

Weekly Notebook

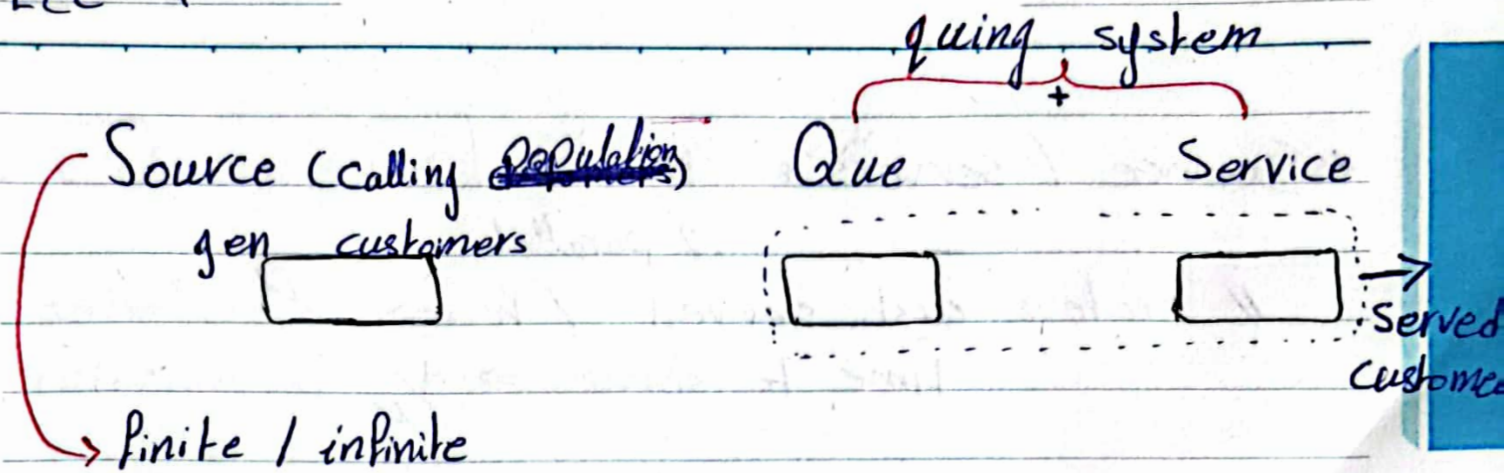
OR2

First
Semester
2024



10/10/23

Lec 1



Que : 1. single / 1. finite / 1. arrival rate
2. multiple / 2. infinite / 2. inter-arrival rate

→ I care about rate $\Rightarrow$ customers $\Rightarrow$ rate = λ $\frac{\text{arrival}}{\text{unit of time (h)}}$

$\frac{1}{\lambda} \Rightarrow$ time between arrivals (interarrival rate)(time)

how we act due to ques :

jockey	Butt	Reneg
moves from one to one	Doesn't enter	enters then leaves

Que discipline how, order I'll call people to service
FIFO FILO LIFO rash order (emergency)
Random

* Continuous & range (time)

descrete & # of cust

time $\Rightarrow$ exponential

Poisson $\Rightarrow$ # of events in an interval & arrival rate (customers)

* Service / servers : 1. series

2 parallel

// rate : cust served / h $\Rightarrow \mu$

time to serve $\Rightarrow 1/\mu$

* Survival time : departure vs arrival // que

of people in the que sys $\geq$ // u u u u que $\geq$ service
الرجاء

改善

KAIZEN

TEAM

Sun, 15/10

Lec 2

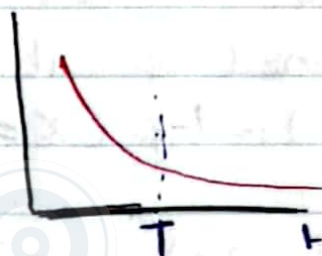
λ [S] [Q] [S] μ

arrival $\rightarrow$ poisson

interarrival $\rightarrow$ exponential

Exponential distribution

$$P(t) = \lambda e^{-\lambda t} \quad t \geq 0$$



after integrating we get

$$P(t < T) = 1 - e^{-\lambda T}$$

$$P(T_1 < t < T_2) = e^{-\lambda T_1} - e^{-\lambda T_2}$$

$$P(t > T) = e^{-\lambda T} = 1 - P(t < T)$$

(t) is time between 2 successive

Poisson distribution (discrete)

(λ) is # of arrivals / time

of arrivals is random variable (متغير)

$$P(X = x) = \frac{(\lambda t)^x}{x!} e^{-\lambda t} \quad x = 0, 1, 2, 3, \dots$$

sometimes we might have decimals since it's avg

9/2h = 4.5/h even tho it's discrete

* Ex 2)) $\lambda = 3$ $-\frac{3}{60} \times 10$

$$a) P(t < 10) = 1 - e^{-3 \times \frac{10}{60}} = 1 - e^{-0.5}$$

if we want to solve using poisson, we can't assume $x=1$ since we have no idea how many arrivals in the given period so it's $P(x=1) + P(x=2) + \dots$
 $= P(x \geq 1) = 1 - P(x=0)$

$$b) P(t > 30) = e^{-3(0.5)}$$

using poisson it's $P(x=0)$ since none in the interval the first comes after

$$c) 0$$

$$d) 1 - e^{-3 \times \frac{1}{60}}$$

e) $1 \rightarrow 2$ as if we had 0 arrivals in the first min

$$f) 0 \rightarrow 2 \text{ or } < 2$$

Lec

$$\lambda = 30/h$$

- Ex 3 a) poisson since we want # not time

$$p(x=60) = \frac{(\lambda t)^x}{x!} e^{-\lambda t}$$
$$= \frac{(30 \cdot 2)^{60}}{60!} e^{-60}$$

* Poisson :

(mean) expected value	$E(x) = \lambda$
variance	$V(x) = \lambda$
Std dev	$= \sqrt{\lambda t}$

if he asks
if it's poisson
we calculate
mean & var if
they're close then

b) $E(x) = 30 \cdot 4 = 120$ cup

c) $\sqrt{120} = 10.95$

d) exponential then it's true between
 $p(1 < t < 3)$

$$e^{-\frac{30}{60}} - e^{-\frac{30 \cdot 3}{60}} = e^{-0.5} - e^{-1.5}$$

* exponential :

$$E(T) = \frac{1}{\lambda}$$

$$V(T) = \frac{1}{\lambda^2}$$

λ is always # of arrivals per hr
* when given time it's $\frac{1}{\lambda}$

18.4 pure birth - Death models

pure birth & arrival

a) $\frac{1}{\lambda} = 12 \text{ min}$

$$\lambda = \frac{\text{birth}}{12 \text{ min}} \cdot \frac{60 \text{ min}}{1 \text{ h}} \cdot \frac{24 \text{ h}}{1 \text{ d}} \cdot \frac{365 \text{ d}}{1 \text{ year}}$$

$$\text{b) } p(X=0) = \frac{(\lambda t)^x}{x!} e^{-\lambda t} \Rightarrow p(X=0) = \frac{(120)^0}{0!} e^{-120} = e^{-120}$$

or

$$p(X \geq 1) = e^{-\lambda t} = e^{-120}$$

c) due to lack of memory we only care about the 10 left in 1 h

$$p(X=10) = \frac{(5 \times 1)^{10}}{10!} e^{-5}$$

من بقی

Pure death & only departals occurs

N : # of customers at time = 0 & this will decrease by time: $(N-1, N-2, \dots, n-3, 2, 1, 0)$

truncated poisson dist (since $x = 0 \dots N$)

* $P(t)$: prob that we have n customers in the sys (que + service) at specific time ($0 \leq n \leq N$)

الموجود في النظام

$$P(n) = \frac{(\mu t)^{N-n}}{(N-n)!} e^{-\mu t}, \quad n = 0, 1, 2, \dots, N$$

بما أن N و n متغيران

Ex 5: 3 on avg

$$P_0 = 1 - \sum_{n=1}^N P_n$$

so now Ex 5.8

$$P_n (n \leq 5) = P_0 + P_1 + P_2 + P_3 + P_4 + P_5$$

so we have to calculate the whole

tho we don't need it here

starting from	sold	stock
$(N-n)$	n	
0		$N=18$
1		17
2		16
		15
		14
		13
		12
		11
		10
		9
		8
		7
		6
		5
		4
		3
		2
		1
		0

for day one = $\mu t = 3 \times 1 = 3$

$$2 = 6$$

$$3 = 4$$

$$1$$

$$1$$



assume we are dealing with day 3

for day 3

$$P_1 = \frac{9^{17} e^{-9}}{17!}$$

N-n	n	P _n
0	18	.000123

1	17	.001106
---	----	---------

2	16	.004998
---	----	---------

$$P_5 = \frac{9^{13} e^{-9}}{13!} = .050375$$

3	15	.01499
---	----	--------

4	14	.03373
---	----	--------

5	13	.06072
---	----	--------

$$P_0 = 1 - .994578 = .005422$$

6	12	.09109
---	----	--------

7	11	.1171
---	----	-------

$$P_H (X \leq 5) = P_0 + P_1 + P_2 + P_3 + P_4 + P_5$$

8	10	.13175
---	----	--------

9	9	.13175
---	---	--------

$$= .12433 \text{ for day 3}$$

10	8	.11858
----	---	--------

we have to repeat this for

11	7	.09702
----	---	--------

the 7 days

12	6	.07276
----	---	--------

13	5	.050375
----	---	---------

14	4	.03238
----	---	--------

15	3	.01973
----	---	--------

16	2	.010424
----	---	---------

17	1	.005486
----	---	---------

18	0	
----	---	--

$$b) E(n) = \sum_{n=0}^{18} n p_n$$

المسألة
النسبة

Expected value

$$E(x) = \sum x p(x)$$

$$\sum x p(x)$$

المسألة
sum product

$$= 0 p_0 + 1 p_1 + 2 p_2 + \dots + 18 p_{18} = 0.664$$



مستقلا ان rate ممكن يتغير حسب n اي عندى مودايقا ثابت λ_n

المحتملين ومانى فى small interval Δt



18.5 General poisson models (arrivals & departures)

$n(t)$ State of the system: # of customers in the system

p_n : prob that we have n customers in the system

$$p_n(t) =$$

Steady state: يعني كل sys فى الـ warm up و SS و افترج بنزل

p_n فبا steady state ما بغير في الوقت حتمى زي ثابت

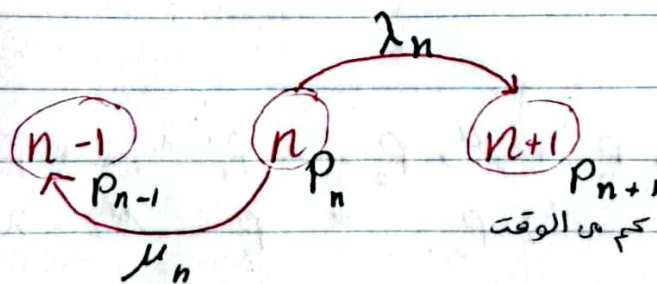
λ_n μ_n n كم في الوقت بكون ب state n * احتمالي بكون p_n عندى مالق n customers

Transition rate diagram

$p_5 = 0.3$ يعني 30% في الوقت عندى 5

prob of more than 1 event during small time interval $h = \text{zero as } h \rightarrow 0$

