Markov Chains

Chapter 17: Definition of a Markov Chain

- **Markovian property (Markov Process)**

  Definition:

  A stochastic process is a Markov Process if a future state depends only on the immediately preceding state.

Given that the current state is known, the conditional probability of the next state depends only on the current state and no way on the past .

- For discrete- state space and discrete-time stochastic process:

  Conditional probability for the next state ($x_{tn} = x_n$) given

  The current state ($x_{tn-1} = x_{n-1}$)

  All state prior to the current state $x_{tn-2} = x_{n-2}, \ldots . x_{t0} = x_0$

$$\Pr\{(x_{tn} = x_n | x_{tn-1} = x_{n-1}, \ldots ., x_{t0} = x_0\} = \Pr\{(x_{tn} = x_n | x_{tn-1} = x_{n-1,}\}$$

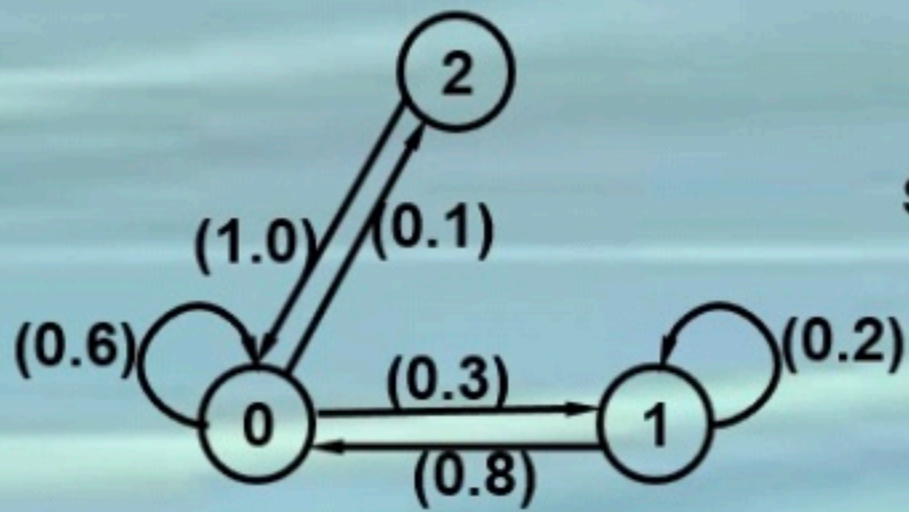
Probabilistic Operations Research

Chapter 17: Definition of a Markov Chain

- **Definition: A discrete-time Markov chain (Markov chain) is a stochastic process with the following characteristics**
  1. **A discrete state space**
  2. **Markovian property**
  3. **The one-step transition probabilities, p_{ij}, from time n to time n+1 remain constant overtime (termed stationary transition probabilities)**
- **Three- state Example**

Transition matrix,

$$
P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}
\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.8 & 0.2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}
$$



State transition Diagram for a simple Markov chain

- **Transition Probabilities**

Probability of moving from state *i* at time *t-1* to state *j* at time *t* is known as the one-step transition probability (p_{ij})

$P\{(x_t = j \mid x_{t-1} = i\}$, $i = 1, 2 \ldots., j = 1, 2, \ldots.., t = 0, 1, 2, \ldots.$

$$\sum_{j=1}^{n} p_{ij} = 1, \ \forall \mathrm{i} = 1, 2, \ldots \mathrm{n}$$

$$p_{ij} \geq 0, \forall (\mathrm{i}, \mathrm{j}) = 1, 2, \ldots \mathrm{n}$$

The Matrix notation is a convenient way to summarize the one step transition probabilities

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1n} \\ p_{21} & p_{21} & p_{21} & \cdots & p_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ p_{n1} & p_{n2} & p_{n3} & \cdots & p_{nn} \end{pmatrix}$$

Chapter 17: Definition of a Markov Chain

Example (Machine Maintenance): **The condition of a machine at the time of the monthly preventive maintenance is poor, fair, or good. For month t, the stochastic process for this situation can be represented as ;**

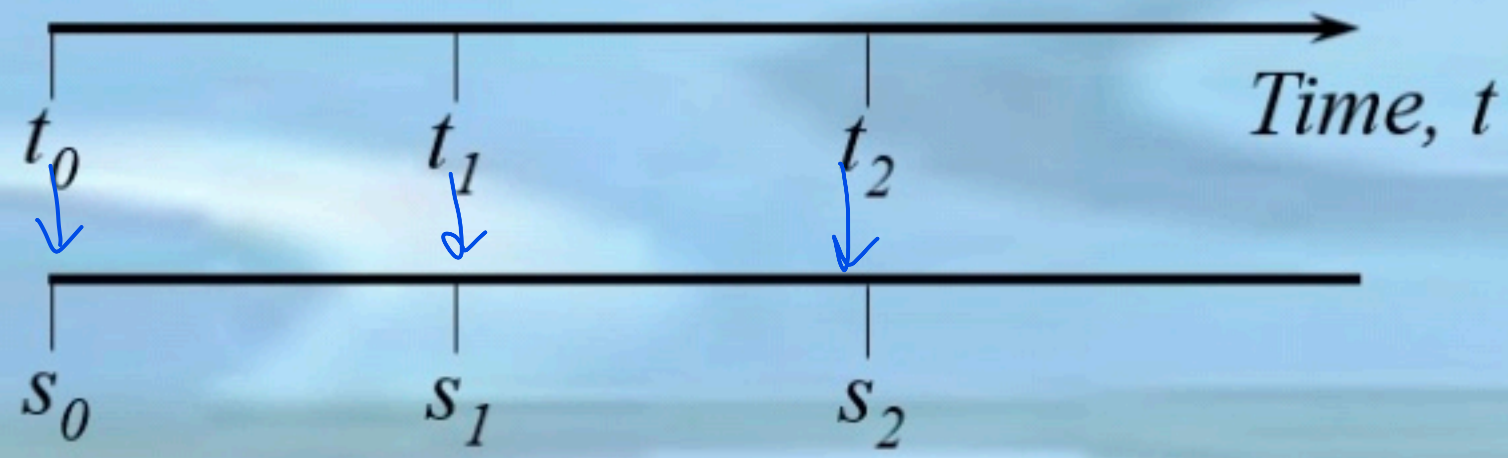
$$X_t = \begin{Bmatrix} 0, \text{if the condition is poor} \\ 1, \text{if the condition is fair} \\ 2, \text{if the condition is good} \end{Bmatrix}, t = 1, 2, \ldots$$

The random variable X_t is finite

Markov Chains

Chapter 17: Definition of a Markov Chain

System State, s: Describes the attributes at some point in time



Automated Teller Machine (ATM) Example

State ,s, at time t

State Space **S**

$\mathbf{S} = \{s_0 , s_1 , s_2 \ldots..\}$

$\mathbf{S} = \{3 , 2, 3 \ldots\}$ infinite

Markov Chains

Chapter 17: Definition of a Markov Chain

Job Shop Example: **Jobs arrive randomly at a shop at the rate of 5 jobs per hour. The arrival process follows a Poisson distribution, which, theoretically, allows any number of jobs to arrive at the shop during the time interval (0, t). The infinite-state processes describing the number of arriving jobs is $X_t = \{0, 1, 2, \ldots\}, t > 0$.**

Definition

Let x_t be a random variable that characterizes the state of the system at a discrete points of time $t = 1, 2, \ldots$. The family of random variables $\{X_t\}$ form a stochastic process with a finite or infinite number of state

- Definition: a stochastic process is a collection of random variables $\{x_t = \{x_1, x_2 \ldots x_t\}\}$, where t is a time index that takes values from a given set T

Probabilistic Operations Research

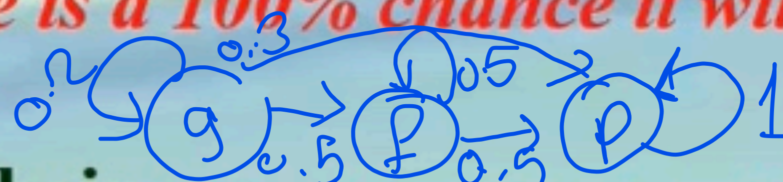
Example 17.1-3: The Gardener Problem

Every year, during the March-through-September growing season, a gardener uses a chemical test to check soil condition. Depending on the outcome of the soil test, productivity for the new season can be one of three states: (1) good, (2) fair, and (3) poor. Over the year, the gardener has observed that last year's soil condition impacts current year's productivity.

If this year's soil condition is good, there is a 0.2 chance it will not change next year, and a 0.5 chance it will be fair.

If this year's soil condition is fair, there is a 0.5 chance it will not change next year, and a 0.5 chance it will be poor.

If this year's soil condition is poor, there is a 100% chance it will not change next year.



Described this situation as a Markov chain (Construct the one step transition matrix)

Example 17.1-3: The Gardener Problem

three states: (1) good, (2) fair, and (3) poor. *If this year's soil condition is good, there is a 0.2 chance it will not change next year, and a 0.5 chance it will be fair.*

If this year's soil condition is fair, there is a 0.5 chance it will not change next year, and a 0.5 chance it will be poor.

If this year's soil condition is poor, there is a 100% chance it will not change next year.

