = \frac{1}{e^{3t}} \begin{bmatrix} e^{2t} & -e^{2t} \\ -e^{t} & 2e^{2t} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-t} \end{bmatrix}$$

$$y^{(p)} = \begin{bmatrix} 2e^{t} & e^{2t} \\ e^{t} & e^{2t} \end{bmatrix} \int \begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-t} \end{bmatrix} \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix} dt$$

Chapter 5 (power series Method)

the solution of the DE will be a series

Analytic →
- polynomial
- exp / sin x / cos x
- rational functions (الكسور النسبية) → ماعدا الأرقام اللي بتصفر المقام

$$\frac{\text{poly}}{\text{poly}}$$

مثلاً

$$\frac{1}{x^2-3x+2}$$ analytic at all $\mathbb{R}$ exept at $x=1$ $x=2$

Now if $A(x)y'' + B(x)y' + C(x)y = 0$

how do we find something called ordinary points?

we find singular points
- اللي بصفروا المقام بالكسور النسبية
- اللي بصفروا ال $A(x)$

then the ordinary points will be

$\mathbb{R} - \{\text{singular points}\}$

طبعًا هون A,B,C كلهم analytic

ex $y'' + xy' + (x^2+2)y = 0$

$A(x) = 1$

$B(x) = x$

$C(x) = (x^2+2)$

No singular points

the ordinary points are all $x \in \mathbb{R}$

ex $(x-1)y'' + xy' + \frac{1}{x}y = 0$

$A(x) = x-1$ $\quad x-1=0 \quad \boxed{x=1}$

$B(x) = x$

$C(x) = \frac{1}{x}$ $\quad \boxed{x=0}$

singular points $x=1$, $x=0$

ordinary points $\mathbb{R} - \{0,1\}$

ex $x^2 y'' + \frac{1}{x}y' + 2xy = 0$

$A(x) = x^2$ $\quad x^2=0 \quad x=0$

$B(x) = \frac{1}{x}$ $\quad x=0$

$C(x) = 2x$

singular points $(x=0)$

ordinary points $\mathbb{R} - \{0\}$

ordinary points

$\swarrow$ $A(x)/B(x)/C(x)$ are analytic

$\searrow$ not singular points

if x_0 was an ordinary point

and $A(x)y'' + B(x)y' + C(x)y = 0$

then there will be two independent solutions as a power series (taylor series)

$$y = \sum_{n=0} a_n (x - x_0)^n$$

Remark

$$y' = \sum_{n=1} n a_n (x - x_0)^{n-1}$$

$$y'' = \sum_{n=2} (n-1)(n) a_n (x - x_0)^{n-2}$$

انتبه عالحدود !

ex $\sum_{n=2} n(n-1)(x-x_0)^{n-2}$ $\quad n \longrightarrow n+2$

$$\sum_{n+2=2} (n+2)(n+2-1)(x-x_0)^{n+2-2}$$

اذا ضفنا بالداخل

نطرح بال index الحدود

$$\sum_{n=0} (n+1)(n+2)(x-x_0)^n$$

if $\sum_{n=0} a_n x^n = \sum_{n=0} b_n x^n$ then $a_n = b_n$ for all $n \geq 0$

if $\sum_{n=0} a_n x^n = 0$ then $a_n = 0 \quad \forall n \geq 0$

ex

$$2a_1 + 6a_2 x + \sum_{n=2} \left[(n+1)(n+2)a_{n+2} - a_{n-2}\right]x^n = 0$$

$2a_1 = 0$

$6a_2 = 0$

$(n+1)(n+2)a_{n+2} - a_{n-2} = 0$ for all $n \geq 2$

① Find a power series solution of

$y'' + x^2 y = 0$ about $x_0 = 0$

ordinary points of the DE are $x \in \mathbb{R}$

so x_0 is an ordinary point

let $y = \sum_{n=0} a_n x^n$ be a solution

$$y' = \sum_{n=1} n a_n x^{n-1}$$

$$y'' = \sum_{n=2} n(n-1) a_n x^{n-2}$$

$$\sum_{n=2} n(n-1)a_n x^{n-2} + x^2 \sum_{n=0} a_n x^n = 0$$

$$\sum_{n=2} n(n-1)a_n x^{n-2} + \sum_{n=0} a_n x^{n+2} = 0$$

أول اشي بدنا نوحد القوة

$$\sum_{n=0} (n+1)(n+2)a_{n+2} x^n + \sum_{n=2} a_{n-2} x^n = 0$$

ثاني اشي بدنا نوحد ال index

$$2a_2 + 6a_3 x + \sum_{n=2} (n+1)(n+2)a_{n+2} x^n + \sum_{n=2} a_{n-2} x^n = 0$$