ببطلوا يعني التاني

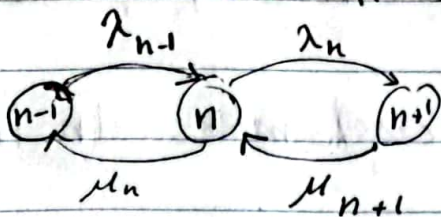


بي اي بيا لاه وقت قليله

كم في الوقت

Total transitions into n

المجموع الانتقالات التي تدخل في n



$$\lambda_{n-1} p_{n-1} + \mu_{n+1} p_{n+1}$$

Tot " out of n

$$\lambda_n p_n + \mu_n p_n$$

now in steady state, Balance Eqn

Flow rate in = Flow rate out

(Tot transitions into n = tot " out of n)

$$\lambda_{n-1} p_{n-1} + \mu_{n+1} p_{n+1} = \lambda_n p_n + \mu_n p_n$$

$$= p_n (\lambda_n + \mu_n)$$

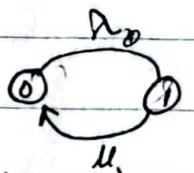
Ex $n=1$



$$\lambda_0 p_0 + \mu_2 p_2 = p_1 \lambda_1 + p_1 \mu_1$$

$$\lambda_0 p_0 + \mu_2 p_2 = p_1 (\mu_1 + \lambda_1) \text{ on next page}$$

* For 0 $\mu_1 = \lambda_0 p_0 \Rightarrow p_1 = \frac{\lambda_0 p_0}{\mu_1}$



P_1 from state 0

$$(\mu_1 + \lambda_1) \frac{\lambda_0}{\mu_1} P_0 = \lambda_0 P_0 + \frac{\lambda_0 \lambda_1}{\mu_1} P_0$$

$$P_0 \cancel{\lambda_0} + P_2 \mu_2 = \cancel{\lambda_0 P_0} + \frac{\lambda_0 \lambda_1}{\mu_1} P_0$$

$$P_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} P_0 \quad \#$$

again we have λ_n arrivals rate & μ_n departure rate

For state 2: 

$$P_1 \lambda_1 + P_3 \mu_3 = \underbrace{P_2 \mu_2 + P_2 \lambda_2}_{P_2 (\mu_2 + \lambda_2)}$$

So $P_3 = \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} P_0$ 3 terms starting from 0 ending (less) with 3

eventually:

$$P_n = \frac{\lambda_0 \lambda_1 \dots \lambda_{(n-1)}}{\mu_1 \mu_2 \dots \mu_n} P_0$$

$$P_0 = 1 - \sum_{n=1}^{\infty} P_n$$

Ex 6.8 1+2+3, "Lil 5 added from problems

$$\lambda = 10 \text{ cust/h}$$

$$\cancel{\frac{1}{12}} \Rightarrow \frac{1}{\mu} = 12 \text{ min} \Rightarrow \mu = \frac{1 \text{ cust}}{12 \text{ min}} \times \frac{60 \text{ min}}{1 \text{ hr}} = \boxed{\frac{5 \text{ cust}}{\text{hr}}}$$

We have steady state so we are ind from time, & μ, λ depend on customers in the system, yet we are told nothing about λ so it is 10 for all $[\lambda = 10]$

But service isn't const, it depends on # of customers $\mu = \begin{cases} 5 & n=1, 2, 3 \\ 10 & n=4, 5, 6 \\ 15 & n=7, 8, \dots \end{cases}$

$$\text{now, } p_1 = \frac{\lambda}{\mu_1} p_0 = \frac{10}{5} p_0 = \boxed{2 p_0}$$

$$p_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} p_0 = \frac{10 \times 10}{5 \times 5} p_0 = \frac{100}{25} p_0 = \boxed{4 p_0}$$

$$p_3 = \frac{1000}{125} p_0 = \boxed{8 p_0}$$

$$p_n = \frac{10000}{125 \times 10} = \boxed{8 p_0}$$

$$p_5 = \frac{10^{8^3}}{125 \times 10^2} = \boxed{8 p_0}$$

$$p_6 = \boxed{8 p_0}$$

$$p_7 = \frac{10^7}{125 \times 10^3 \times 15} = 8p_0 + \frac{10}{15} = \frac{80}{15} p_0 = \left[\frac{16}{3} p_0 \right] \text{ and so on}$$

$$p_0 = 1 - \sum_{n=1}^{\infty} p_n$$

$$2p_0 + 4p_0 + 8p_0 + 8p_0 + 8p_0 + 8p_0 + 8 \cdot \frac{2}{3} p_0 + 8 \left(\frac{2}{3} \right)^2 p_0 + 8 \left(\frac{2}{3} \right)^3 p_0 + \dots$$

$$1 + p + p^2 + p^3 + \dots, p < 1 = \boxed{\frac{1}{1-p}} \quad \text{Geo}$$

$$\text{So } 2p_0 + 4p_0 + 8p_0 + 8p_0 + 8p_0 + 8p_0 \left[1 + \frac{2}{3} + \left(\frac{2}{3} \right)^2 + \left(\frac{2}{3} \right)^3 + \dots \right]$$

$$= 30p_0 + 8p_0 \left[\frac{1}{1 - \frac{2}{3}} \right] = 54p_0 \quad \#$$

$$\text{Now } p_0 + \sum_{n=1}^{\infty} p_n = 1 \Rightarrow p_0 + 54p_0 = 1 \Rightarrow 55p_0 = 1$$

$$\text{So } \boxed{p_0 = \frac{1}{55}} \quad \& \text{ we get the rest by substituting}$$

$$p_1 = 2p_0 = \frac{2}{55}, \quad p_2 = 4p_0 = \frac{4}{55}$$

$$2) p_1 + p_2 + p_3 + \dots = p_0 + 2p_0 + 4p_0 + 8p_0 = \frac{15}{55} = .2727$$

الغیر الاولیٰ

علیہ خلاف میں اصل

و لا كذا p متساوي يكون غير
uniform dist

3) n	Idle		Busy
0	3	p_0	0
1	2	$p_1 + p_2 + p_3 = 14p_0$	1
2	2		1
3	2		1
4	1	$p_4 + p_5 + p_6 = 24p_0$	2
5	1		2
6	1		2
7	0	$p_7 + p_8 + p_9 + \dots$	3
$\vdots$	$\vdots$	$1 - \sum_{n=0}^{\infty} p_n = \frac{16}{55}$	$\vdots$

$$E(n) = \sum_{n=0}^{\infty} n \cdot p_n \Rightarrow E(\text{Idle cost}) = 0 \left(\frac{1}{55} \right) + 1(14p_0) + 2(24p_0) + 3p_0$$

$$= \frac{55}{55} = 1 \text{ #}$$

4) we write the event & corresponding probability في الجدول

$$5) \frac{110}{55} = 2 \rightarrow \text{avg busy} = 2 \leftarrow \text{avg idle} = 1 \text{ يا بالمثل}$$

$$E(\text{Busy}) = 0 \left(\frac{1}{55} \right) + 1 \left(\frac{14}{55} \right) + 2 \left(\frac{24}{55} \right) + 3 \left(\frac{16}{55} \right)$$

$$= \frac{110}{55}$$

18.6 Kendall Lee notation $(a/b/c) : (d/e/f)$
 $(M/D/3) : (FCFS/\infty/\infty)$

a & arrivals dist

b 8 Departures u

cost of parallel servers 1-2

de Que discipline

ex max num allowed in the sys $q + r$

Pe size of calling source

St. St performance measures :

1) L_s : Expected # of customers in the system $\rightarrow$ (q. 4.5)

$$L_s = E(n) = \sum_{n=0}^{\infty} n p_h^{-n}$$

2) $L_{\frac{7}{8}}$: u u u u u u Que

$$L_z = \sum_{n=c+1}^{\infty} (n-c) \rho_n$$

C: # parallel servers

$n < c \rightarrow \text{No Q}$

$n = c \rightarrow No @$

$$n > c \rightarrow Q \checkmark$$

3) $\bar{C}$: ~~Exp~~ Exp # of busy servers

$$\bar{C} = L_s - L_f = p$$

4) W_s : Waiting time in the sys =

$$W_s = W_q + \text{Service time} = W_q + 1/\mu$$

littles law

$$L_s = \lambda W_s$$

Flow rate & residence time
Sometimes we use λ or $\lambda_{\text{effective}}$

$$L_q = \lambda W_q$$

Ex 8 $C=2$, $\lambda = 8 \text{ cust/h}$, $p_0 = .4$, $p_1 = .3$, $p_2 = .2$
 $p_3 = .1$ as $\sum p_i = 1$

$$L_s = 1 \cdot 1 + 2 \cdot 1 + 3 \cdot 1 + 4 \cdot 1 = (0 \cdot .4) + (1 \cdot .3) + (2 \cdot .2) + (3 \cdot .1)$$

$$L_q = 0.1 = \left(\sum_3 1 \cdot (.1) = .1 \right)$$

$$p = .4 = (L_s - L_q)$$

$$W_s = \text{From littles law} = \frac{L_s}{\lambda} = \frac{1}{8} = .125$$

$$W_q = \frac{L_q}{\lambda} = \frac{.1}{8} = .0125$$

$$1/\mu = W_s - W_q = .1125 \text{ hr} = 6.75 \text{ min}$$

or. From $p = \frac{\lambda}{\mu}$

$$\lambda \approx 18.6 - 18 : \lambda = 6 \text{ cars/hr.}$$

$$\mu = \frac{1 \text{ car}}{30 \text{ min}} \times \frac{60 \text{ min}}{\text{hr}} = 2 \text{ cars/hr} \text{ but it depends on } n$$

$$\text{So it's not const. } \mu_n = \begin{cases} 2n & , n=1, 2, 3, 4, 5 \\ 10 & , n=6, 7, 8 \end{cases}$$

٨ - لو عندي سيارة وحدة و بارك واحد شغال في بالستة بطوع 2



لو ٤ و ٥ و ٦ و ٧ و ٨ و ٩ و ١٠ و ١١ و ١٢ و ١٣ و ١٤ و ١٥ و ١٦ و ١٧ و ١٨ و ١٩ و ٢٠ و ٢١ و ٢٢ و ٢٣ و ٢٤ و ٢٥ و ٢٦ و ٢٧ و ٢٨ و ٢٩ و ٣٠ و ٣١ و ٣٢ و ٣٣ و ٣٤ و ٣٥ و ٣٦ و ٣٧ و ٣٨ و ٣٩ و ٤٠ و ٤١ و ٤٢ و ٤٣ و ٤٤ و ٤٥ و ٤٦ و ٤٧ و ٤٨ و ٤٩ و ٥٠ و ٥١ و ٥٢ و ٥٣ و ٥٤ و ٥٥ و ٥٦ و ٥٧ و ٥٨ و ٥٩ و ٦٠ و ٦١ و ٦٢ و ٦٣ و ٦٤ و ٦٥ و ٦٦ و ٦٧ و ٦٨ و ٦٩ و ٧٠ و ٧١ و ٧٢ و ٧٣ و ٧٤ و ٧٥ و ٧٦ و ٧٧ و ٧٨ و ٧٩ و ٨٠ و ٨١ و ٨٢ و ٨٣ و ٨٤ و ٨٥ و ٨٦ و ٨٧ و ٨٨ و ٨٩ و ٩٠ و ٩١ و ٩٢ و ٩٣ و ٩٤ و ٩٥ و ٩٦ و ٩٧ و ٩٨ و ٩٩ و ١٠٠

a) we will find from $p_0 \rightarrow p$

$$p_1 = \frac{\lambda_0}{\mu_1} p_0 = \frac{6}{2} p_0 = 3 p_0$$

$$p_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} p_0 = \frac{6 \times 6}{2 \times 4} = \frac{9}{2} p_0$$

$$p_3 = \frac{9}{2} \times \frac{6}{6} = \frac{9}{2} p_0$$

$$p_4 = \frac{9}{2} \times \frac{6}{8} = \frac{27}{8} p_0$$

$$p_5 = \frac{27}{8} \times \frac{6}{10} = \frac{81}{40} p_0$$

$$p_6 = \frac{81}{40} \times \frac{3}{5} = \frac{243}{200} p_0$$

$$p_7 = \frac{729}{1000} p_0$$

$$p_8 = \frac{2187}{5000} p_0$$

now we want to find $p_0 = 1 - \sum_{n=1}^{n=8} p_n$

$p_0 = 1 - 19.7814 p_0 = \boxed{.04812}$ now we substitute in the previous page

b)) $\lambda_{eff} = \lambda - \lambda_{lost}$ { $\lambda_{lost} = \lambda \cdot p_8 = 6 \cdot .2$