Described this situation as a Markov chain **(Construct the one step transition matrix)**

State of the system next year

$$P^1 = \begin{matrix}\text{State of the}\\ \text{system this}\\ \text{year}\end{matrix} \left(\begin{array}{cc} & \begin{array}{ccc} 1good & 2\text{fair} & 3\text{poor} \end{array} \\ \begin{array}{c} 1\text{good} \\ 2\text{fair} \\ 3\text{poor} \end{array} & \begin{pmatrix} 0.2 & 0.5 & 0.3 \\ 0.0 & 0.5 & 0.5 \\ 0.0 & 0.0 & 1.0 \end{pmatrix} \end{array} \right)$$

Chapman-Kolomogorov mathematics (equations)

- Initial probability matrix P^0, it is an identity matrix
- One step transition probability matrix P^1 and
- n-step transition probability matrix (n = 2, 3, 4, 5, 6 …. n)

State of the system next year

$$P^0 = \begin{array}{c|ccc} \text{State of the system this year} & 1\text{ good} & 2\text{ fair} & 3\text{ poor} \\ \hline 1\text{ good} & 1.0 & 0.0 & 0.0 \\ 2\text{ fair} & 0.0 & 1.0 & 0.0 \\ 3\text{ poor} & 0.0 & 0.0 & 1.0 \end{array}$$

State of the system next year

$$P^1 = \begin{array}{c|ccc} \text{State of the system this year} & 1\text{ good} & 2\text{ fair} & 3\text{ poor} \\ \hline 1\text{ good} & 0.2 & 0.5 & 0.3 \\ 2\text{ fair} & 0.0 & 0.5 & 0.5 \\ 3\text{ poor} & 0.0 & 0.0 & 1.0 \end{array}$$

$$P^1 = P^0 P$$

$$P^2 = P^1 P$$

$$P^3 = P^2 P$$

.

.

$$P^n = P^{n-1} P$$

and

$$P^n = P^{n-m} P^m$$

Example 17.2-3: The Gardener Problem

The following matrix applies to the gardener problem with fertilizer (Example 17.1-3)

State of the system next year

$$P^1 = \begin{array}{c} \text{State of the} \\ \text{system this} \\ \text{year} \end{array} \begin{array}{c|ccc} & 1good & 2fair & 3poor \\ \hline 1good & 0.30 & 0.60 & 0.10 \\ 2fair & 0.10 & 0.60 & 0.30 \\ 3poor & 0.05 & 0.40 & 0.55 \end{array}$$

The initial condition of the soil is good. Determine the absolute probabilities of the three states of the system after 1, 2, and 3 gardening season.

$$a^{(1)} \text{good} \to j = g \begin{array}{ccc} g & f & p \\ \end{array} [0.3 \quad 0.6 \quad 0.1]$$

$$a^{(2)} \text{good} \to j = [a^{(1)} \text{good} \to j][P] = g [0.3 \quad 0.6 \quad 0.1] \cdot \begin{array}{c} \\ g \\ f \\ p \end{array} \begin{array}{ccc} g & f & p \\ 0.30 & 0.60 & 0.10 \\ 0.10 & 0.60 & 0.30 \\ 0.05 & 0.40 & 0.55 \end{array} = g [0.155 \quad 0.58 \quad 0.265]$$

Example 17.2-3: The Gardener Problem

The following matrix applies to the gardener problem with fertilizer (Example 17.1-3)

State of the system next year

$$P^1 = \begin{array}{c} \text{State of the} \\ \text{system this} \\ \text{year} \end{array} \left(\begin{array}{c|ccc} & 1good & 2fair & 3poor \\ 1good & & & \\ 2fair & & & \\ 3poor & & & \end{array} \right) \quad \begin{array}{c} \\ \begin{pmatrix} 0.30 & 0.60 & 0.10 \\ 0.10 & 0.60 & 0.30 \\ 0.05 & 0.40 & 0.55 \end{pmatrix} \end{array}$$

The initial condition of the soil is good. Determine the absolute probabilities of the three states of the system after 1, 2, 4, 8, and 16 gardening season.

$$\begin{array}{l} \qquad\qquad\qquad\quad\;\; g \qquad f \qquad p \\ a^{(1)}\ \text{good} \rightarrow j = g\begin{bmatrix} 0.3 & 0.6 & 0.1 \end{bmatrix} : \text{State}_{good} \text{ vector in } [p]^1 \end{array}$$

Example 17.2-3: The Gardener Problem

The following matrix applies to the gardener problem with fertilizer (Example 17.1-3)

State of the system next year

$$P^1 = \underset{\text{year}}{\overset{\text{State of the}}{\text{system this}}} \begin{array}{c|ccc} & 1good & 2fair & 3poor \\ \hline 1good & 0.30 & 0.60 & 0.10 \\ 2fair & 0.10 & 0.60 & 0.30 \\ 3poor & 0.05 & 0.40 & 0.55 \end{array}$$

The initial condition of the soil is good. Determine the absolute probabilities of the three states of the system after 1, 2, 4, 8, and 16 gardening season.

$$P^2 = \begin{pmatrix} 0.155 & 0.580 & 0.265 \\ 0.105 & 0.540 & 0.355 \\ 0.05825 & 0.49 & 0.4275 \end{pmatrix}$$

$$a^{(2)}\text{ good} \to j = g \begin{array}{ccc} g & f & p \\ [0.155 & 0.58 & 0.265] \end{array} : \text{State}_{\text{good}} \text{ vector in } [p]^2$$

Example 17.2-3: The Gardener Problem

The following matrix applies to the gardener problem with fertilizer (Example 17.1-3)

State of the system next year

$$P^1 = \text{State of the system this year} \begin{array}{c|ccc} & 1good & 2fair & 3poor \\ \hline 1good & 0.30 & 0.60 & 0.10 \\ 2fair & 0.10 & 0.60 & 0.30 \\ 3poor & 0.05 & 0.40 & 0.55 \end{array}$$

The initial condition of the soil is good. Determine the absolute probabilities of the three states of the system after 1, 2, 4, 8, and 16 gardening season.

$$P^4 = \begin{pmatrix} 0.1068 & 0.5330 & 0.3603 \\ 0.1023 & 0.5265 & 0.3713 \\ 0.0995 & 0.5219 & 0.3786 \end{pmatrix}$$

$$a^{(4)} \text{good} \rightarrow j = g \begin{array}{ccc} g & f & p \\ [0.1068 & 0.5330 & 0.3603] \end{array} : \text{State}_{good} \text{ vector in } [p]^4$$

Example 17.2-3: The Gardener Problem

The following matrix applies to the gardener problem with fertilizer (Example 17.1-3)

State of the system next year

$$P^1 = \text{State of the system this year} \quad \begin{array}{c|ccc} & 1good & 2fair & 3poor \\ \hline 1good & 0.30 & 0.60 & 0.10 \\ 2fair & 0.10 & 0.60 & 0.30 \\ 3poor & 0.05 & 0.40 & 0.55 \end{array}$$

$P^1 P = P^2$

The initial condition of the soil is good. Determine the absolute probabilities of the three states of the system after 1, 2, 4, 8, and 16 gardening season.

$$P^8 = \begin{pmatrix} 0.101753 & 0.525514 & 0.372733 \\ 0.101702 & 0.525435 & 0.372863 \\ 0.101669 & 0.525384 & 0.372863 \end{pmatrix}$$

$$a^{(8)}\ \text{good} \rightarrow j = g\begin{array}{c} \\ \left[\right. \end{array}\!\!\begin{array}{ccc} g & f & p \\ 0.101753 & 0.525514 & 0.372733 \end{array}\!\!\begin{array}{c} \\ \left.\right] \end{array} : \text{State}_{good} \text{ vector in } [p]^8$$

Example 17.2-3: The Gardener Problem

The following matrix applies to the gardener problem with fertilizer (Example 17.1-3)

State of the system next year

$$P^1 = \begin{array}{c} \text{State of the} \\ \text{system this} \\ \text{year} \end{array} \left(\begin{array}{c|ccc} & 1\textit{good} & 2\text{fair} & 3\text{poor} \\ 1\text{good} & 0.30 & 0.60 & 0.10 \\ 2\text{fair} & 0.10 & 0.60 & 0.30 \\ 3\text{poor} & 0.05 & 0.40 & 0.55 \end{array} \right)$$

The initial condition of the soil is good. Determine the absolute probabilities of the three states of the system after 1, 2, 4, 8, and 16 gardening season.

$$P^{16} = \begin{pmatrix} 0.101659 & 0.52454 & 0.372881 \\ 0.101659 & 0.52454 & 0.372881 \\ 0.101659 & 0.525354 & 0.372881 \end{pmatrix}$$

$$a^{(16)}\ \text{good} \rightarrow j = g \begin{array}{c} \\ \left[\right. \end{array}\!\!\begin{array}{ccc} g & f & p \\ 0.101659 & 0.52454 & 0.372881 \end{array}\!\!\begin{array}{c} \\ \left. \right] \end{array} : \text{State}_{\text{good}}\ \text{vector in}\ [p]^{16}$$

Example 17.2-3: The Gardener Problem

The following matrix applies to the gardener problem with fertilizer (Example 17.1-3)

State of the system next year

$$P^1 = \text{State of the system this year} \quad \begin{array}{c} \\ \text{1good} \\ \text{2fair} \\ \text{3poor} \end{array} \begin{array}{c} \begin{array}{ccc} \text{1good} & \text{2fair} & \text{3poor} \end{array} \\ \begin{pmatrix} 0.30 & 0.60 & 0.10 \\ 0.10 & 0.60 & 0.30 \\ 0.05 & 0.40 & 0.55 \end{pmatrix} \end{array}$$

If the initial condition of the soil is good- that is

1. Write down the initial transition probability vector from the good state to all other possible states (after 0 step).

$$a^{(0)} \text{ good} \rightarrow j = g \begin{array}{c} \begin{array}{ccc} g & f & p \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \end{array}$$