$$2a_2 + 6a_3 x + \sum_{n=2} \left[(n+1)(n+2)a_{n+2} + a_{n-2}\right] x^n = 0$$

$2a_2 = 0 \qquad a_2 = 0$

$6a_3 = 0 \qquad a_3 = 0$

$$\boxed{(n+1)(n+2)a_{n+2} + a_{n-2} = 0} \qquad n \geq 2$$

recurrence relation

$$a_{n+2} = \frac{-a_{n-2}}{(n+1)(n+2)} \qquad n \geq 2$$

$a_2 = a_3 = 0$

$(n=2) \qquad a_4 = \dfrac{-a_0}{12}$

$(n=3) \qquad a_5 = \dfrac{-a_1}{20}$

هلأ بنرجع لل solution

$$y = \sum_{n=0} a_n x^n$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a^5 x^5 \ldots$$

$$= a_0 + a_1 x + 0 + 0 - \frac{a_0}{12} x^4 - \frac{a_1 x^5}{20} + \ldots$$

$$= a_0 \underbrace{\left[1 - \frac{x^4}{12} \ldots\right]}_{y_1} + a_1 \underbrace{\left[x - \frac{x^5}{20} \ldots\right]}_{y_2}$$

---

② Find a power series solution of

$y'' - xy = 0$ about $x_0 = 2$

about
at
near
are the same

ordinary points $x \in \mathbb{R}$

$\therefore \ x_0 = 2$ is an ordinary point

$$y = \sum_{n=0} a_n (x-2)^n$$

$$y' = \sum_{n=1} n a_n (x-2)^{n-1}$$

$$y'' = \sum_{n=2} n(n-1) a_n (x-2)^{n-2}$$

$$\sum_{n=2} n(n-1)a_n(x-2)^{n-2} - x\sum_{n=0} a_n(x-2)^n = 0$$

$$(x-2)+2 \quad \sum_{n=0} a_n(x-2)^n$$

$$\sum_{n=0} a_n(x-2)^{n+1} + \sum_{n=0} 2a_n(x-2)^n$$

$$\sum_{n=2} n(n-1)a_n(x-2)^{n-2} - \sum_{n=0} a_n(x-2)^{n+1} - \sum_{n=0} 2a_n(x-2)^n = 0$$

$$\sum_{n=0}(n+1)(n+2)a_{n+2}(x-2)^n - \sum_{n=1} a_{n-1}(x-2)^n - \sum_{n=0} 2a_n(x-2)^n = 0$$

$$2a_2 - 2a_0 + \sum_{n=1}\left[(n+1)(n+2)a_{n+2} - a_{n-1} - 2a_n\right](x-2)^n = 0$$

$$2a_2 - 2a_0 = 0$$

$$a_2 = a_0$$

$$a_{n+2} = \frac{a_{n-1}+2a_n}{(n+1)(n+2)} \qquad n \geq 1$$

$$(n=1) \qquad a_3 = \frac{a_0+2a_1}{(2)(3)} = \frac{a_0}{6} + \frac{2a_1}{6} = \frac{a_0}{6} + \frac{a_1}{3}$$

$$(n=2) \qquad a_4 = \frac{a_1+2a_2}{(3)(4)} = \frac{a_1}{12} + \frac{2a_0}{12} = \frac{a_1}{12} + \frac{a_0}{6}$$

$$y = \sum_{n=0} a_n (x-2)^n$$

$$= a_0 + a_1(x-2) + a_2(x-2)^2 + a_3(x-2)^3 \ldots$$

$$= a_0 + a_1(x-2) + a_0(x-2)^2 + \frac{a_0}{6}(x-2)^3 + \frac{a_1}{3}(x-2)^3 \ldots$$