$6 - .12 = 5.88 \text{ cars/hr}$

$= .12 \text{ car/hr}$ بالاعانة كم سيارة يعني وبتروح لانك ما ح تقدر تدخل السيارات لما عني 8 جوا عيون ب امتقان

λ_{eff} يعني كم سيارة هي الي بييجوا فعلا بنخلوها

So actually the reason behind having λ & λ_{eff} is that i'm limited with the # of cars that enters, not ∞

~~C))~~ avg # of cars in the lot ~~we must use λ_{eff}~~

$$L_s = \sum_{n=0}^{n=8} n p_n = 0 p_0 + 1 p_1 + 2 p_2 + 3 p_3 + 4 p_4 + 5 p_5 + 6 p_6 + 7 p_7 + 8 p_8$$

$$= 3.1286 \text{ cars}$$

we must use d)) $W_q = W_s - 1/\mu$ to calculate $W_s = L_s / \lambda_{eff} = .53$ the actually entered

$.53 - 1/2 = \boxed{.0326 \text{ hr}}$

او $L_q = L$

$L_q = \sum_{n=0}^{n=8} (n-1) p_n$

كوبي محصور ب (park) $\sum_{n=0}^8$ ف الله لانك بس الي خطاوا وبتروح 8 بدون مع لسانك ال λ_{eff}

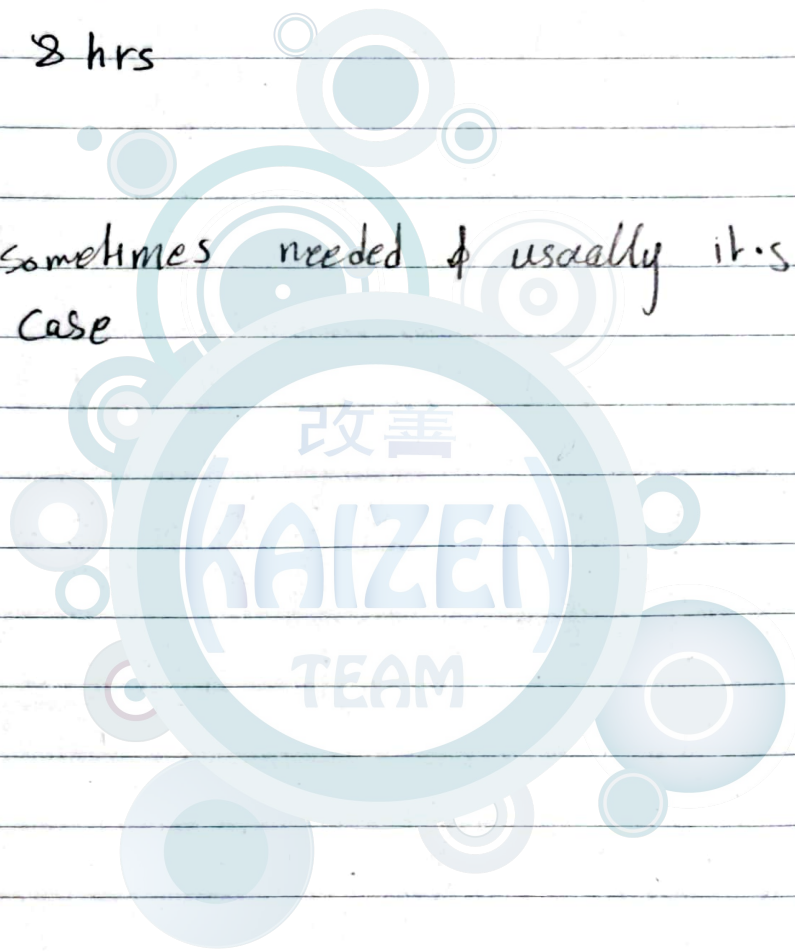
$$e) \bar{C} = L_s - L_q = 2.94 \text{ (or } \frac{\lambda_{eff}}{\mu} \text{)}$$

یعنی یہ صاف حساب ہے
الجزیہ کے منسوب

$$P) \frac{\text{used}}{5} = \frac{\bar{C}}{5} = \frac{2.94}{5}$$

$$F) \lambda_{lost} \neq 8 \text{ hrs}$$

μ_n is sometimes needed & usually it's a multi servers case



The single server models

1) $(M/M/1) : (GD/\infty/\infty)$

$$\boxed{\lambda_n = \lambda} \Rightarrow \boxed{\rho = \frac{\lambda}{\mu}} = \frac{10 \text{ cars/hr}}{20 \text{ cars/hr}} \text{ for example}$$

$$\rho_n = \frac{\lambda_0}{\mu_1} \dots \frac{\lambda_{n-1}}{\mu_n} \quad \rho_0 = \rho^n \rho_0$$

$$\boxed{\rho_0 = 1 - \rho} \quad \rho_0 \frac{1}{1 - \rho} = 1 \Rightarrow \rho_0 = 1 - \rho$$

$$\boxed{\rho_n = \rho^n (1 - \rho)} \quad \rho < 1$$

$$\boxed{E(n) = L_s = \sum_{n=0}^{\infty} n \rho_n = \frac{\rho}{1 - \rho}}$$

فرضاً $\rho < 1$ ففرضنا $\rho < 1$ ورياضياً، ففرضنا $\rho < 1$ growing faster than ρ or $\frac{10 \text{ cars/hr}}{5 \text{ cars/hr}}$

$$\boxed{L = L_s - L_{\text{being served}} = L_s - \bar{c} = L_s - \rho}$$

steady state for $\rho < 1$

$$L_s = L_q + \bar{c} \Rightarrow$$

$$\boxed{L_s = W_s \cdot \lambda}$$

$$\boxed{L_q = W_q \cdot \lambda}$$

λ	1	5	4
ρ	.1	.5	.9
L_s	.1111	1	9
L_q	.01111	.5	8.1
W_s	.1111 hr	.2	1 hr
W_q	.01111 hr	.1	.9

Ex sports $\lambda = 105 \text{ cust/hr}$ $\mu = \frac{1 \text{ cust}}{30 \text{ sec}} \times \frac{60 \text{ sec}}{\text{min}} \times \frac{60 \text{ min}}{\text{hr}} = 120 \text{ cust/hr}$

Service rate $= \mu = 120 \text{ cust/hr}$

The utilization $= \rho = \frac{105}{120} = .875$

$L_s = \frac{\rho}{1-\rho} = 7 \text{ cust}$

$L_q = 7 - \rho = 6.125 \text{ cust}$

$W_s = \frac{L_s}{\lambda} = \frac{7}{105} = .0667 \text{ hr} \approx 4 \text{ mins}$

$W_q = \frac{L_q}{\lambda} = \frac{6.125}{105} = .058 \approx 3.48 \text{ mins}$

Key words & single server

Ex $\lambda = 10$ cars/hr.

$$\mu = \frac{1 \text{ cust}}{6 \text{ min}} \times \frac{60}{\text{hr}} = 10$$

1) p teller is idle = $P_0 = 1 - \rho = 1 - \frac{10}{15} = .333$

2) $L_q = L_s - \rho = 2 - \frac{10}{15} = 1.33$ $L_s = \frac{\rho}{1 - \rho} = 2$

3) $W_s = L_s / \lambda = .2 \text{ hr}$

4) $\mu < \rho = 15 < 10$ ^{$\lambda = 10$ cars/hr} $\Rightarrow$ assuming M is a single server M is busy all the time (utilization is 100%) but here he is only busy for $\rho = .667$ of the time

$\rho = \rho_{\text{busy}}$

which means if I take μ I'm serving 15 cust/hr but I only have 10 cust arriving $\lambda = 10$ $\mu > \lambda$ $\Rightarrow$ 1 server

Ex 18.6-2: one bay facility: 1 server

$\lambda = 4$ cars/hr, $\mu = 6$ cars/hr $\Rightarrow \rho = .667$

1) $\rho = .667$

3) $L_q = 1.33$ car

2) $L_s = 2$ cars

4) $W_s = .5 \text{ hr}$

5) $W_q = .333 \text{ hr}$

box (service)

$P_0 \checkmark \quad P_1 \checkmark \quad P_2 \checkmark$

لا يدخل

$P_{entering}$

$P_{not\ entering} = 1 - (\quad)$

(M/M/1 server): (GD general discipline / ∞ # in the sys / ∞ calling pop)

a) prob that the ^{arriving} car will enter ^{the facility} $P_0 + P_1 + P_2 = 70\%$
 " not " $1 - (\quad) = 30\%$

$$P_0 = 1 - \rho = .3333$$

$$\rho = .09881$$

$$P_1 = .2222$$

$$P_2 = .0659$$

$$P_2 = .1481$$

$$P_3 = .0434$$

$$P_3 = .0929$$

$$P_4 = .014$$

ممكن
اجمع
بذل
فر
احد
الفرق
العدد

additional: if 3 parking spaces will enter $= P_0 + P_1 + P_2 + P_3 \approx 80\%$
 لا اكبر من 3 makes sense

8) depends on my desire, here we chose to cover 40%
 so

$$P_0 + P_1 + P_2 + \dots + P_5 \geq .4$$

$$(1-\rho) + (1-\rho)\rho + (1-\rho)\rho^2 + \dots + (1-\rho)\rho^5 \geq .4$$

not ρ

so it is Geo series but not to ρ

$$(1-p) [1 + p + p^2 + p^3 + \dots + p^s] \geq .9$$

$$(1-p) \left[\frac{1-p^{s+1}}{1-p} \right] \geq .9 \quad \left(\begin{array}{l} \text{أولى صون بخط بالالة} \\ \text{الطرية solve} \end{array} \right)$$

$$-p^{s+1} \geq -.1 \Rightarrow p^{s+1} \leq .1$$

هناك احط $\ln$ و

$\ln$ اي في اقل من 1

يكون الب ف بقلب

الشارة

~~$\ln$~~

$$\ln(p^{s+1}) \geq \ln .1$$

$$(s+1) \ln p \geq \ln .1$$

$$(s+1) \geq \frac{\ln .1}{\ln p} \Rightarrow s+1 \geq 5.679$$

$$\ln .6667$$

او بطالع ال p و بقل اجمع

لحد اومر الي بي ياه

بي مذكور بيض احدثك

ف ح انظر

$$s \geq 4.679 \approx 5$$

$$\frac{\ln .1}{\ln p}$$

ما الي دخل بالي برا

$$2) (M/M/1) : (GD/N/\infty)$$

$$\lambda_n \begin{cases} \lambda & n=0, 1, \dots, N-1 \\ 0 & n=N, N+1, \dots \end{cases}$$

$$\mu_n = \mu \quad n=0, 1, \dots$$

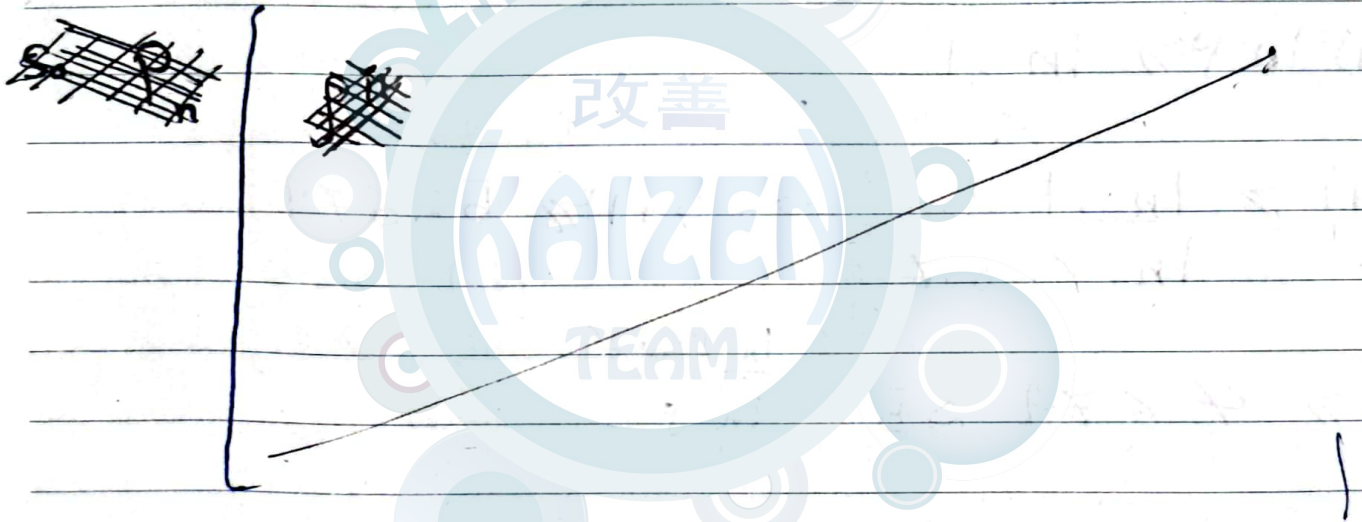
$$\rho = \frac{\lambda}{\mu}$$

ρ could be >1 <1 $=1$

we are limited $N \leq \infty$

$$p_n = \begin{cases} \rho^n p_0 & n \leq N \\ 0 & n > N \end{cases}$$

$$p_0 = \begin{cases} \frac{1-\rho}{1-\rho^{N+1}} & \rho \neq 1 \\ \frac{1}{N+1} & \rho = 1 \end{cases}$$