2. Write down the transition probability vector from the good state to all other possible states after 1 step.

Example 17.2-3: The Gardener Problem

1. **Write down the initial transition probability vector from the good state to all other possible states (after 0 step).**
2. **Write down the transition probability vector from the good state to all other possible states after 1 step.**

$$a^{(0)}\ \text{good} \to j = g\begin{array}{c} \\ \\ \end{array}\overset{\begin{array}{ccc} g & f & p \end{array}}{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}$$

$$a^{(1)}\ \text{good} \to j = \left[a^{(0)}\ \text{good} \to j\right][P] = g\overset{\begin{array}{ccc} g & f & p \end{array}}{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}} \cdot \left(\begin{array}{cc} & \begin{array}{ccc} g & f & p \end{array} \\ \begin{array}{c} g \\ f \\ p \end{array} & \begin{pmatrix} 0.30 & 0.60 & 0.10 \\ 0.10 & 0.60 & 0.30 \\ 0.05 & 0.40 & 0.55 \end{pmatrix} \end{array} \right) = g\overset{\begin{array}{ccc} g & f & p \end{array}}{\begin{bmatrix} 0.3 & 0.6 & 0.1 \end{bmatrix}}$$

3. **Compute the transition probability vector from the good state to all other possible states after 2 step.**

$$a^{(1)}\ \text{good} \to j = g\overset{\begin{array}{ccc} g & f & p \end{array}}{\begin{bmatrix} 0.3 & 0.6 & 0.1 \end{bmatrix}}$$

$$a^{(2)}\ \text{good} \to j = \left[a^{(1)}\ \text{good} \to j\right][P] = g\overset{\begin{array}{ccc} g & f & p \end{array}}{\begin{bmatrix} 0.3 & 0.6 & 0.1 \end{bmatrix}} \cdot \left(\begin{array}{cc} & \begin{array}{ccc} g & f & p \end{array} \\ \begin{array}{c} g \\ f \\ p \end{array} & \begin{pmatrix} 0.30 & 0.60 & 0.10 \\ 0.10 & 0.60 & 0.30 \\ 0.05 & 0.40 & 0.55 \end{pmatrix} \end{array} \right) = g\overset{\begin{array}{ccc} g & f & p \end{array}}{\begin{bmatrix} 0.155 & 0.58 & 0.265 \end{bmatrix}}$$

Example 17.2-3: The Gardener Problem

4. Compute the transition probability vector from the good state to all other possible states after 3 step.

$$a^{(2)}\,\text{good} \rightarrow j = \begin{array}{c|ccc} & g & f & p \\ \hline g & 0.155 & 0.58 & 0.265 \end{array}$$

$$a^{(3)}\,\text{good} \rightarrow j = \left[a^{(2)}\,\text{good} \rightarrow j\right][P] = \begin{array}{cccc} & g & f & p \\ g & [0.155 & 0.58 & 0.265] \end{array} . \left(\begin{array}{cccc} & g & f & p \\ g & 0.30 & 0.60 & 0.10 \\ f & 0.10 & 0.60 & 0.30 \\ p & 0.05 & 0.40 & 0.55 \end{array}\right) = \begin{array}{cccc} & g & f & p \\ g & [0.1178 & 0.5470 & 0.3353] \end{array}$$

5. Compute the transition probability vector from the good state to all other possible states after 4 step.

$$a^{(4)}\,\text{good} \rightarrow j = \left[a^{(3)}\,\text{good} \rightarrow j\right][P] = \begin{pmatrix} 0.1178 & 0.547 & 0.3353 \end{pmatrix} \begin{pmatrix} 0.30 & 0.60 & 0.10 \\ 0.10 & 0.60 & 0.30 \\ 0.05 & 0.40 & 0.55 \end{pmatrix} = \begin{array}{cccc} & \text{good} & \text{fair} & \text{poor} \\ \text{good} & (0.1068 & 0.5330 & 0.3603) \end{array}$$

$$a^{n} \cdot p^{1} = a^{n+1}$$

Markov Chains

Chapter 17: Absolute and n-step transition probabilities

Example 17.2-3: The Gardener Problem

4. A Spreadsheet for computing the transition probability vector from the good state to all other possible states after infinite number of step (19 step).

Note that after step 10 the vector does not changed

Season	Good	Fair	Poor	Sum
0	1.0000	0.0000	0.0000	1
1	0.3000	0.6000	0.1000	1
2	0.1550	0.5800	0.2650	1
3	0.1178	0.5470	0.3353	1
4	0.1068	0.5330	0.3603	1
5	0.1033	0.5279	0.3687	1
6	0.1022	0.5263	0.3715	1
7	0.1019	0.5257	0.3724	1
8	0.1018	0.5255	0.3727	1
9	0.1017	0.5255	0.3728	1
10	0.1017	0.5254	0.3729	1
11	0.1017	0.5254	0.3729	1
12	0.1017	0.5254	0.3729	1
13	0.1017	0.5254	0.3729	1
14	0.1017	0.5254	0.3729	1
15	0.1017	0.5254	0.3729	1
16	0.1017	0.5254	0.3729	1
17	0.1017	0.5254	0.3729	1
18	0.1017	0.5254	0.3729	1
19	0.1017	0.5254	0.3729	1

After many steps, the system reaches the steady-state condition, where:

$$a^{(2)}\ \text{good} \rightarrow j = g\begin{array}{c} \begin{array}{ccc} g & f & p \end{array} \\ \begin{bmatrix} 0.155 & 0.58 & 0.265 \end{bmatrix} \end{array}$$

$$ss^{(?)}\ \text{good} \rightarrow j = g\begin{array}{c} \begin{array}{ccc} g & f & p \end{array} \\ \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix} \end{array} \cdot \left(\begin{array}{cc} & \begin{array}{ccc} g & f & p \end{array} \\ \begin{array}{c} g \\ f \\ p \end{array} & \begin{pmatrix} 0.30 & 0.60 & 0.10 \\ 0.10 & 0.60 & 0.30 \\ 0.05 & 0.40 & 0.55 \end{pmatrix} \end{array} \right) = g\begin{array}{c} \begin{array}{ccc} g & f & p \end{array} \\ \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix} \end{array}$$

Steady

steady

6. Compute the steady state transition probability vector from the good state to all other possible states

$$\pi P = \pi$$

$$\pi_1 = 0.3\pi_1 + 0.1\pi_2 + 0.05\pi_3$$

$$\pi_2 = 0.6\pi_1 + 0.6\pi_2 + 0.4\pi_3$$

$$\pi_3 = 0.1\pi_1 + 0.3\pi_2 + 0.55\pi_3$$

$$\pi_1 + \pi_2 + \pi_3 = 1$$

Prob.

$$\pi_1 = \pi_{good} = 0.1017,$$

$$\pi_2 = \pi_{fair} = 0.5254,$$

$$\pi_3 = \pi_{poor} = 0.3729$$

$$\pi_{Good \rightarrow j} = \lim_{n \rightarrow \infty} a^{(n)}_{Good \rightarrow j}$$

IE 0960552 Probabilistic Operations Research

The *steady state probabilities* are defined as:

$$\pi_j = \lim_{n \to \infty} a_j^{(n)}, \; j = 0, 1, 2 \ldots.$$

These probabilities are independent of $[a_j^{(0)}]$, can be determined from the equation:

$$\boxed{\begin{aligned} \boldsymbol{\pi} &= \boldsymbol{\pi}\mathbf{P}^{1} \\ \sum_j \boldsymbol{\pi}_j &= 1 \end{aligned}}$$

The expected number of transitions before the system returns to a state j for the first time is known as *the mean first return time or the mean recurrence time*, computed from the equation:

$$\boxed{\mu_{jj} = \frac{1}{\pi_j}, \; j = 0, 1, 2 \ldots.}$$

Markov Chains

Chapter 17: Steady state probabilities and return time

For the gardener problem with fertilizer, The mean first return times (season) are computed as

$$\pi_{good} = 0.1017,\ \pi_{fair} = 0.5254,\ \pi_{poor} = 0.3729$$

$$\mu_{11} = \mu_{good\to good} = \frac{1}{0.1017} = 9.83,\ \mu_{22} = \mu_{fair\to fair} = \frac{1}{0.5254} = 1.9,\ \mu_{33} = \mu_{poor\to poor} = \frac{1}{0.3729} = 2.68$$

The garden needs 2 bags of fertilizer if the soil is good. The amount is increased by 25% if the soil is fair and 60% if the soil is poor. The cost of the fertilizer is $50 per bag. Estimate the seasonal expected cost of fertilizer

bags x cost of bag x steady state prob.

$$= 2\times\$50\times\pi_1 + (1.25\times2)\times\$50\times\pi_2 + (1.6\times2)\times\$50\times\pi_3$$

$$= 2\times\$50\times0.1017 + (1.25\times2)\times\$50\times0.5254 + (1.6\times2)\times\$50\times0.3729$$

$$= \$135.51$$

A simpler way to determine the mean first for all the states in an m-transition matrix, P, is to use the following

$$\left\| \mu_{ij} \right\| = \left(\mathbf{I} - \mathbf{N}_j \right)^{-1} \mathbf{1}, \; j \neq i$$

- **I**: *(m-1) identity matrix*
- $\mathbf{N_j}$: *transition matrix P less its j^{th} row and j^{th} column of target state j*
- **1** *(m-1) column vector with all elements equal to 1*