$$= a_0\left[1 + (x-2)^2 + \frac{(x-2)^3}{6} \ldots\right] + a_1\left[(x-2) + \frac{(x-2)^3}{3} \ldots\right]$$

$$A(x)y'' + B(x)y' + C(x)y = 0$$

if $A(x_0)=0$ then x_0 is a singular point

regular

$$\lim_{x\to x_0} \frac{B(x)}{A(x)}(x-x_0) < \infty \quad (p_0)$$

$$\lim_{x\to x_0} \frac{C(x)}{A(x)}(x-x_0)^2 < \infty \quad (q_0)$$

irregular

if one of these limits doesn't exist then it is irregular

indicial equation

$$r(r-1) + p_0 r + q_0 = 0$$

Find all regular singular points

① $(x^2-4x+3)y'' + 4xy' + 2xy = 0$

$A(x_0) = 0$

هلا هم بس singular

$x^2 - 4x + 3 = 0 \longrightarrow \quad x = 1 \quad x = 3$

At $(x_0 = 1)$

$$\lim_{x\to1} \frac{4x(x-1)}{x^2-4x+3} = \lim_{x\to1} \frac{4x\cancel{(x-1)}}{(x-3)\cancel{(x-1)}} = -2 \quad P_0$$

$$\lim_{x\to1} \frac{2x(x-1)^2}{x^2-4x+3} = \lim_{x\to1} \frac{2x(x-1)^2}{(x-3)\cancel{(x-1)}} = 0 \quad q_0$$

At $(x_0 = 3)$

$$\lim_{x\to3} \frac{4x(x-3)}{x^2-4x+3} = \lim_{x\to3} \frac{4x\cancel{(x-3)}}{\cancel{(x-3)}(x-1)} = 6 \quad P_0$$

$$\lim_{x\to3} \frac{2x(x-3)^2}{x^2-4x+3} = \lim_{x\to3} \frac{2x(x-3)^2}{\cancel{(x-3)}(x-1)} = 0 \quad q_0$$

both $x_0 = 1, 3$ are regular singular points

$\{1, 3\}$

Find the indicial equation at ($x_0 = 1$)

$r(r-1) + P_0 r + q_0 = 0$

$r(r-1) - 2r = 0$

$r^2 - 3r = 0$

don't find the Roots

② $x(x+2)^2 y'' + (x+1) y' + 2xy = 0$

$A(x_0) = 0$

$x(x+2)^2 = 0$

$x_0 = 0 \quad x_0 = -2$

At ($x_0 = 0$)

$$\lim_{x \to 0} \frac{(x+1)\cancel{x}}{\cancel{x}(x+2)^2} = \frac{1}{4} \quad P_0$$

$$\lim_{x \to 0} \frac{(2\cancel{x})\, x^2}{\cancel{x}(x+2)^2} = 0 \quad q_0$$

$x_0 = 0$ is a regular singular point

At ($x_0 = -2$)

$$\lim_{x \to -2} \frac{(x+1)\cancel{(x+2)}}{x(x+2)^{\cancel{2}}} = dne$$

$$\lim_{x \to -2} \frac{(2\cancel{x})\cancel{(x+2)^2}}{\cancel{x}\cancel{(x+2)^2}} = 2$$

$\rightarrow$ irregular

regular singular points $\{0\}$

Find the indicial equation at $(x_0 = 0)$

$r(r-1) + p_0 r + q_0 = 0$

$r(r-1) + \frac{r}{4} = 0$

$r^2 - \frac{3}{4} r = 0$

Frobenius Method

deals with regular singular points

يوجد
فرقيه
رئيسيه
بسيه اكل
الناتج عن ordin..
والناتج عن regul

here $y = \sum_{n=0} a_n (x - x_0)^{n+r}$

$y' = \sum_{n=0} (n+r) a_n (x - x_0)^{n+r-1}$

① هنا at least one solution
هناك حليه عالاكيد

$y'' = \sum_{n=0} (n+r-1)(n+r) a_n (x - x_0)^{n+r-2}$

② هنا index
بعد الاشتقاق
تبقى $n = 0$

solve

① $2x^2 y'' + (x^2 - x) y' + y = 0$ near $(x_0 = 0)$

$A(0) = 0$ means $x_0 = 0$ is a singular point and regular because we are meant to solve