so we have effective since we are limited

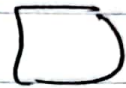
$$\lambda_{\text{lost}} = \lambda p_n \Rightarrow \lambda_{\text{eff}} = \lambda - \lambda_{\text{lost}}$$

$$L_s = \begin{cases} \rho \left[\frac{1-(N+1)\rho^N + N\rho^{N+1}}{(1-\rho)(1-\rho^{N+1})} \right], & \rho \neq 1 \\ N/2, & \rho = 1 \end{cases}$$

$$L_s = w_s \lambda_{eff}$$

$$L_q = w_q \cdot \lambda_{eff}$$

$$w_s = w_q + 1/\mu$$



Ex 18.6-4 + problem 18-69
4 parking spaces



a) p_0

b) w_q

c) $4 - L_s$

d) p_5

e) $w_s = 10 \text{ min}$

$$(M/M/c) : (GD/\infty/\infty)$$

$$p_n = \begin{cases} \frac{\rho^n}{n!} p_0, & n < c \\ \frac{\rho^n}{c! c^{n-c}} p_0, & n \geq c \end{cases}$$

بستگی (توسط عدد servers)

$$\rho = \lambda / \mu, \frac{\rho}{c} < 1 \text{ (حالت پایدار) } \Rightarrow \text{عادی بودن اوقات ماندن عادی لازم است (} \rho < c \text{) } \Rightarrow \text{عادی بودن شرط } \frac{\rho}{c}$$

$$p_0 = \frac{1}{\left[\sum_{h=0}^{c-1} \frac{\rho^h}{h!} \right] + \frac{\rho^c}{c!} \left(\frac{1}{1 - \frac{\rho}{c}} \right)}$$

$$L_q = \frac{\rho^{c+1}}{(c-1)! (c-\rho)^2} p_0$$

$$L_s = L_q + \rho$$

$$W_s = L_s / \lambda$$

$$W_q = L_q / \lambda$$

Ex 8 Sports City

$$\lambda = 105 \text{ cust/hr} \quad \mu = \frac{1}{30 \text{ sec}} \times \frac{60}{\text{min}} \times 60 = 120 \text{ cust/hr}$$

~~P~~ $W_s = 4 \text{ min}$, single server

W_s multiple

$$P_0 = .391$$

$$P = .875$$

$$L_q = .207 \text{ cust}$$

$$L_s = 1.08196 \text{ cust}$$

$$W_q = .11826 \text{ min}$$

$$W_s = .618 \text{ min}$$

Same que

$\lambda = 18.6 - 5 = 2$ companies

2 cabs 2 cabs

So for the first sys $(M/M/2) : (GD/\infty/\infty)$

$\lambda = 8$ calls/hr $\mu = \frac{1}{12} \text{ cust/min} = \frac{5}{2} \text{ sys}$

after the merge $(M/M/4) : (GD/\infty/\infty)$

$\lambda = 16$ calls/hr $\mu = \frac{1}{12} \text{ cust/min}$

For the first sys

$\rho = 1.6$

$P_0 = .111 \Rightarrow L_q = 2.8416 \Rightarrow L_s = 4.4416$

$W_s = .5552 \text{ hr}$ $W_q = 21.33 \text{ min}$

For the second sys

$\rho = 3.2$

$P_0 = .0273 \Rightarrow L_q = 2.385 \text{ cust} \Rightarrow L_s = 5.586$

$W_q = 8.94 \text{ min}$ $W_s = 20.94 \text{ min}$

so 2nd sys is better since W_s is less

$\rho < 1$ $\rho < 1$
 $\frac{\rho}{c} < 1$ $\frac{\rho}{c} < 1$
 $\rho < c$ $\rho < c$

14th Nov
Tuesday

$(M/M/c) : (GD/N/\infty)$

$$\lambda_n = \begin{cases} \lambda & n = 0, 1, \dots, N-1 \\ 0 & n \geq N \end{cases}$$

$$\mu_n = \begin{cases} n \mu & n = 0, 1, \dots, c-1 \\ c \mu & c \leq n \leq N \end{cases}$$

1 server

μ : service rate per server

μ 's service rate for the whole sys

$$P_n = \begin{cases} \frac{\rho^n}{n!} P_0 & n = 0, \dots, c-1 \\ \frac{\rho^n}{c! c^{n-c}} P_0 & n = c, \dots, N \end{cases}$$

$$P_0 = \begin{cases} \frac{1}{\left[\left(\sum_{n=0}^{c-1} \frac{\rho^n}{n!} \right) + \frac{\rho^c}{c!} \cdot \frac{1 - \left(\frac{\rho}{c}\right)^{N-c+1}}{1 - \frac{\rho}{c}} \right]} & \frac{\rho}{c} \neq 1 \\ \frac{1}{\left[\sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c}{c!} \cdot (N - c + 1) \right]} & \frac{\rho}{c} = 1 \end{cases}$$

$$L_q = \begin{cases} \frac{\rho^{c+1}}{(c-1)! (c-\rho)^2} \left[1 - \left(\frac{\rho}{c}\right) - (N-c+1) \left(1 - \frac{\rho}{c}\right) \left(\frac{\rho}{c}\right)^{N-c} \right] \rho & \text{when } \frac{\rho}{c} \neq 1 \\ \frac{\rho^c (N-c) (N-c+1)}{2 c!} \rho & , \frac{\rho}{c} = 1 \end{cases}$$

$$\lambda_{\text{eff}} = \lambda - \lambda_{\text{lost}} = \lambda - \lambda \rho_N = \lambda (1 - \rho_N)$$

$$L_s = L_q + \frac{\lambda_{\text{eff}}}{\mu}$$

$$W_s = \frac{L_s}{\lambda_{\text{eff}}}$$

$$W_q = \frac{L_q}{\lambda_{\text{eff}}}$$

Ex 18.6-6

we care about waiting time in sys
for the customer, ϕ que
I care about all including L_z & L_s

$$\lambda = 16$$

$$4+6 \quad N=10$$

$$\lambda_n = 16$$

$$\mu_n = 5$$

$$c = 4$$

$$(M/M/4) : (GD/10/\infty)$$

$$\rho = 3.2, \quad \rho_c = .8$$

to assess we want to check w_s & w_q & L_z & L_s

$$\rho_o = 0.03121$$

$$L_z = \boxed{1.154} = \left(\frac{(3.2)^5}{(4)(.8)^2} \left[1 - \left(\frac{3.2}{4} \right)^7 - 7 \left(1 - \frac{3.2}{4} \right) \left(\frac{3.2}{4} \right)^6 \right] \right) \rho_o$$

$$L_s = 4.2398$$

$$w_s = 0.2648$$

$$w_q = .0748$$

Thursday
16 - Nov

$(M/M/\infty) \cdot (GD/\infty/\infty)$, self service

machines لا حاجة لخدمة العملاء
no que , no w_q لا حاجة لخدمة العملاء

$$\lambda_n = \lambda \quad n=0, 1, \dots$$

$$\mu_n = n\mu \quad n=0, 1, \dots$$

$$p_n = \frac{\lambda^n e^{-\lambda}}{n!} \quad n=0, 1, \dots \text{ (like poisson)}$$

$$E(n) = \rho = L_s \quad \text{exp \# of cost in the sys}$$

$$W_s = L_s / \lambda = 1/\mu$$

$$L_q = 0 \quad , \quad W_q = 0$$

problem 18-92

$$\lambda = 100 \text{ / hr} \quad 1/\mu = 30 \text{ min} \Rightarrow \mu = \frac{1}{30} \times 60 \times 8 = 16 \text{ per hr}$$

a) $L_s = \rho = 6.25 \text{ seats}$

b) $p_{n \geq 7} = 1 - p_{n < 7} = 1 - [p_0 + p_1 + \dots + p_6] = .436$

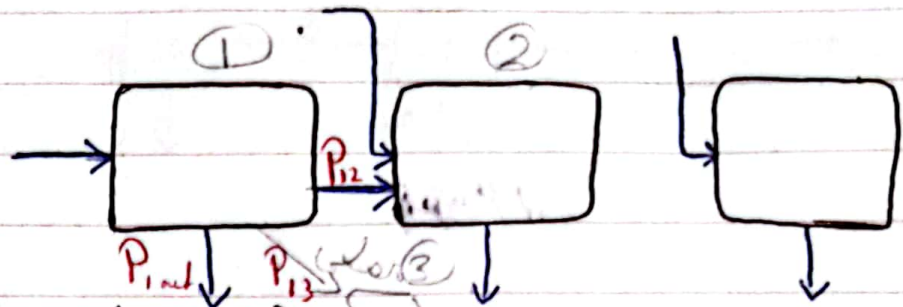
c) $p_0 = \frac{(6.25)^0 e^{-6.25}}{0!} = .00193$

Jackson network

19 - Nov
Sunday

$$p_{12} = 1 - (p_{out} + p_{13})$$

بقدر انتقال معمم مواد
individuals
models



$(M/M/1) : (GD/\infty/\infty)$

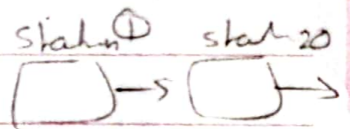
lack of memory

جackson واحد
open system
تدفق و خروج

We care about min the obj function

EDD earliest due date SPT shortest processing time
but here we are random

Example: 1st station $(M/M/1) (GD/\infty/\infty)$



$$\lambda = 54 \text{ car/hr} / \mu = 60 \text{ car/hr}$$

$$L_s = \frac{\rho}{1-\rho} = \frac{.9}{1-.9} = 9$$

2nd station $(M/M/3) (GD/\infty/\infty)$

$$L_q = L_s - \rho = 9 - .9 = 8.1$$

$$W_q = \frac{L_q}{\lambda} = \frac{8.1}{54} = .15 \text{ hr} = 9 \text{ mins}$$

1st to be in steady
 $\rho < 1$
2nd $\rho/c < 1$

$$\lambda = 54 \quad \mu = 20 \quad \rho = 2.7$$

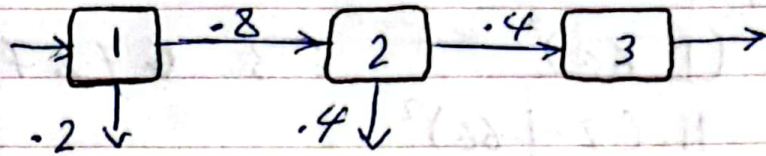
$$\rho = .02491$$

$$L_q = 7.35 \text{ cars}$$

$$W_q = .136 \text{ hr} = 8.17 \text{ mins}$$

$$W_{q1} + W_{q2} = 9 + 8.17 = 17.17 \text{ mins}$$

Example



$$\lambda = 10 \text{ cust/min}$$

$$\mu = \text{Station ①} = \frac{1}{10} \times 60 = 6 \text{ cust/min}$$

$$\text{Station ②} = \frac{1}{3} \times 60 = 20 \text{ cust/min}$$

$$\text{Station ③} = \frac{1}{70} \times 60 = .857$$

$$\lambda = \text{st. ①} = 600$$

$$\text{st. ②} = 480$$

$$\text{st. ③} = 192$$

a) Find c → from the condition $\frac{\rho}{c} < 1$

$$\text{For station ①} \quad \rho = 1.667 \Rightarrow c > 1.667 \Rightarrow c = 2$$

$$\text{②} \quad \rho = .4 \Rightarrow c > .4 \Rightarrow c = 1$$

$$\text{③} \quad \rho = 3.733 \Rightarrow c > 3.733 \Rightarrow c = 4$$

Even if $\frac{\rho}{c}$ is integer, we take the next integer
 now finding L_s, W_s, L_q, W_q