Example 17.5-1

Consider the gardener Markov chain with fertilizers once again

State of the system next year

$$P^1 = \text{State of the system this year} \begin{array}{c|ccc} & 1\textit{good} & 2\text{fair} & 3\text{poor} \\ \hline 1\text{good} & 0.30 & 0.60 & 0.10 \\ 2\text{fair} & 0.10 & 0.60 & 0.30 \\ 3\text{poor} & 0.05 & 0.40 & 0.55 \end{array}$$

$$\left\|\mu_{ij}\right\| = \left(\mathbf{I} - \mathbf{N}_j\right)^{-1} \mathbf{1}, \ j \neq i$$

- **I**: *(m-1) identity matrix*
- $\mathbf{N_j}$: *transition matrix P less its j^{th} row and j^{th} column of target state j*
- **1** *(m-1) column vector with all elements equal to 1*

Example 17.5-1

Consider the gardener Markov chain with fertilizers once again

State of the system next year

$$P^1 = \text{State of the system this year} \begin{array}{c|ccc} & 1\text{good} & 2\text{fair} & 3\text{poor} \\ 1\text{good} & 0.30 & 0.60 & 0.10 \\ 2\text{fair} & 0.10 & 0.60 & 0.30 \\ 3\text{poor} & 0.05 & 0.40 & 0.55 \end{array}$$

Consider the passage from states 2 and 3 to state 1, thus j = 1, i = 2, 3

$$\begin{bmatrix} \mu_{21} \\ \mu_{31} \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.60 & 0.30 \\ 0.40 & 0.55 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 12.50 \\ 13.34 \end{bmatrix}$$

It takes 12.5 season on average to pass from fair to good soil, and 13.34 season to pass from bad to good soil

$$\|\mu_{ij}\| = (\mathbf{I} - \mathbf{N}_j)^{-1}\,\mathbf{1},\ j \neq i$$

- **I**: *(m-1) identity matrix*
- $\mathbf{N}_j$: *transition matrix P less its j^{th} row and j^{th} column of target state j*
- **1** *(m-1) column vector with all elements equal to 1*

Example 17.5-1

Consider the gardener Markov chain with fertilizers once again

$$P^1 = \text{State of the system this year} \quad \begin{array}{c|ccc} & \multicolumn{3}{c}{\text{State of the system next year}} \\ & 1\,good & 2\,fair & 3\,poor \\ \hline 1\,good & 0.30 & 0.60 & 0.10 \\ 2\,fair & 0.10 & 0.60 & 0.30 \\ 3\,poor & 0.05 & 0.40 & 0.55 \end{array}$$

Similarly

Consider the passage from states 1 and 3 to state 2, thus j = 2, i = 1, 3

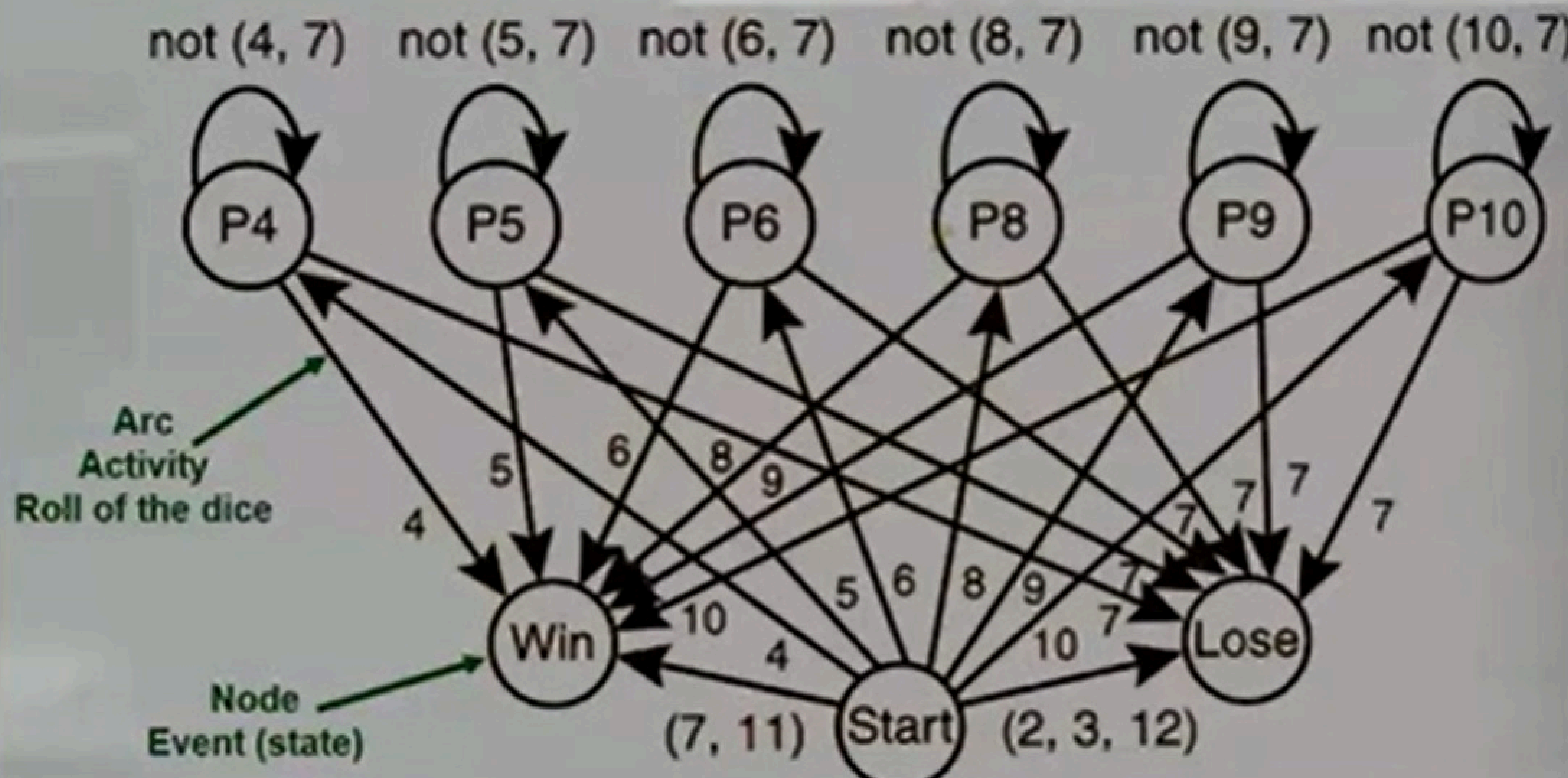
$$\begin{bmatrix} \mu_{12} \\ \mu_{32} \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.30 & 0.10 \\ 0.05 & 0.55 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.77 \\ 2.41 \end{bmatrix}$$

It takes 1.77 season on average to pass from good to fair soil, and 2.41 season to pass from poor to fair soil

Discrete-Time Markov Chain (DTMC)
The Game of Craps

- The player rolls a pair of dice and sums the numbers showing. A total of 7 or 11 on the first roll wins for player, whereas a total of 2, 3, or 12 loses. Any other number is called the point. The player then rolls the dice again. If she rolls the point number, she wins. If she throws a 7, she loses. Any other number requires another roll. The process continues until either a 7 or the point is thrown.

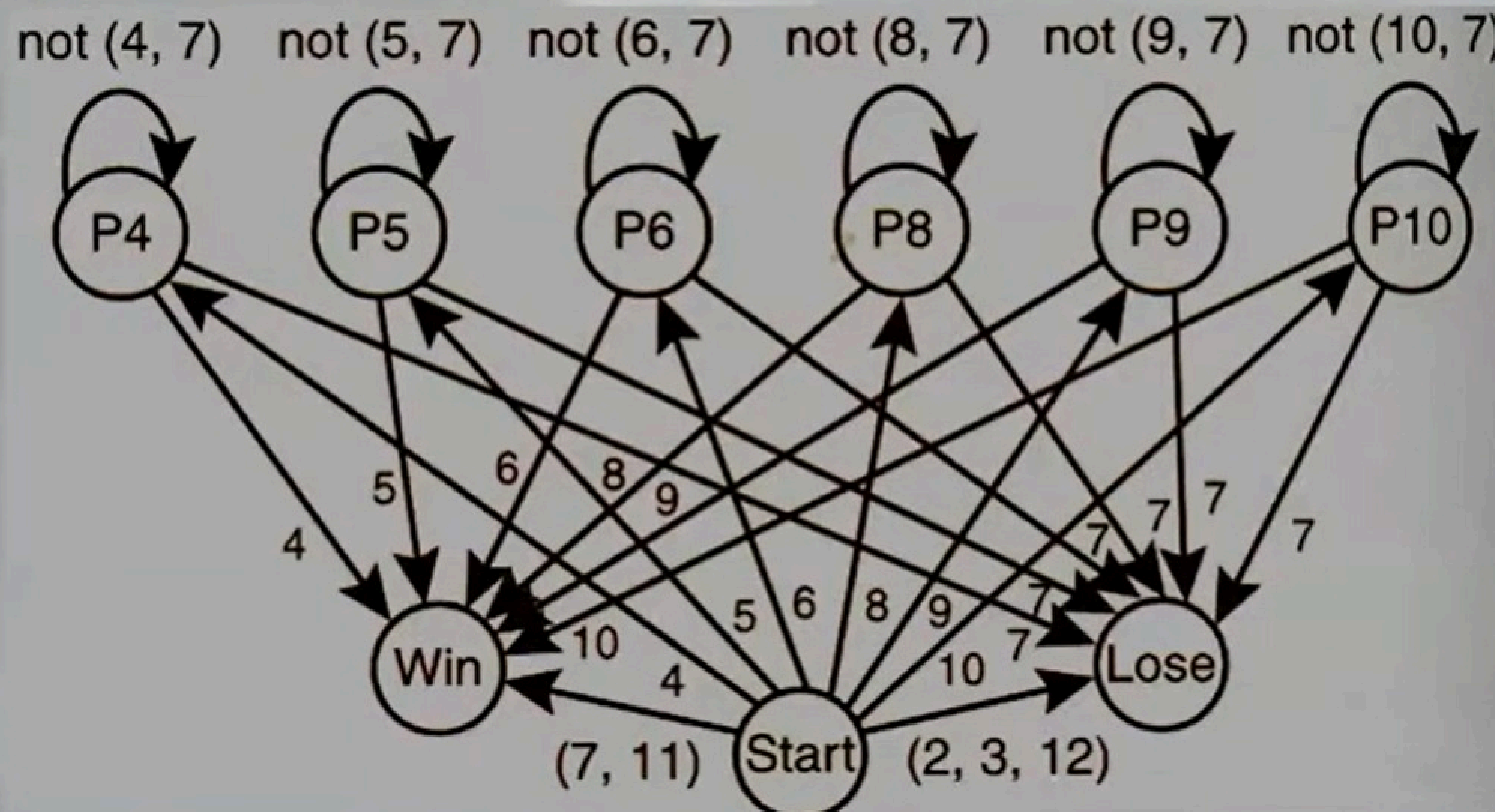
  Construct the state transition network and the transition matrix for the game.