let $y = \sum_{n=0} a_n x^{n+r}$ $a_0 \neq 0$ be a solution

$$y' = \sum_{n=0} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0} (n+r-1)(n+r) a_n x^{n+r-2}$$

$$\sum_{n=0} 2(n+r-1)(n+r) a_n x^{n+r} + \sum_{n=0} (n+r) a_n x^{n+r+1}$$

$$- \sum_{n=0} (n+r) a_n x^{n+r} + \sum_{n=0} a_n x^{n+r} = 0$$

$$\sum_{n=0} 2(n+r-1)(n+r) a_n x^{n+r} + \sum_{n=1} (n+r-1) a_{n-1} x^{n+r}$$

$$- \sum_{n=0} (n+r) a_n x^{n+r} + \sum_{n=0} a_n x^{n+r} = 0$$

$$2(r-1)(r) a_0 x^r - r a_0 x^r + a_0 x^r = 0$$

$$[2(r-1)(r) - r + 1] a_0 = 0$$

$$a_0 \neq 0$$

$$2r^2 - 2r - r + 1 = 0$$

$$2r^2 - 3r + 1 = 0$$

$$[2(n+r-1)(n+r) a_n + (n+r-1) a_{n-1} - (n+r) a_n + a_n] = 0$$

$n \geq 1$

$$\left[2(n+r-1)(n+r) - (n+r)+1\right] a_n + (n+r-1)\, a_{n-1} = 0$$

$$\left[2(n+r-1)(n+r) - (n+r-1)\right] a_n + (n+r-1)\, a_{n-1} = 0$$

$$\cancel{(n+r-1)}\left(2(n+r)-1\right) a_n = -\cancel{(n+r-1)}\, a_{n-1}$$

$$a_n = \frac{-a_{n-1}}{2(n+r)-1} \qquad n \geq 1$$

reccrunce relation

$2r^2 - 3r + 1 = 0 \qquad (2r-1)(r-1)$ $\qquad r = 1$, $r = \frac{1}{2}$

$(r=1)$ $\qquad a_n = \frac{-a_{n-1}}{2n+1} \qquad n \geq 1$

$n=1 \qquad a_1 = \frac{-a_0}{3}$

$n=2 \qquad a_2 = \frac{-a_1}{5} = \frac{a_0}{15}$

$$y = \sum_{n=0} a_n x^{n+1}$$

$$y = a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 \ldots$$

$$= a_0 x - \frac{a_0}{3} x^2 + \frac{a_0}{15} x^3 \ldots$$

$$y = a_0 \underbrace{\left[x - \frac{x^2}{3} + \frac{x^3}{15} \cdots \right]}_{y_1}$$

$\left(r = \frac{1}{2}\right)$

$$a_n = \frac{-a_{n-1}}{2n} \qquad n \geq 1$$

$n=1$ $$a_1 = \frac{-a_0}{2}$$

$n=2$ $$a_2 = \frac{-a_1}{4} = \frac{a_0}{8}$$

$$y = \sum_{n=0} a_n x^{n+\frac{1}{2}}$$

$$= a_0 x^{\frac{1}{2}} + a_1 x^{\frac{3}{2}} + a_2 x^{\frac{5}{2}} + a_3 x^{\frac{7}{2}} \cdots$$

$$= a_0 x^{\frac{1}{2}} - \frac{a_0}{2} x^{\frac{3}{2}} + \frac{a_0}{8} x^{\frac{5}{2}} \cdots$$

$$= a_0 \left[\underbrace{\sqrt{x} - \frac{x^{\frac{3}{2}}}{2} + \frac{x^{\frac{5}{2}}}{8} \cdots}_{y_2} \right]$$