~~For station~~

$$P_0 = \frac{1}{\sum_{n=0}^{\infty} \frac{(1.667)^n}{n!}} = \frac{1.667}{1!} + \frac{(1.667)^2}{2!} \left(\frac{1}{1 - \frac{1.667}{2}} \right) = .0428$$

for $c=2$

$$L_z = \frac{(1.66)^3}{11(2-1.66)^2} = 3.7878$$

$$L_s = L_z + p = 5.4545$$

$$w_s = \frac{L_s}{\lambda} = 0.5454$$

$$w_f = \frac{L_z}{\lambda} = 0.37878$$

for ~~2~~ $c=1$

$$L_s = .667$$

$$w_s = .0833$$

$$L_z = .2667$$

$$w_f = .0333$$

改善

KAIZEN

TEAM

for $c=4$

$$L_s = 15.7565$$

$$w_s = 4.9239$$

$$L_z = 12.02256$$

$$w_f = 3.75706$$

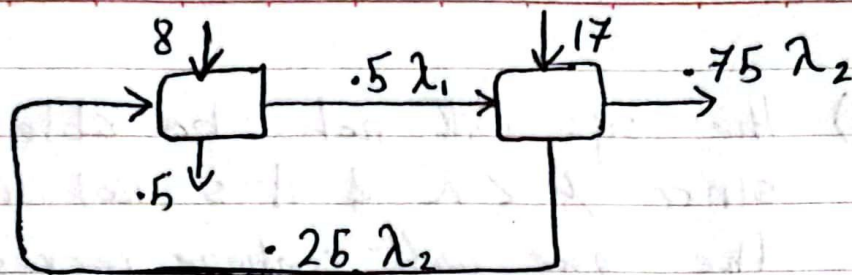
Thursday
23. Nov

Example
(M/M/2): (GD/∞/∞)

$$C = 2$$

$$\mu_1 = 20$$

$$\mu_2 = 30$$



Reminders we are dealing with steady state, so we are ind. of time.

∅ we don't care about the very first/last moments, we care about avg when the sys has stabilised ∪

$$\lambda_1 = 8 + 0.25 \lambda_2$$

$$\lambda_2 = 17 + 0.5 \lambda_1$$

by solving for eq 1 & 2 we get stations
∅ maybe $c=3$
∅ 3 equations
∅ so on.

$$\lambda_1 = 14, \quad \lambda_2 = 24$$

1) server 1 is idle (M/M/1): (GD/∞/∞)

$$P_0 = 1 - \rho = 1 - 0.7 = 0.3 \text{ so } 30\% \text{ of the time}$$

2) L_s for each server (M/M/1): (GD/∞/∞)

$$E(n) = L_s = \frac{\rho}{1 - \rho} = \frac{7}{3} \text{ for 1st server}$$

$$= 4 \text{ " 2nd "}$$

3) W_s for the sys

4) The sys will not be able to stabilize
since $\mu < \lambda$ & it's not limited so
the que will continue increasing (exploding)

~~7/2~~ $\frac{\lambda}{\mu} > 1$



Ch. 17 Markov chains

28. NOV
Tuesday

of cut / state

System: Attributes "Aspects" characteristics
evolve "change" over time

* State of the sys: # of customers in the queue sys
(not const, random variable & that's why
we used to call it "expected")

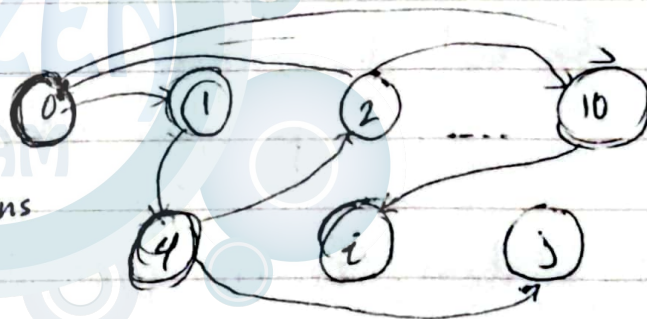
$$X_1 = \begin{cases} 0 & \text{poor machine} \\ 1 & \text{fair " } \\ 2 & \text{good} \end{cases}$$

$$X_1 = \begin{cases} 0 & \text{Rainy} \\ 1 & \text{cloudy} \\ 2 & \text{sunny} \\ 3 & \text{R \& C} \end{cases}$$

* Transition governed by prob theory, so my
system is stochastic

State transition diagram

- we used to just have limited trans
but now in general & I'm free



of states $\rightarrow$ finite
 $\rightarrow$ infinite

State at time 0, (doesn't mean on 1st station here I'm free)

$$X_t = \{X_{t_0}, X_{t_1}, X_{t_2}, \dots, X_{t_n}\}$$

* We are dealing
with time as const
intervals & as discrete
DTMC

conditional prob

$P(X_1 = 2 \mid X_0 = 10)$ means I'm on time 0, what is the prob that I'll go to state 2 at time 1

given that state

in state 1

state 10

state 2

$$P(X_{t+1} = j \mid X_t = i) = P_{ij} \quad * \text{One step trans prob}$$

$P(X_{t+1} = 2 \mid X_t = 10) = P_{10,2}$ is stationary ما يستقر مع الزمن

time homogeneous لو بقي ارفى في كل وقت سواء $t=0$ او $t=1000$

if I go directly from 10 to 2 or I went thro other stations in between it is the same result

Future present past

$$\text{So } P(X_{t+1} = j \mid X_t = i, X_{t-1} = X_{t-1}, X_{t-2} = X_{t-2}, \dots, X_0 = X_0)$$

$$= P(X_{t+1} = j \mid X_t = i) \text{ this is an evidence of lack of memory property}$$

in matrix notation is the one step trans matrix (imp)

$$P = \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1n} \\ P_{21} & P_{22} & \dots & P_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & \dots & P_{nn} \end{bmatrix}$$

i j

n

so P is nxn matrix

اي وقت
را او
رحب
اي
منيت
مستقبل

P that today is sunny

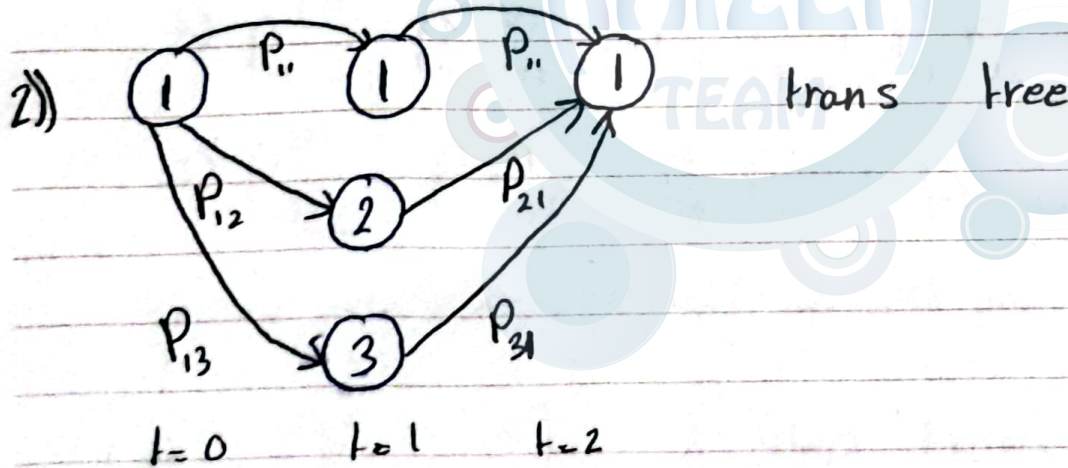
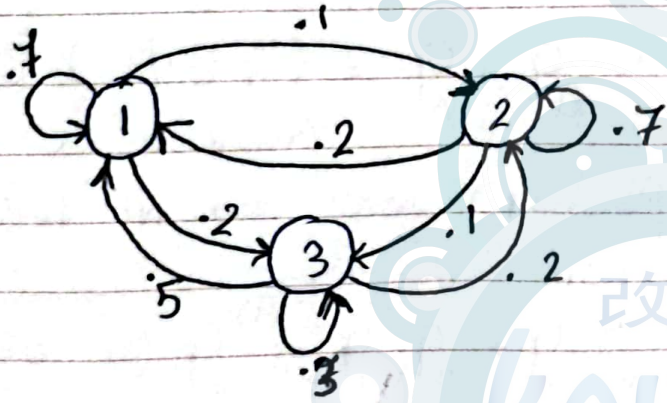
& tomorrow is sunny

Ex: state 1 → sunny 2 → Cloudy 3 → Snowfall

$$P = \begin{bmatrix} .7 & .1 & .2 \\ .2 & .7 & .1 \\ .5 & .2 & .3 \end{bmatrix}$$

⇒ row sum must = 1

column sum doesn't have to = 1



trans in 2 steps $P_{ij}^{(2)} \Rightarrow P_{ii}^{(2)}$



stationary distribution

$$P_{ii} = P_{i1} \cdot P_{1i} + P_{i2} \cdot P_{2i} + P_{i3} \cdot P_{3i} \quad \# \text{ The multiplication is}$$

The multiplication is for independent events (no relationship) (may)
 The addition is for mutually exclusive (either, or no way together)

$$P(A \cap B) = P_A \cdot P_B$$

$$P(A \cup B) = P_A + P_B - P(A \cap B)$$

union (or)

But in our case (mutually exclusive) $\Rightarrow (A \cap B) = P(\emptyset) = 0$

so a) $P_{ii}^{(2)} = P_{i1} \cdot P_{1i} + P_{i2} \cdot P_{2i} + P_{i3} \cdot P_{3i} = 0.61$

b) $P_{i2}^{(2)} = P_{i1} \cdot P_{12} + P_{i2} \cdot P_{22} + P_{i3} \cdot P_{32} = 0.18$

c) $P_{i3}^{(2)} = P_{i1} \cdot P_{13} + P_{i2} \cdot P_{23} + P_{i3} \cdot P_{33} = 0.21$

Sum = 1

because eventually i'll end up with one of these 3 options

$$P^2 = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} P_{11}^2 & P_{12}^2 & P_{13}^2 \\ P_{21}^2 & P_{22}^2 & P_{23}^2 \\ P_{31}^2 & P_{32}^2 & P_{33}^2 \end{bmatrix} \end{matrix}$$