Discrete-Time Markov Chain (DTMC)
The Game of Craps

Sum of the numbers showing	2	3	4	5	6	7	8	9	10	11	12
Discrete Probability of sum	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

- Probability of throwing a 7 is 6/36 = 0.167
- Probability of throwing a 11 is 2/36 = 0.056
- Probability of throwing a 7 or 11 is 6/36 + 2/36 = 0.222 = probability of win from first time = p_{SW}

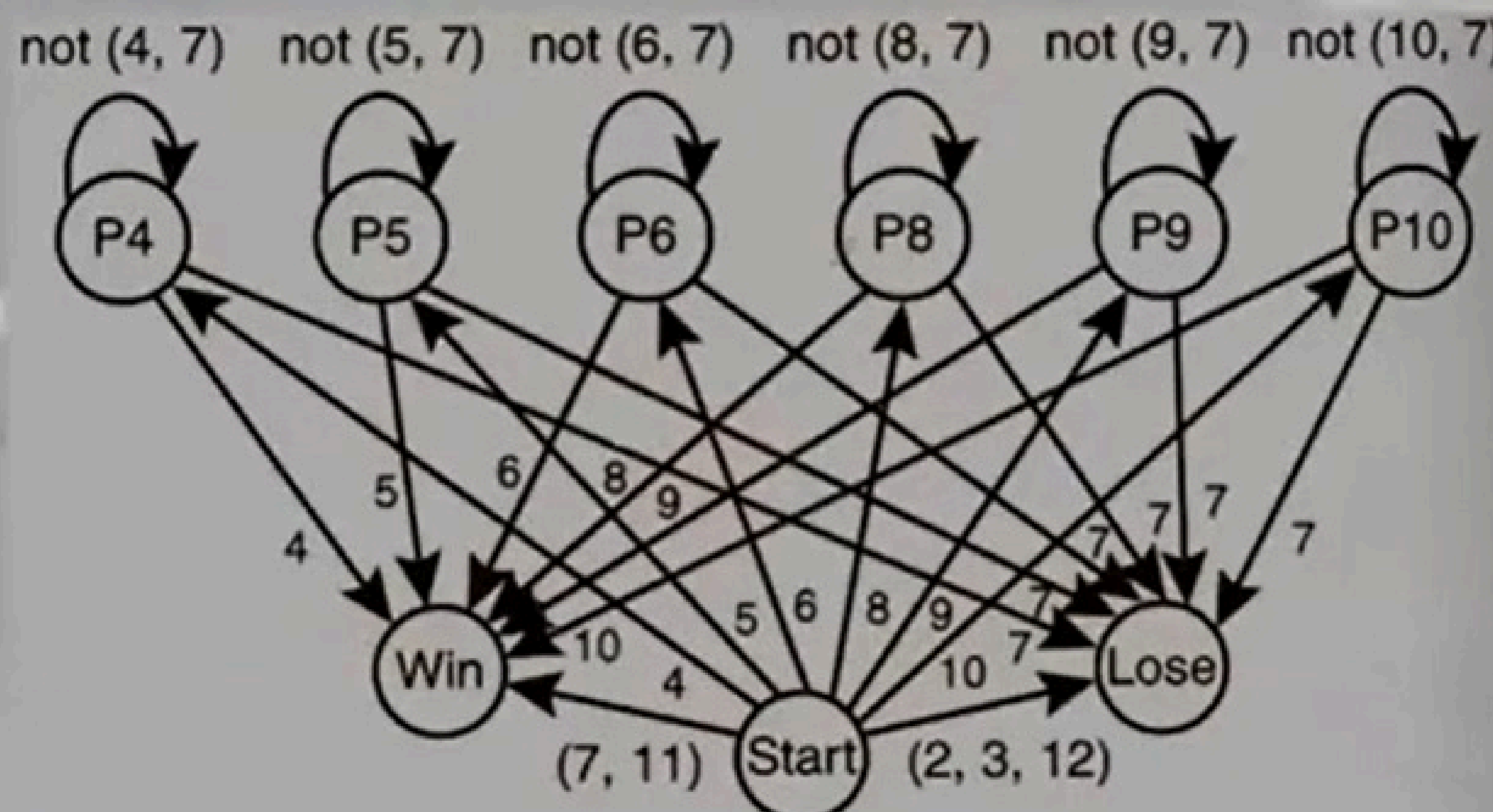


Discrete-Time Markov Chain (DTMC)
The Game of Craps

Sum of the numbers showing	2	3	4	5	6	7	8	9	10	11	12
Discrete Probability of sum	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

If the current state is S

- Probability of throwing a 7 is 6/36 = 0.167
- Probability of throwing a 11 is 2/36 = 0.056
- Probability of throwing a 7 or 11 is 6/36 + 2/36 = 0.222 = probability of win = p_{SW}
- Probability of Lose = p_{SL} = Pr{2} or pr{3} or pr{12} = 1/36 + 2/36 + 1/36 = 4/36 = 0.111
- $p_{S,P4}$ = 3/36 = 0.083 Similarly: $p_{S,P5}$ = 4/36 = 0.111, $p_{S,P6}$ = 0.139, $p_{S,\ P8}$ = 0.139, $P_{S,\ P9}$ = 0.111, $p_{S,\ P10}$ = 0.083



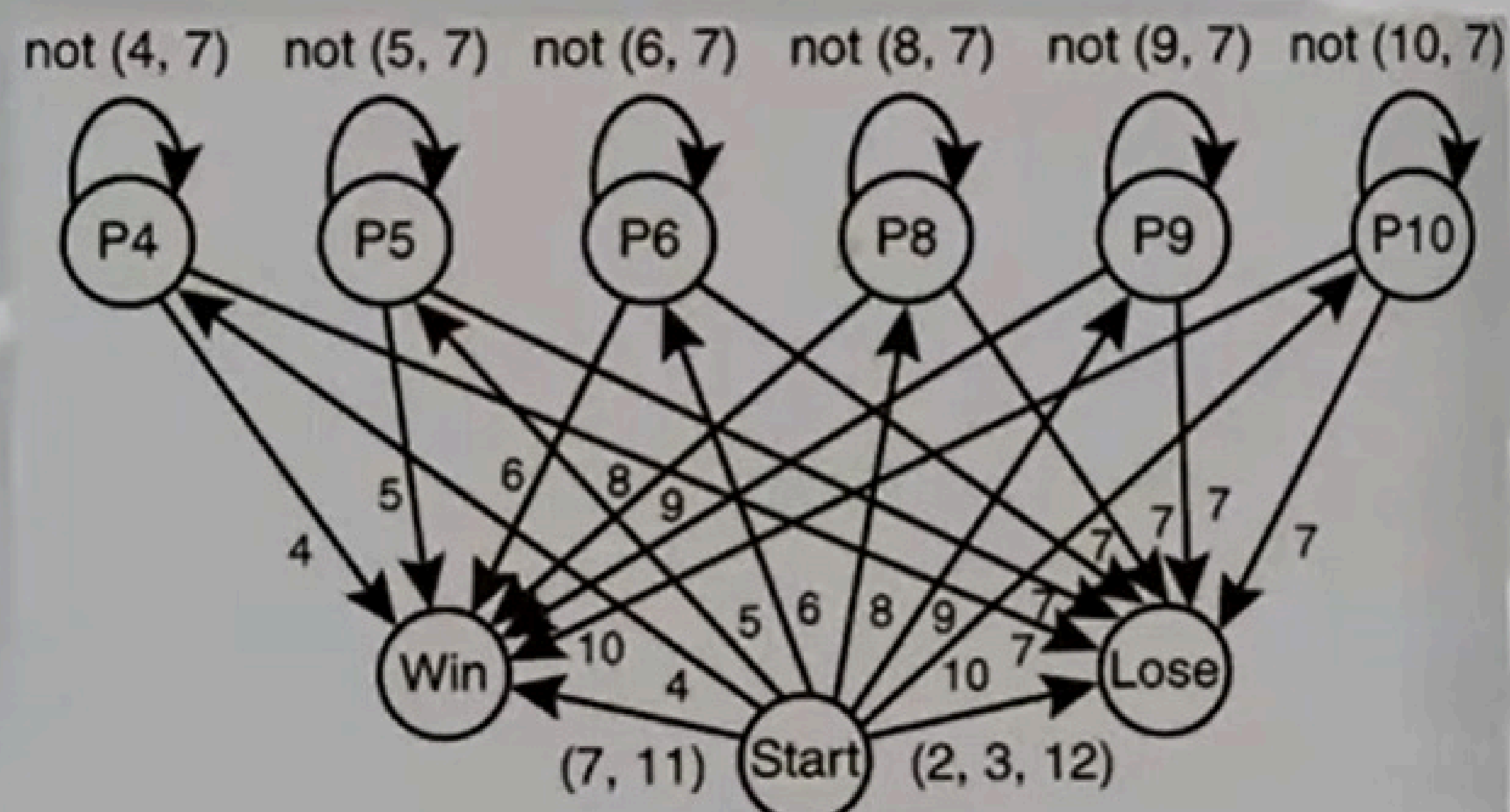
$$\begin{array}{c|ccccccccc} & S & W & L & P4 & P5 & P6 & P8 & P9 & P10 \\ \hline S & 0.00 & 0.222 & 0.111 & 0.083 & 0.111 & 0.139 & 0.139 & 0.111 & 0.083 \end{array} = \text{Star tvector} = q_{start}$$