هاد مختلفة عن اللي قبل

كل recurrance أعطانا solution واحد

$$a_n = \frac{-a_{n-1}}{2n}$$

$$a_1 = \frac{-a_0}{2 \cdot 1}$$

$$a_2 = \frac{-a_1}{2 \cdot 2} = \frac{a_0}{2 \cdot 2 \quad 2 \cdot 1} = \frac{a_0}{2^2 \quad 2 \cdot 1}$$

$$a_3 = \frac{-a_2}{2 \cdot 3} = \frac{-a_0}{2^3 \quad 3 \cdot 2 \cdot 1}$$

so the general form for this reccrunce relation

is $$a_n = \frac{(-1)^n a_0}{2^n \; n!}$$

$\boxed{a_0 \neq 0}$

regular singular

$r_1 > r_2$

$r_1 - r_2 \notin \mathbb{Z}$

$$y_1 = \sum_{n=0} a_n x^{n+r_1}$$

$$y_2 = \sum_{n=0} a_n x^{n+r_2}$$

$r_1 > r_2$

$r_1 - r_2 \in \mathbb{Z}$

$$y_1 = \sum_{n=0} a_n x^{n+r_1}$$

$$y_2 = c \ln x \, y_1 + \sum_{n=0} a_n x^{n+r_2}$$

c may be equal to zero

يعني هنا y_2 قد تحتوي على ln وقد لا تحتوي

$r_1 = r_2 = r$

$$y_1 = \sum_{n=0} a_n x^{n+r}$$

$$y_2 = y_1 \ln x + \sum_{n=1} a_n x^{n+r}$$

هنا دائمًا y_2 فيه ln

② $xy'' - xy' + y = 0 \quad$ at $\quad x_0 = 0$

$\boxed{1}$ $A(x_0) = 0$

$x = 0 \qquad$ it is singular

$$\lim_{x \to 0} \frac{-x}{x}(x) = 0 \quad P_0$$

$$\lim_{x \to 0} \frac{x^2}{x} = 0 \quad q_0$$

$\boxed{2}$ $r(r-1) + P_0 r + q_0 = 0$

$r(r-1) = 0$

$r = 0 \qquad r = 1$

لما شني

$r_1 - r_2 \in \mathbb{Z}$

$y_1 = \sum_{n=0} a_n x^{n+r_1}$

$y_2 = c\,y_1 \ln x + \sum_{n=0} a_n x^{n+r_2}$

$\boxed{3}$ $y_1 = \sum_{n=0} a_n x^{n+1}$

$y_1' = \sum_{n=0} (n+1) a_n x^n$

$y_1'' = \sum_{n=1} n(n+1) a_n x^{n-1}$

$$\sum_{n=1} n(n+1) a_n x^n - \sum_{n=0} (n+1) a_n x^{n+1} + \sum_{n=0} a_n x^{n+1} = 0$$

$$\sum_{n=0} (n+1)(n+2)\, a_{n+1}\, x^{n+1} + \sum_{n=0} [(-1-n)+1] a_n x^{n+1} = 0$$

$$\sum_{n=0} \left[(n+1)(n+2)\, a_{n+1} - (n)\, a_n\right] x^{n+1} = 0$$

$$(n+1)(n+2)\, a_{n+1} - n\, a_n = 0 \qquad n \geqslant 0$$

$$a_{n+1} = \frac{(n)\, a_n}{(n+1)(n+2)} \qquad n \geqslant 0$$

$$a_1 = 0$$

$$a_2 = \frac{a_1}{(2)(3)} = 0$$

$$a_n = 0 \qquad n \geqslant 0$$

$$y_1 = a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 \ldots$$

$$y_1 = a_0 x$$

$$y_2 = Cx \ln x + \sum_{n=0} a_n x^n$$

x_0 is an ordinary point		$y=\sum_{n=0} a_n (x-x_0)^n$
x_0 is regular singular point	$r_1 > r_2$, $r_1 - r_2 \notin \mathbb{Z}$	$y_1=\sum_{n=0} a_n (x-x_0)^{n+r_1}$ $y_2=\sum_{n=0} a_n (x-x_0)^{n+r_2}$
	$r_1 > r_2$, $r_1 - r_2 \in \mathbb{Z}$	$y_1=\sum_{n=0} a_n (x-x_0)^{n+r_1}$ $y_2 = c y_1 \ln x + \sum_{n=0} a_n (x-x_0)^{n+r_2}$
	$r_1 = r_2 = r$	$y_1=\sum_{n=0} a_n (x-x_0)^{n+r}$ $y_2 = y_1 \ln x + \sum_{n=1} a_n (x-x_0)^{n+r}$