2 step trans matrix

so we can get P^n n step trans matrix by

$$P^3 = P^2 \cdot P^1$$

$$P^n = P^{n-1} \cdot P = P^n$$

$$P^n = P^{n-m} \cdot P^m$$

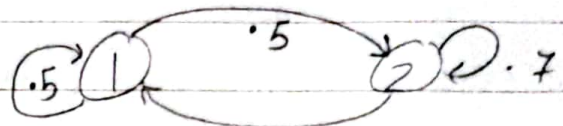
Tuesday
5th. Dec

one step trans matrix

$$P = \begin{bmatrix} .5 & .5 \\ .3 & .7 \end{bmatrix}$$

from state 1 $\rightarrow$ 1
from state 2 $\rightarrow$ 1

Find $P^{(2)}$, $P^{(4)}$, $P^{(8)}$



$$P^{(2)} = \begin{bmatrix} .4 & .6 \\ .36 & .64 \end{bmatrix}$$

$$P^{(4)} = \begin{bmatrix} .376 & .624 \\ .374 & .63 \end{bmatrix}$$

$$P^{(8)} = \begin{bmatrix} .375 & .62 \\ .374 & .63 \end{bmatrix}$$

$$P^{(16)} = \begin{bmatrix} .37 & .62 \\ .37 & .63 \end{bmatrix}$$

I'm ending up in 1 no matter
from where I started

more $P^{(n)}$

we notice that probability converges to
a certain value here (steady state)
after some time (in some sys only)

likely I'm ending
up in state 2
due to greater p
(was obvious a bit
from the first matrix)

$$P_{ij}^{(n)} = \pi_j \quad \text{as } n \rightarrow \text{infinity}$$

$$P = \begin{bmatrix} .6 & .4 \\ .5 & .5 \end{bmatrix}$$

$$P^{(2)} = \begin{bmatrix} .56 & .44 \\ .55 & .45 \end{bmatrix}$$

$$P^{(4)} = \begin{bmatrix} .5556 & .4444 \\ .5555 & .4445 \end{bmatrix}$$

π_1

π_1 , no matter where I start from I'll end up in 1 with
this prob

initial state prob vector :

$$a^{(0)} = \{ a_j^{(0)}, j=1, 2, \dots, n \}$$

$$= \{ a_1^{(0)}, a_2^{(0)}, a_3^{(0)} \}$$

$$a^0 = \{ 1, 0, 0 \}$$

$$a^{(1)} = a^{(0)} p^{(1)} = [\text{vec}] [\text{matrix}]$$

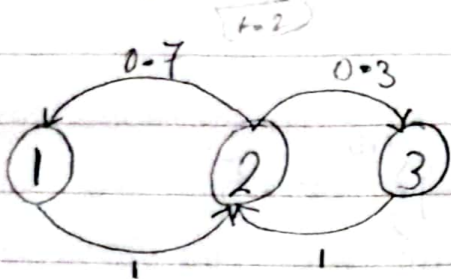
$$a^{(2)} = a^{(1)} p^{(2)} = a^{(0)} p^{(2)}$$

$$\text{so } a^{(n)} = a^{(0)} p^{(n)}$$

$\hookrightarrow n^{\text{th}}$ state prob vec

Thursday
7th. Dec

$$P_{ij}^{(n)} \xrightarrow{n \rightarrow \infty} \pi_j \quad \text{always?} \quad \text{no}$$



$$P_{22}^{(1)} = 0$$

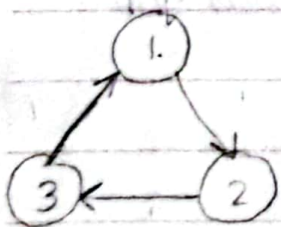
$$P_{22}^{(2)} = 0.7 + 0.3 = 1$$

$$P_{22}^{(3)} = 0$$

$$P_{22}^{(4)} = 1$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0.7 & 0 & 0.3 \\ 0 & 1 & 0 \end{bmatrix}$$

* We conclude that $P_{22}^{(n)} = \begin{cases} 0, & n = \text{odd} \\ 1, & n = \text{even} \end{cases}$ i will never converge i'm oscillating



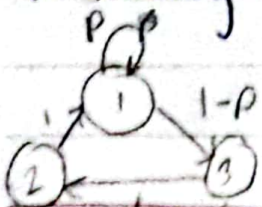
here periodic states, so $P_{ij}^{(n)} \xrightarrow{n \rightarrow \infty} \pi_j$ doesn't apply

period 3 (أقل رقم trans يوصل للبدية) here = 3!

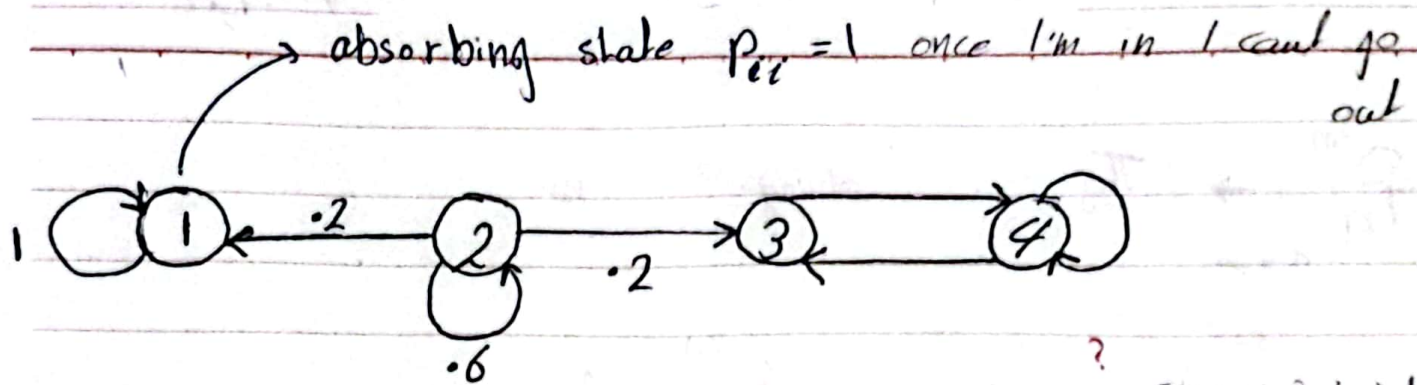
So $P_{ij}^{(n)} \rightarrow \pi_j$ applies on a periodic markov

Chans (not periodic)

* having $P_{ii} > 0$ self transition makes the chain aperiodic (لا يكون له دور)



* periodic: I have fixed # of steps to go back



$$P_{11}^{(n)} = 1$$

$$P_{31}^{(n)} = 0$$

since $.2 \rightarrow .2$ to 3

$$P_{21}^{(n)} = .5$$

on the long term

$$P_{22} \text{ will } = 0$$

* it matters from where I start here in order to end up in 1 so I ain't going to reach st. st

* Reachable (accessible) : state j is reachable from state i $P_{ij}^{(n)} > 0$ for some n

2 states (i, j) communicate when i is reachable from j & state j is reachable from i

reachable ← ممكن الوصول إليه

Irreducible M.C : a Markov chain where all states communicate with each other (I will end up with 1 M.C)

Class "closed set"

So to apply st. st. $P_{ij}^{(n)} = \pi_j$ we have to have

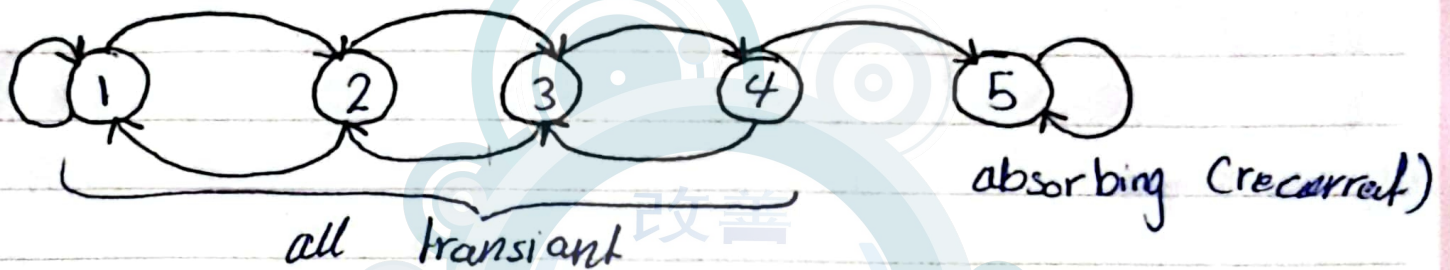
1) irreducible + 2) aperiodic Ergodic

all are recurrent

Sunday
10. Dec

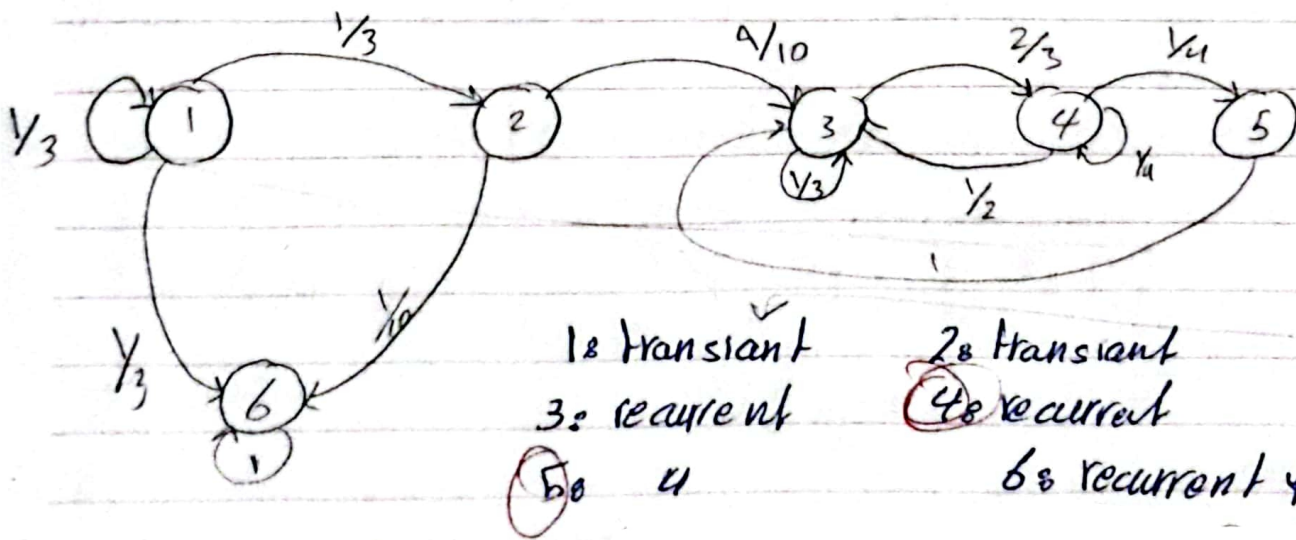
recurrent state: if upon entering the state the process will definitely return to this state

transient state: a state is trans if there's transition out of it to some state (i) to some state (j) from which state (i) can never be reached back again
recurrent $\Rightarrow$ $\bar{S} \rightarrow S$



$$P = \begin{bmatrix} 1/3 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 1/10 & 0 & 0 \\ 0 & 0 & 1/2 & 1/4 & 1/4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

identify the state



$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}, \quad P^2 = P \cdot P = \begin{bmatrix} p_{11}^{(2)} & p_{12}^{(2)} \\ p_{21}^{(2)} & p_{22}^{(2)} \end{bmatrix}$$

$$P^{(n+1)} = P^n \cdot P = \begin{bmatrix} p_{11}^n & p_{12}^n \\ p_{21}^n & p_{22}^n \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

taking that n is a very large num $P^n = \begin{bmatrix} \pi_1 & \pi_2 \\ \pi_3 & \pi_4 \end{bmatrix}$

$$\begin{bmatrix} \pi_1 & \pi_2 \\ \pi_1 & \pi_2 \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 \\ \pi_1 & \pi_2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \quad \begin{matrix} 2 \text{ unknowns} & 2 \text{ eq} \end{matrix}$$

Example $P = \begin{bmatrix} .6 & .4 \\ .2 & .8 \end{bmatrix}$

$$\begin{bmatrix} \pi_1 & \pi_2 \\ \pi_1 & \pi_2 \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 \\ \pi_1 & \pi_2 \end{bmatrix} \begin{bmatrix} .6 & .4 \\ .2 & .8 \end{bmatrix}$$

$$\left. \begin{aligned} \pi_1 &= .6\pi_1 + .2\pi_2 & .4\pi_1 &= .2\pi_2 \\ \pi_2 &= .4\pi_1 + .8\pi_2 & .2\pi_2 &= .4\pi_1 \end{aligned} \right\} \text{redundant}$$

one is

So i always will have 1 equation missing (sum 1)

∴ 11 is

$$\pi_1 + \pi_2 = 1$$

$$\text{So } \pi_j = \sum_{i=1}^n \pi_i p_{ij}$$

$$\sum \pi_j = 1$$

$$\pi_1 = .333$$

$$\pi_2 = .667$$

* π_j is st. st probability
(percentage of time it'll be in the state)

Exs

$$\begin{bmatrix} .3 & .6 & .1 \\ .1 & .6 & .3 \\ .05 & .4 & .55 \end{bmatrix}$$

Find

$$\pi_1 = .1016$$

$$\pi_2 = .525$$

$$\pi_3 = .373$$

$$[\pi_1 \ \pi_2 \ \pi_3] \begin{bmatrix} .3 & .6 & .1 \\ .1 & .6 & .3 \\ .05 & .4 & .55 \end{bmatrix} \Rightarrow \begin{aligned} \pi_1 &= .3\pi_1 + .1\pi_2 + .05\pi_3 \\ \pi_2 &= .6\pi_1 + .6\pi_2 + .4\pi_3 \\ \pi_3 &= .1\pi_1 + .3\pi_2 + .55\pi_3 \\ \pi_1 + \pi_2 + \pi_3 &= 1 \end{aligned}$$

since Rand. var time (days/hours/...)