Discrete-Time Markov Chain (DTMC)
The Game of Craps

Sum of the numbers showing	2	3	4	5	6	7	8	9	10	11	12
Discrete Probability of sum	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

If the current state is P4

- Probability of Win is the pr {4} = $p_{P4,W}$ = 3/36 = 0.083
- Probability of Lose = $p_{P4,L}$ = Pr{7} = 6/36 = 0.167
- $p_{P4,4}$ = 1- pr {4}- pr {7} = 1- $p_{P4,W} - p_{P4,L}$ = 1- 0.083 – 0.167 = 0.75
- $p_{P4,f}$ = 0.0 for all f = { S, P5, P6, P8, P9, P10}



$$\begin{array}{c} \\ \underline{P4} \end{array}\begin{array}{ccccccccc} S & W & L & P4 & P5 & P6 & P8 & P9 & P10 \\ \end{array}$$
$$P4\,[0.0 \quad 0.083 \quad 0.167 \quad 0.750 \quad 0.000 \quad 0.000 \quad 0.000 \quad 0.000 \quad 0.000] = \text{P4 vector} = q_{P4}(\text{Roll } 1)$$

The Game of Craps
Identify absorbing and transient states ?

- Similarly: the collection of the transition probabilities forms the shown transition matrix P which, along with the state definitions, completely describes the Markov chain

$$
P = \begin{array}{c} \\ S \\ W \\ L \\ P4 \\ P5 \\ P6 \\ P8 \\ P9 \\ P10 \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} S & W & L & P4 & P5 & P6 & P8 & P9 & P10 \end{array} \\
\begin{bmatrix}
0 & 0.222 & 0.111 & 0.083 & 0.111 & 0.139 & 0.139 & 0.111 & 0.083 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.083 & 0.167 & 0.75 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.111 & 0.167 & 0 & 0.722 & 0 & 0 & 0 & 0 \\
0 & 0.139 & 0.167 & 0 & 0 & 0.694 & 0 & 0 & 0 \\
0 & 0.139 & 0.167 & 0 & 0 & 0 & 0.694 & 0 & 0 \\
0 & 0.111 & 0.167 & 0 & 0 & 0 & 0 & 0.722 & 0 \\
0 & 0.083 & 0.167 & 0 & 0 & 0 & 0 & 0 & 0.750
\end{bmatrix}
\end{array}
$$

- Win and Lose are absorbing states signaling that the game is Over. Probability of 1 of the main diagonal for these two states.
- The other states are transient states.
- Probability that a particular absorbing state will ultimately be reached.

The Game of Craps
Find the Absorbing State Probabilities ?

$$
P = \begin{array}{c} \\ S \\ W \\ L \\ P4 \\ P5 \\ P6 \\ P8 \\ P9 \\ P10 \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} S & P4 & P5 & P6 & P8 & P9 & P10 & W & L \end{array} \\
\begin{bmatrix}
0 & 0.083 & 0.111 & 0.139 & 0.139 & 0.111 & 0.083 & 0.222 & 0.111 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0.75 & 0 & 0 & 0 & 0 & 0 & 0.083 & 0.167 \\
0 & 0 & 0.722 & 0 & 0 & 0 & 0 & 0.111 & 0.167 \\
0 & 0 & 0 & 0.694 & 0 & 0 & 0 & 0.139 & 0.167 \\
0 & 0 & 0 & 0 & 0.694 & 0 & 0 & 0.139 & 0.167 \\
0 & 0 & 0 & 0 & 0 & 0.722 & 0 & 0.111 & 0.167 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.750 & 0.083 & 0.167
\end{bmatrix}
\end{array}
$$

$$Q = [I - S]^{-1} T$$

$$
Q = \begin{array}{c}
\begin{array}{cc} \text{Win} & \text{Lose} \end{array} \\
\begin{bmatrix}
0.493 & 0.507 \\
0.333 & 0.667 \\
0.400 & 0.600 \\
0.455 & 0.545 \\
0.455 & 0.545 \\
0.400 & 0.600 \\
0.333 & 0.667
\end{bmatrix}
\end{array}
\begin{array}{c} \\ \text{Start} \\ P4 \\ P5 \\ P6 \\ P8 \\ P9 \\ P10 \end{array}
$$

- the probability that The system will pass to absorbing state j if it begins in transient state i.

Operations Research - II

The Game of Craps
Find the Absorbing State Probabilities ?

P =

	S	P4	P5	P6	P8	P9	P10	W	L
S	0	0.083	0.111	0.139	0.139	0.111	0.083	0.222	0.111
W	0	0	0	0	0	0	0	1	0
L	0	0	0	0	0	0	0	0	1
P4	0	0.75	0	0	0	0	0	0.083	0.167
P5	0	0	0.722	0	0	0	0	0.111	0.167
P6	0	0	0	0.694	0	0	0	0.139	0.167
P8	0	0	0	0	0.694	0	0	0.139	0.167
P9	0	0	0	0	0	0.722	0	0.111	0.167
P10	0	0	0	0	0	0	0.750	0.083	0.167

P =

	S	P4	P5	P6	P8	P9	P10	W	L
S	0	0.083	0.111	0.139	0.139	0.111	0.083	0.222	0.111
P4	0	0.75	0	0	0	0	0	0.083	0.167
P5	0	0	0.722	0	0	0	0	0.111	0.167
P6	0	0	0	0.694	0	0	0	0.139	0.167
P8	0	0	0	0	0.694	0	0	0.139	0.167
P9	0	0	0	0	0	0.722	0	0.111	0.167
P10	0	0	0	0	0	0	0.750	0.083	0.167
W	0	0	0	0	0	0	0	1	0
L	0	0	0	0	0	0	0	0	1

$$P = \begin{bmatrix} S & T \\ 0 & I \end{bmatrix}$$

$$P = \begin{bmatrix} \text{Transient} & \text{Absorbing} \\ \text{Zeros} & \text{Identity} \end{bmatrix}$$

$$Q = [I - S]^{-1} T$$

Q =

	Win	Lose
Start	0.493	0.507
P4	0.333	0.667
P5	0.400	0.600
P6	0.455	0.545
P8	0.455	0.545
P9	0.400	0.600
P10	0.333	0.667

- the probability that The system will pass to absorbing state j if it begins in transient state i.

Operations Research - II

The Game of Craps
Multi Step Transitions Probabilities

- Is the probability that the system will be in state j after n periods given the state of the current period
- The Transition Matrix P provides direct information about one step transition probabilities.
- P can be used to calculate the probabilities for transitions involving more than one steps (n)
- The n step transition matrix $P^{(n)}$ gives the n step transition probability from each state to every other state
- In general the n step transition matrix is define as:

  $P^{(n)} = P^{(n-1)}\,P$ (Derived from Chapman- Kolmogorov (C-K) equation)
- The n step transition probability $p^{(n)}_{ij}$ is element of $P^{(n)}$
- **After five rolls of the game of Craps the 5- step Transition matrix is:**

  $P^{(5)} = P^{(4)}\,P = P^{(3)}\,PP = P^{(2)}\,PPP = PPPPP$

Where:

$$P^{1} = \begin{array}{c} \\ S \\ W \\ L \\ P4 \\ P5 \\ P6 \\ P8 \\ P9 \\ P10 \end{array}\begin{array}{c} \begin{array}{ccccccccc} S & W & L & P4 & P5 & P6 & P8 & P9 & P10 \end{array} \\ \left[\begin{array}{ccccccccc} 0 & 0.222 & 0.111 & 0.083 & 0.111 & 0.139 & 0.139 & 0.111 & 0.083 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.083 & 0.167 & 0.75 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.111 & 0.167 & 0 & 0.722 & 0 & 0 & 0 & 0 \\ 0 & 0.139 & 0.167 & 0 & 0 & 0.694 & 0 & 0 & 0 \\ 0 & 0.139 & 0.167 & 0 & 0 & 0 & 0.694 & 0 & 0 \\ 0 & 0.111 & 0.167 & 0 & 0 & 0 & 0 & 0.722 & 0 \\ 0 & 0.083 & 0.167 & 0 & 0 & 0 & 0 & 0 & 0.750 \end{array}\right]\end{array}\begin{array}{l} \\ q_{S,S}(1) \\ q_{W,S}(1) \\ q_{L,S}(1) \\ q_{P4,S}(1) \\ q_{P5,S}(1) \\ q_{P6,S}(1) \\ q_{P8,S}(1) \\ q_{P9,S}(1) \\ q_{P10,S}(1) \end{array}$$

1. **Write down the initial transition probability vector from the good state to all other possible states (after 0 step).**

$$a_{Start}(0) = S\begin{array}{c} \\ \left[\begin{array}{ccccccccc} S & W & L & P4 & P5 & P6 & P8 & P9 & P10 \\ 1.00 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \end{array}\right] \end{array} \text{Roll } 0$$

2. **Write down the transition probability vector from the start state to all other possible states after 1 step.**

$$a^{(1)} = a^{(0)}.P^1 = (1,0,0,0,0,0,0,0,0)P = \begin{array}{c|ccccccccc} & S & W & L & P4 & P5 & P6 & P8 & P9 & P10 \\ \hline S & 0 & 0.2222 & 0.1111 & 0.0833 & 0.1111 & 0.1389 & 0.1389 & 0.1111 & 0.0833 \\ W & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ L & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ P4 & 0 & 0.0833 & 0.1667 & 0.75 & 0 & 0 & 0 & 0 & 0 \\ P5 & 0 & 0.1111 & 0.1667 & 0 & 0.7222 & 0 & 0 & 0 & 0 \\ P6 & 0 & 0.1389 & 0.1667 & 0 & 0 & 0.694 & 0 & 0 & 0 \\ P8 & 0 & 0.1389 & 0.1667 & 0 & 0 & 0 & 0.6944 & 0 & 0 \\ P9 & 0 & 0.1111 & 0.1667 & 0 & 0 & 0 & 0 & 0.7222 & 0 \\ P10 & 0 & 0.0833 & 0.1667 & 0 & 0 & 0 & 0 & 0 & 0.750 \end{array}$$

$$= S\left[\begin{array}{ccccccccc} S & W & L & P4 & P5 & P6 & P8 & P9 & P10 \\ 0.00 & 0.2222 & 0.1111 & 0.0833 & 0.1111 & 0.1389 & 0.1389 & 0.1111 & 0.0833 \end{array}\right]$$

3. Compute the transition probability vector from the start state to all other possible states after 2 step.

$$a^{(2)} = a^{(1)}.P^{1} = (0,0.2222,0.1111,0.0833,0.1111,0.1389,0.1389,0.1111,0.0833)P = \begin{array}{c|ccccccccc} & S & W & L & P4 & P5 & P6 & P8 & P9 & P10 \\ \hline S & 0 & 0.2222 & 0.1111 & 0.0833 & 0.1111 & 0.1389 & 0.1389 & 0.1111 & 0.0833 \\ W & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ L & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ P4 & 0 & 0.0833 & 0.1667 & 0.75 & 0 & 0 & 0 & 0 & 0 \\ P5 & 0 & 0.1111 & 0.1667 & 0 & 0.7222 & 0 & 0 & 0 & 0 \\ P6 & 0 & 0.1389 & 0.1667 & 0 & 0 & 0.694 & 0 & 0 & 0 \\ P8 & 0 & 0.1389 & 0.1667 & 0 & 0 & 0 & 0.6944 & 0 & 0 \\ P9 & 0 & 0.1111 & 0.1667 & 0 & 0 & 0 & 0 & 0.7222 & 0 \\ P10 & 0 & 0.0833 & 0.1667 & 0 & 0 & 0 & 0 & 0 & 0.750 \end{array}$$

$$= \begin{array}{c|ccccccccc} & S & W & L & P4 & P5 & P6 & P8 & P9 & P10 \\ \hline S & 0.00 & 0.299 & 0.222 & 0.063 & 0.080 & 0.096 & 0.096 & 0.080 & 0.063 \end{array}$$

[illegible]

4. Compute the transition probability vector from the start state to all other possible states after 3 step.

$$a^{(3)} = a^{(2)}.P^1 = \begin{array}{c} \\ S \end{array} \begin{array}{ccccccccc} S & W & L & P4 & P5 & P6 & P8 & P9 & P10 \\ [0.00 & 0.299 & 0.222 & 0.063 & 0.080 & 0.096 & 0.096 & 0.080 & 0.063] \end{array}[P]$$

$$= \begin{array}{c} \\ S \end{array} \begin{array}{ccccccccc} S & W & L & P4 & P5 & P6 & P8 & P9 & P10 \\ [0.354 & 0.302 & 0.047 & 0.058 & 0.080 & 0.067 & 0.067 & 0.058 & 0.047] \end{array}$$

5. Compute the transition probability vector from the start state to all other possible states after 4 step.

$$a^{(4)} = a^{(3)}.P^1 = \begin{array}{c} \\ S \end{array} \begin{array}{ccccccccc} S & W & L & P4 & P5 & P6 & P8 & P9 & P10 \\ [0.354 & 0.302 & 0.047 & 0.058 & 0.080 & 0.067 & 0.067 & 0.058 & 0.047] \end{array}[P]$$

$$=$$

Markov Chains

Transition probability vector from the start state to all other possible states after infinite number of step (23 step).

Rolls	start	W	L	P4	P5	P6	P8	P9	P10	Sum
0	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
1	0.0000	0.2222	0.1111	0.0833	0.1111	0.1389	0.1389	0.1111	0.0833	0.9999
2	0.0000	0.2994	0.2222	0.0625	0.0802	0.0965	0.0965	0.0802	0.0625	0.9999
3	0.0000	0.3544	0.3020	0.0469	0.0579	0.0670	0.0670	0.0579	0.0469	0.9999
4	0.0000	0.3937	0.3592	0.0351	0.0418	0.0465	0.0465	0.0418	0.0351	0.9999
5	0.0000	0.4217	0.4004	0.0264	0.0302	0.0323	0.0323	0.0302	0.0264	0.9999
6	0.0000	0.4418	0.4300	0.0198	0.0218	0.0224	0.0224	0.0218	0.0198	0.9999
7	0.0000	0.4562	0.4514	0.0148	0.0158	0.0156	0.0156	0.0158	0.0148	0.9999
8	0.0000	0.4665	0.4668	0.0111	0.0114	0.0108	0.0108	0.0114	0.0111	0.9999
9	0.0000	0.4739	0.4779	0.0083	0.0082	0.0075	0.0075	0.0082	0.0083	0.9999
10	0.0000	0.4792	0.4859	0.0063	0.0059	0.0052	0.0052	0.0059	0.0063	0.9999
11	0.0000	0.4830	0.4917	0.0047	0.0043	0.0036	0.0036	0.0043	0.0047	0.9999
12	0.0000	0.4857	0.4959	0.0035	0.0031	0.0025	0.0025	0.0031	0.0035	0.9999
13	0.0000	0.4877	0.4990	0.0026	0.0022	0.0017	0.0017	0.0022	0.0026	0.9999
14	0.0000	0.4891	0.5012	0.0020	0.0016	0.0012	0.0012	0.0016	0.0020	0.9999
15	0.0000	0.4902	0.5028	0.0015	0.0012	0.0008	0.0008	0.0012	0.0015	0.9999
16	0.0000	0.4909	0.5039	0.0011	0.0008	0.0006	0.0006	0.0008	0.0011	0.9999
17	0.0000	0.4914	0.5048	0.0008	0.0006	0.0004	0.0004	0.0006	0.0008	0.9999
18	0.0000	0.4918	0.5054	0.0006	0.0004	0.0003	0.0003	0.0004	0.0006	0.9999
19	0.0000	0.4921	0.5058	0.0005	0.0003	0.0002	0.0002	0.0003	0.0005	0.9999
20	0.0000	0.4923	0.5062	0.0004	0.0002	0.0001	0.0001	0.0002	0.0004	0.9999
21	0.0000	0.4924	0.5064	0.0003	0.0002	0.0001	0.0001	0.0002	0.0003	0.9999
22	0.0000	0.4926	0.5066	0.0002	0.0001	0.0001	0.0001	0.0001	0.0002	0.9999
23	0.0000	0.4926	0.5067	0.0001	0.0001	0.0000	0.0000	0.0001	0.0001	0.9999
24	0.0000	0.4927	0.5068	0.0001	0.0001	0.0000	0.0000	0.0001	0.0001	0.9999
25	0.0000	0.4927	0.5069	0.0001	0.0000	0.0000	0.0000	0.0000	0.0001	0.9999
26	0.0000	0.4928	0.5069	0.0001	0.0000	0.0000	0.0000	0.0000	0.0001	0.9999
27	0.0000	0.4928	0.5070	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.9999
28	0.0000	0.4928	0.5070	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.9999
29	0.0000	0.4928	0.5070	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.9999
30	0.0000	0.4928	0.5070	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.9999
31	0.0000	0.4928	0.5070	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.9999
32	0.0000	0.4928	0.5070	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.9999
33	0.0000	0.4928	0.5070	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.9999
34	0.0000	0.4928	0.5070	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.9999
35	0.0000	0.4928	0.5071	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.9999
36	0.0000	0.4928	0.5071	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.9999