* μ_{jj} = expected # of trans before the sys returns back to the state j for the 1st time = $\frac{1}{\pi_j}$

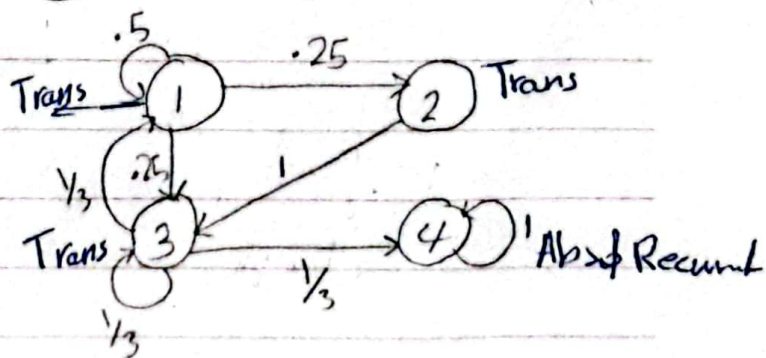
* When Ergodic (irreducible & aperiodic) that $\pi_j = \sum_{i=1}^n \pi_i p_{ij}$
 $\sum_{j=1}^n \pi_j = 1$

Problems 17-10

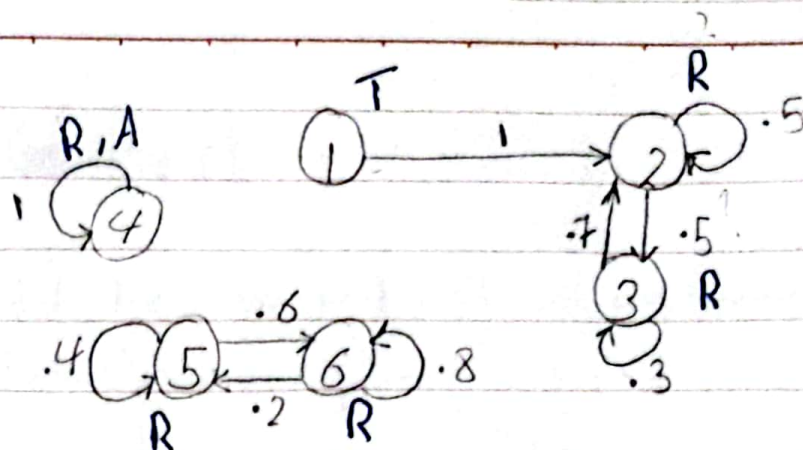
a) periodic = , period = 3



b) aperiodic

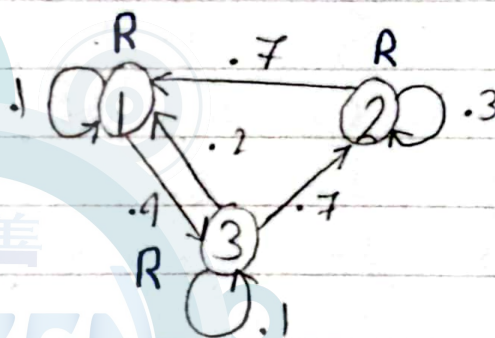


c) aperiodic



d) aperiodic (ergodic so we can find st. st & recurrence time)

$$\begin{bmatrix} .1 & 0 & .9 \\ .7 & .3 & 0 \\ .2 & .7 & .1 \end{bmatrix}$$



$$[\pi_1 \quad \pi_2 \quad \pi_3] = [\pi_1 \quad \pi_2 \quad \pi_3] \begin{bmatrix} .1 & 0 & .9 \\ .7 & .3 & 0 \\ .2 & .7 & .1 \end{bmatrix}$$

$$\pi_1 = .1\pi_1 + .9\pi_3$$

$$\pi_2 = .7\pi_1 + .3\pi_2$$

$$\pi_1 + \pi_2 + \pi_3 = 1$$

$$\pi_1, \pi_2, \pi_3 = \frac{1}{3}, \quad M_w = 3$$

Notes: st. st doesn't mean sys will stop, it's dynamic

Thursday
14th. Dec

problem 17.13 (modified)

(Since he gave me (det the exp prob) then we need st. st prob)

today sunny $\rightarrow$ Tomorrow Sunny .8
cloudy .2

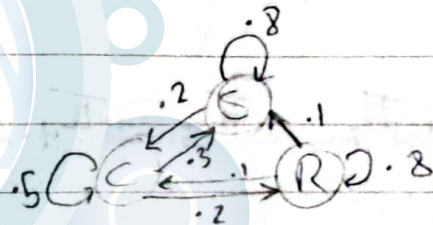
cloudy $\rightarrow$ Sunny .3
today rainy $\rightarrow$ rainy .2
cloudy .5
sunny .1

$$\begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix} \begin{bmatrix} .8 & .2 & 0 \\ .3 & .5 & .2 \\ .1 & .1 & .8 \end{bmatrix}$$

$$\pi_1 = .8\pi_1 + .3\pi_2 + .1\pi_3 \quad (1)$$

$$\pi_2 = .2\pi_2 + .5\pi_3 \quad (2)$$

$$\pi_1 + \pi_2 + \pi_3 = 1$$



a) we get $\pi_1 = .5$ / $\pi_2 = .25$ / $\pi_3 = .25$

b) Revenues : $S = 2000 \$$ / $C = 1600 \$$ / $R = 400 \$$

$$\text{Exp revenue} = (.5)(2000) + (.25)(1600) + (.25)(400) = 1500 \$$$

Problem: 17. 14 € 12 \$ mex / 17 \$ 11 / 11 \$ chi / 13 \$ Thai

$$a \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{bmatrix} \begin{bmatrix} .7 & .1 & .1 & .1 \\ .1 & .7 & .1 & .1 \\ .1 & .1 & .7 & .1 \\ .1 & .1 & .1 & .7 \end{bmatrix}$$

$$\pi_1 = .7\pi_1 + .1\pi_2 + .1\pi_3 + .1\pi_4 \rightarrow$$

$$\pi_2 = .1\pi_1 + .7\pi_2 + .1\pi_3 + .1\pi_4$$

$$\pi_3 = .1\pi_1 + .1\pi_2 + .7\pi_3 + .1\pi_4$$

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$$

$$\text{we get } \pi_1, \pi_2, \pi_3, \pi_4 = .25$$

$$b) .25 * (12) + .25(17) + .25(11) + .25(13) = 13.25 \$$$

$$c) \frac{1}{\pi} = 4 \text{ each 4 days i turn back to the first meal}$$

(4 transitions in unit time [day] here)

$$a \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix} \begin{bmatrix} .7 & .1 & .2 \\ .2 & .7 & .1 \\ .5 & .2 & .3 \end{bmatrix}$$

$$.7\pi_1 + .2\pi_2 + .5\pi_3 = \pi_1$$

$$.1\pi_1 + .7\pi_2 + .2\pi_3 = \pi_2$$

$$\pi_1 + \pi_2 + \pi_3 = 1$$

$$-.3\pi_1 + .2\pi_2 + .5\pi_3 = 0$$

$$.1\pi_1 - .3\pi_2 + .2\pi_3 = 0$$

$$\pi_1 + \pi_2 + \pi_3 = 1$$

b) cloudy followed by sunny 3 days later (no sun before full 3 days)

so on slides ✓ ✗ ✗ ✓ ✓ ✗ ✗ ✗ ✓

$$f_{ij}^{(n)} = p_{ij}^{(n)} - \sum_{k=1}^{n-1} f_{ij}^{(k)} \cdot p_{jj}^{(n-k)}$$

(prob of first passage from i to j in n time steps)

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)}$$

يعني احتمال اول زنى و باقى عدد

$$f_{ij} = 1 \rightarrow \text{recurrent}$$

الحب رح اطلع من i و اولي j

$$f_{ij} < 1 \text{ Trans}$$

$$\begin{cases} f_{jj} = 1 \rightarrow \text{recurrent} \\ f_{jj} < 1 \rightarrow \text{transient} \end{cases}$$

First passage prob

prob that a sys leaves state i & enters state j for the first time after (n) steps

$$P_{ij}^{(n)} = P_{ij}^{(n)} - \sum_{k=1}^{n-1} f_{ij}^{(k)} P_{ij}^{(n-k)}$$

$$P_{ij} = \sum_{n=1}^{\infty} P_{ij}^{(n)} = P_{ij}^{(1)} + P_{ij}^{(2)} + P_{ij}^{(3)} + \dots + P_{ij}^{(n=100000)}$$

$$P_{ij} = 1$$

Recurrent

$$P_{ij} < 1$$

Trans

Ex 8

$$P_{21}^{(3)} = P_{21}^{(3)} - [P_{21}^{(1)} \cdot P_{11}^{(2)} + P_{21}^{(2)} \cdot P_{11}^{(1)}]$$

recursive eq

$$P_{21}^{(3)} = (.407) - [P_{21}^{(1)} \cdot P_{11}^{(2)} + P_{21}^{(2)} \cdot P_{11}^{(1)}]$$

ال P_{ij} هي "passage" يعني وجوها من i الى j

يعني اي اول مرة طوفت من i الى j

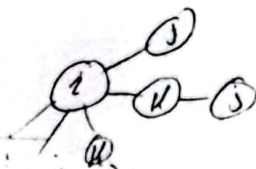
فارقة كان بها اخذت P_{ij}

$$P_{ij}^{(1)} = P_{ij}$$

$$P_{21}^{(2)} = P_{21}^{(2)} - P_{21}^{(1)} \cdot P_{11}^{(1)}$$

$$P_{21}^{(3)} = (.407) - [.2(.61) + .19(.7)] = .152$$

* $\mu_{ii} = \frac{1}{\pi_i}$ (exp # of trans to go from i to i) this is a special case of what's coming



عند احتمال (i) في اول مرة او من حالات ثانيه

$$\mu_{ij} = 1 + \sum_{k \neq j} p_{ik} (\mu_{kj})$$

لانه لو = صي 1

exp # of n $E(n) = \sum_{n=1}^{\infty} n \cdot p_n$

Ex: how many days — : Snowy 3

$k \neq j$ is 2 & 3

$$\mu_{31} = 1 + [p_{32} \mu_{21} + p_{33} \mu_{31}]$$

$$\mu_{21} = 1 + [p_{22} \mu_{21} + p_{23} \mu_{31}]$$

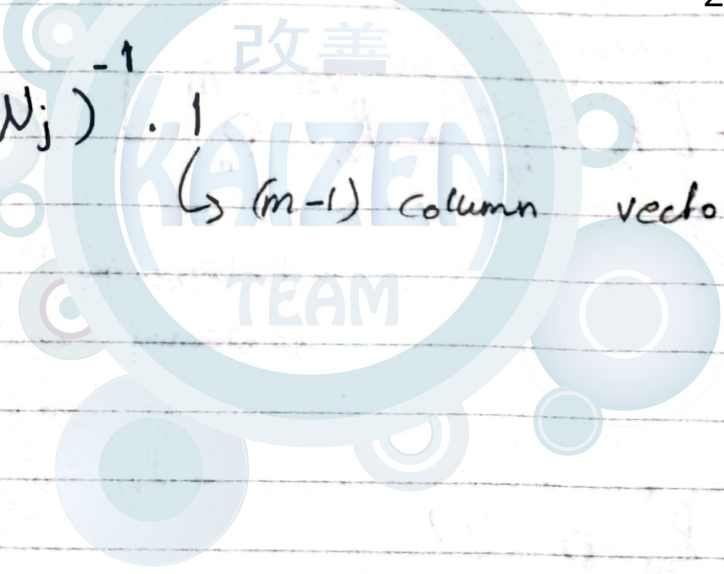
معالجاتي بصيولي و بجلي

4.2105

2.6315

$$\|\mu_{ij}\| = (I - N_j)^{-1} \cdot 1$$

(m-1) column vector



→ exp value of trans to go from i to j

$$\|M_{ij}\| = (I - N_j)^{-1} \cdot 1, \quad j \neq i$$

$$I = (m-1) \times (m-1)$$

$N_j = P$ (one step trans matrix) less its j^{th} column
 j^{th} row

$1 = (m-1)$ column vector with all elements = 1

$$P = \begin{bmatrix} .7 & .1 & .2 \\ .2 & .7 & .1 \\ .5 & .2 & .3 \end{bmatrix} \quad m \times m \quad \mu_{31} \quad \text{Then } \mu_{i1}$$

$$\mu_{21} = 2.6$$

$$\mu_{31} = 4.2$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mu_{11} = \frac{1}{\pi_{11}}$$

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} .7 & .1 \\ .2 & .3 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 3.6842 & .5263 \\ 1.0526 & 1.5789 \end{bmatrix}$$

$$\begin{bmatrix} 3.6842 & .5263 \\ 1.0526 & 1.5789 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.2105 \\ 2.6315 \end{bmatrix} \quad \mu_{21}$$

$$\mu_{31}$$

دی گئی چیز

D doubly stochastic : when all rows & columns sum = 1
 then all π 's are equal = $1/\#$ of states

$$P = \begin{bmatrix} .3 & .6 & .1 \\ .1 & .6 & .3 \\ .05 & .4 & .55 \end{bmatrix} \quad \begin{array}{l} \text{find st. st prob} \\ \mu_{ij} \\ \mu_{ij} \end{array}$$

$$[\pi_1 \quad \pi_2 \quad \pi_3] \begin{bmatrix} .3 & .6 & .1 \\ .1 & .6 & .3 \\ .05 & .4 & .55 \end{bmatrix}$$

solved previously & we got

$$\pi_1 = .1016$$

$$\pi_2 = .525$$

$$\pi_3 = .373$$

$$\text{now to find } \mu_{11} \begin{bmatrix} 7.5 & 5 \\ 6.6666 & 6.6666 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 12.5 \\ 13.3332 \end{bmatrix}$$

$$\mu_{12} \begin{bmatrix} 1.4516 & .3225 \\ .1612 & 2.258 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.7741 \\ 2.4193 \end{bmatrix}$$

$$\mu_{13} \begin{bmatrix} 1.8181 & 2.7272 \\ .4545 & 3.1818 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.5453 \\ 3.6363 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 2 & \dots & m \\ \vdots & & & \end{bmatrix} \quad m \times m$$
 Some states are trans with num t
 " " " " abs " " $(m-t)$

نو حلال اطلع من
 $0 = \text{trans } \lambda_{\text{abs}}$

* Fundamental matrix $(F) = [I - N]^{-1}$

Arithmetic series



So abs states are J, G

$\therefore$ we have 6 states

$$P = \begin{matrix} & \begin{matrix} M_1 & I_{n_1} & M_2 & I_{n_2} & J & G \end{matrix} \\ \begin{matrix} M_1 \\ I_{n_1} \\ M_2 \\ I_{n_2} \\ J \\ G \end{matrix} & \begin{bmatrix} 0 & .95 & 0 & 0 & .05 & 0 \\ .07 & 0 & .9 & 0 & .03 & 0 \\ 0 & 0 & 0 & .95 & .05 & 0 \\ 0 & 0 & .07 & 0 & .03 & .9 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

> 1 because of rework

> 1 because of junk

Now $F = [I - N]^{-1}$

on avg machine 1 is visited 1.07 times, ins, 1.02 times

$$F = \begin{bmatrix} 1.07 & 1.02 & .98 & .93 \\ .07 & 1.07 & 1.03 & .98 \\ 0 & 0 & 1.07 & 1.02 \\ 0 & 0 & .07 & 1.07 \end{bmatrix}$$

Prob a piece will be junked

avg # of visits

$A = F \cdot R$

$$A = \begin{bmatrix} .16 & .24 \\ .12 & .88 \\ .02 & .92 \\ .04 & .96 \end{bmatrix}$$

trans to 1st abs

trans to 2nd abs

Prob will be completed

→ Prob of absorption

$$P_{ij}$$

trans to 2nd abs

return to textbook page 642 for better understanding of the concept :)

Continuous ^{time} Markov chains

A random process $\{X(t), t \geq 0\}$ is continuous time Markov chain if for all $s, t \geq 0$, non -ve integers i, j, k

$$P\{X(t+s)=j \mid X(s)=i, X(u)=k, 0 \leq u < s\}$$

(after this random var we go to j)
given at s time in i

Future
Present
Past

$$P\{X(t+s)=j \mid X(s)=i\}$$

هذا السلسلة والوقت اي بغيره بال state بطل discrete و

ما عندي 2 random var $\rightarrow$ state الى بي احوطها

Follows exp dist (lack of memory)

$$P\{X(t)=j \mid X(0)=i\} = P_{ij}$$

so stationary (time homo)

$$\sum_i P_{ij} = 1$$

P_j = prob that the sys is in state j

$$\sum_j P_j = 1$$

ال sys يكون في واحد من

$$P_{ij}(t+s) = \sum_k P_{ik}(t) P_{kj}(s)$$

(i)
(j)

حالت i
حالت j

$$\mu = \frac{1}{\nu}$$

Whenever a C.T.M.C enters state i it spends an amount of time "dwell time" or "holding time" is expo distributed

(Fraction of the time) with mean $= \frac{1}{\nu_i}$, at the expiration of the holding time the process will make a transition to another state j with prob p_{ij}

* there's no p_{ii} since (expiration of the holding time) so i made a trans

$$\sum_j p_{ij} = 1, j \neq i \Rightarrow p_{ii} = 0$$

ν_i is rate at which the process leaves state i

$\nu_i \cdot p_{ij}$ represent the rate when in state i the process makes a transition to j

$$\nu_i \cdot p_{ij}$$

Δt is small time interval

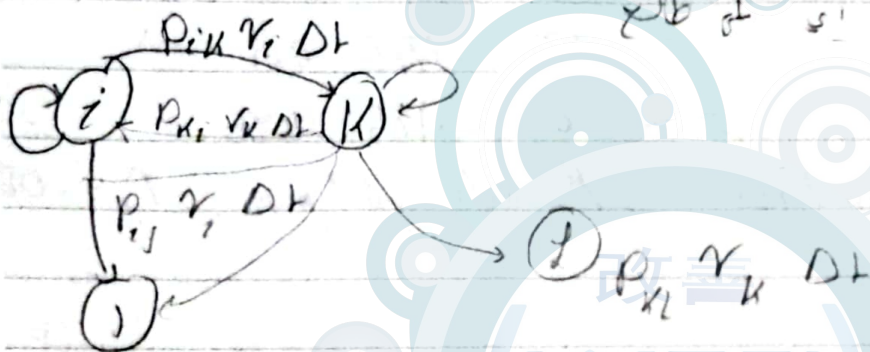
The prob that when the process is in state i a trans to state j ($j \neq i$) will take place in the next small time Δt is

$$= p_{ij} \nu_i \Delta t$$

no transition = $1 - \sum_{j \neq i} p_{ij} \gamma_i \Delta t$ (not expired)

The prob that the process leaves state i

$\sum_{j \neq i} p_{ij} \gamma_i \Delta t$ بتجرى اي شي غير i



Exs ATM & & & & & (M/M/1) (GD/N/1)



بن اعني انه كل ال λ متاويين فعلى μ و λ

rate transition matrix R :

$$R = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0 & \lambda & 0 & 0 & 0 & 0 \\ \mu & 0 & \lambda & 0 & 0 & 0 \\ 0 & \mu & 0 & \lambda & 0 & 0 \\ 0 & 0 & \mu & 0 & \lambda & 0 \\ 0 & 0 & 0 & \mu & 0 & \lambda \\ 0 & 0 & 0 & 0 & \mu & 0 \end{bmatrix} \end{matrix}$$

$\mu_0 = 0$ state 0 to 0

rate: $\nu_i = \lambda_i + \mu_i$

only for that simple M.C

$$P_{ij} \gamma_i \Delta t$$

cause row sum must = 1

γ_i which is $\lambda_i + \mu_i$

prob trans matrix:

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} (1-p_0\lambda\Delta t) & (p_0\lambda\Delta t) & 0 & 0 & 0 & 0 \\ .9 & .1 & 0 & 0 & 0 & 0 \\ .125 & .7775 & 0.1 & 0 & 0 & 0 \\ 0 & .125 & .7775 & 0.1 & 0 & 0 \\ 0 & 0 & .125 & .7775 & 0.1 & 0 \\ 0 & 0 & 0 & .125 & .7775 & 0.1 \\ 0 & 0 & 0 & 0 & .125 & .875 \end{bmatrix} \end{matrix}$$

and so on
so we get

lets take $\lambda = 2$ cust/min, $\mu = 2.5$ cust/min, $\Delta = .05$ min

dia is $1 - DL(M + \lambda)$
بسیع ستر ای

estimated

is

فلو بدي اذق باض $DL = .025$ مثلا وکله بختی

$$P_i = \frac{\lambda_0}{\mu_i} P_0$$

اولو طلب ای قوت بعد ا. دق

(M/M/1)

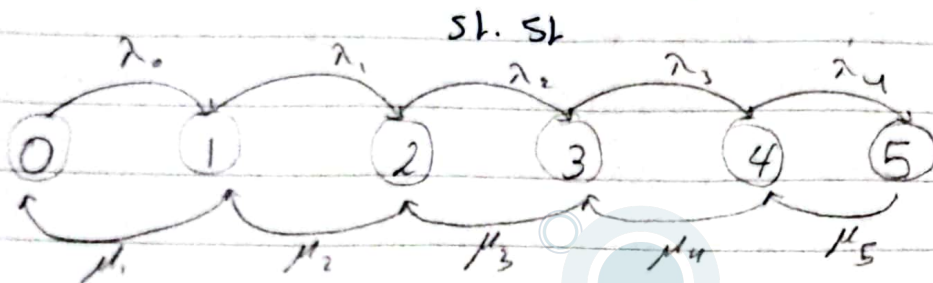
case of (M/M/1)

There's an excel sheet

معدنی بختی .25 و ستر ای

او .025 و بعد ا. ف P^4

Imp Par Final $D = .05$ p^{20}
 $D = .025$ p^{40}



$$p_0 \lambda_0 = p_1 \mu_1 \Rightarrow p_1 = \frac{\lambda_0}{\mu_1} p_0$$

$$p_1 \lambda_1 + p_1 \mu_1 = p_0 \lambda_0 + \mu_2 p_2$$

$$p_2 = \frac{\lambda_1}{\mu_2} p_1 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} p_0 \Rightarrow \text{sum} = 1 \text{ to find } p_0$$

$$p^2 = \begin{bmatrix} .82225 & .1675 & 0 & 0 & 0 & 0 \\ .20969 & .62951 & .155 & .01 & 0 & 0 \\ .015625 & .19437 & .62951 & .155 & .01 & 0 \\ & & .62951 & .155 & .01 & 0 \\ 0 & 0 & 0 & 0 & 0 & .1 \\ 0 & 0 & 0 & .0156 & .2066 & .778 \end{bmatrix}$$

and so on to find the rest