The probability that the system will be in the state p6 after 14 rolls given that the current time is zero and the current state is start is 0.0012

$$
P = \begin{array}{c} \\ S \\ W \\ L \\ P4 \\ P5 \\ P6 \\ P8 \\ P9 \\ P10 \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} S & W & L & P4 & P5 & P6 & P8 & P9 & P10 \end{array} \\
\begin{bmatrix}
0 & 0.2222 & 0.1111 & 0.0833 & 0.1111 & 0.1389 & 0.1389 & 0.1111 & 0.0833 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.0833 & 0.1667 & 0.75 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.1111 & 0.1667 & 0 & 0.7222 & 0 & 0 & 0 & 0 \\
0 & 0.1389 & 0.1667 & 0 & 0 & 0.694 & 0 & 0 & 0 \\
0 & 0.1389 & 0.1667 & 0 & 0 & 0 & 0.6944 & 0 & 0 \\
0 & 0.1111 & 0.1667 & 0 & 0 & 0 & 0 & 0.7222 & 0 \\
0 & 0.0833 & 0.1667 & 0 & 0 & 0 & 0 & 0 & 0.750
\end{bmatrix}
\end{array}
$$

Steps										
Rolls	**start**	**W**	**L**	**P4**	**P5**	**P6**	**P8**	**P9**	**P10**	**Sum**
0	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.[illegible]	1.0000
1	0.0000	0.2222	0.1111	0.0833	0.1111	0.1389	0.1389	0.1111	0.0833	0.9999
2	0.0000	0.2994	0.2222	0.0625	0.0802	0.0965	0.0965	0.0802	0.0625	0.9999
3	0.0000	0.3544	0.3020	0.0469	0.0579	0.0670	0.0670	0.0579	0.0469	0.9999
4	0.0000	0.3937	0.3592	0.0351	0.0418	0.0465	0.0465	0.0418	0.0351	0.9999
5	0.0000	0.4217	0.4004	0.0264	0.0302	0.0323	0.0323	0.0302	0.0264	0.9999
6	0.0000	0.4418	0.4300	0.0198	0.0218	0.0224	0.0224	0.0218	0.0198	0.9999
7	0.0000	0.4562	0.4514	0.0148	0.0158	0.0156	0.0156	0.0158	0.0148	0.9999
8	0.0000	0.4665	0.4668	0.0111	0.0114	0.0108	0.0108	0.0114	0.0111	0.9999
9	0.0000	0.4739	0.4779	0.0083	0.0082	0.0075	0.0075	0.0082	0.0083	0.9999
10	0.0000	0.4792	0.4859	0.0063	0.0059	0.0052	0.0052	0.0059	0.0063	0.9999
11	0.0000	0.4830	0.4917	0.0047	0.0043	0.0036	0.0036	0.0043	0.0047	0.9999
12	0.0000	0.4857	0.4959	0.0035	0.0031	0.0025	0.0025	0.0031	0.0035	0.9999
13	0.0000	0.4877	0.4990	0.0026	0.0022	0.0017	0.0017	0.0022	0.0026	0.9999
14	0.0000	0.4891	0.5012	0.0020	0.0016	0.0012	0.0012	0.0016	0.0020	0.9999
15	0.0000	0.4902	0.5028	0.0015	0.0012	0.0008	0.0008	0.0012	0.0015	0.9999
16	0.0000	0.4909	0.5039	0.0011	0.0008	0.0006	0.0006	0.0008	0.0011	0.9999
17	0.0000	0.4914	0.5048	0.0008	0.0006	0.0004	0.0004	0.0006	0.0008	0.9999

P4 start sums

Ready Accessibility: Investigate 100%

	F	G	H	I	J	K	L	M	N	O	P	Q
34		**31**	0.0000	0.4928	0.5070	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	**0.9999**
35		**32**	0.0000	0.4928	0.5070	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	**0.9999**
36		**33**	0.0000	0.4928	0.5070	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	**0.9999**
37		**34**	0.0000	0.4928	0.5070	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	**0.9999**
38		**35**	**0.0000**	**0.4928**	**0.5071**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.9999**
39		**36**	0.0000	0.4928	0.5071	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	**0.9999**
40		**37**	0.0000	0.4928	0.5071	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	**0.9999**
41		**38**	0.0000	0.4928	0.5071	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	**0.9999**
42		**39**	0.0000	0.4928	0.5071	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	**0.9999**
43		**40**	0.0000	0.4928	0.5071	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	**0.9999**
44		**41**	0.0000	0.4928	0.5071	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	**0.9999**
45		**42**	0.0000	0.4928	0.5071	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	**0.9999**
46												

P4 start sums

Ready Accessibility: Investigate 148%

6	P8	P9	P10
9	0.139	0.111	0.083
	0	0	0
	0	0	0
	0	0	0
	0	0	0
4	0	0	0
	0.694	0	0
	0	0.722	0
	0	0	0.750

Rolls	start	W	L	P4	P5	P6	P8
0	0.000	0.000	0.000	1.000	0.000	0.000	0.000
1	0.000	0.083	0.167	0.750	0.000	0.000	0.000
2	0.000	0.145	0.292	0.563	0.000	0.000	0.000
3	0.000	0.192	0.386	0.422	0.000	0.000	0.000
4	0.000	0.227	0.457	0.316	0.000	0.000	0.000
5	0.000	0.253	0.509	0.237	0.000	0.000	0.000
6	0.000	0.273	0.549	0.178	0.000	0.000	0.000
7	0.000	0.288	0.579	0.133	0.000	0.000	0.000

P4 start sums

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G23 | f_x | 20

	F	G	H	I	J	K	L	M	N	O	P	Q
19		**16**	0.000	0.329	0.661	0.010	0.000	0.000	0.000	0.000	0.000	**1.0**
20		**17**	0.000	0.330	0.663	0.008	0.000	0.000	0.000	0.000	0.000	**1.0**
21		**18**	0.000	0.330	0.664	0.006	0.000	0.000	0.000	0.000	0.000	**1.0**
22		**19**	0.000	0.331	0.665	0.004	0.000	0.000	0.000	0.000	0.000	**1.0**
23		**20**	0.000	0.331	0.666	0.003	0.000	0.000	0.000	0.000	0.000	**1.0**
24		**21**	0.000	0.331	0.666	0.002	0.000	0.000	0.000	0.000	0.000	**1.0**
25		**22**	0.000	0.331	0.667	0.002	0.000	0.000	0.000	0.000	0.000	**1.0**
26		**23**	0.000	0.332	0.667	0.001	0.000	0.000	0.000	0.000	0.000	**1.0**
27		**24**	0.000	0.332	0.667	0.001	0.000	0.000	0.000	0.000	0.000	**1.0**
28		**25**	0.000	0.332	0.667	0.001	0.000	0.000	0.000	0.000	0.000	**1.0**
29		**26**	**0.000**	**0.332**	**0.668**	**0.001**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**1.0**
30		**27**	0.000	0.332	0.668	0.000	0.000	0.000	0.000	0.000	0.000	**1.0**

P4 | start | sums

Ready | Accessibility: Investigate | Average: 2 | Count: 11 | Sum: 22 | 154%

After many steps, the system reaches the steady-state condition, where:

$$\pi_{Good} = \lim_{n\to\infty} a_{Good}^{(n)}$$

6. Compute the steady state transition probability vector from the start state to all other possible states

$$
S\begin{array}{c} \\ \left[\begin{array}{ccccccccc} \overset{S}{\pi_1} & \overset{W}{\pi_2} & \overset{L}{\pi_3} & \overset{P4}{\pi_4} & \overset{P5}{\pi_5} & \overset{P6}{\pi_6} & \overset{P8}{\pi_7} & \overset{P9}{\pi_8} & \overset{P10}{\pi_9} \end{array}\right] \end{array} P =
\begin{array}{c} \\ S \\ W \\ L \\ P4 \\ P5 \\ P6 \\ P8 \\ P9 \\ P10 \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} S & W & L & P4 & P5 & P6 & P8 & P9 & P10 \end{array} \\
\left[\begin{array}{ccccccccc}
0 & 0.2222 & 0.1111 & 0.0833 & 0.1111 & 0.1389 & 0.1389 & 0.1111 & 0.0833 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.0833 & 0.1667 & 0.75 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.1111 & 0.1667 & 0 & 0.7222 & 0 & 0 & 0 & 0 \\
0 & 0.1389 & 0.1667 & 0 & 0 & 0.694 & 0 & 0 & 0 \\
0 & 0.1389 & 0.1667 & 0 & 0 & 0 & 0.6944 & 0 & 0 \\
0 & 0.1111 & 0.1667 & 0 & 0 & 0 & 0 & 0.7222 & 0 \\
0 & 0.0833 & 0.1667 & 0 & 0 & 0 & 0 & 0 & 0.750
\end{array}\right]
\end{array}
$$

$$
= S\left[\begin{array}{ccccccccc} \overset{S}{\pi_1} & \overset{W}{\pi_2} & \overset{L}{\pi_3} & \overset{P4}{\pi_4} & \overset{P5}{\pi_5} & \overset{P6}{\pi_6} & \overset{P8}{\pi_7} & \overset{P9}{\pi_8} & \overset{P10}{\pi_9} \end{array}\right]
$$

$$
= \left[\begin{array}{ccccccccc} \overset{S}{0.000} & \overset{W}{0.4928} & \overset{L}{0.5071} & \overset{P4}{0.000} & \overset{P5}{0.000} & \overset{P6}{0.000} & \overset{P8}{0.000} & \overset{P9}{0.000} & \overset{P10}{0.000} \end{array}\right]
$$

The *steady state probabilities* are defined as:

$$\pi_j = \lim_{n\to\infty} a_j^{(n)},\ j = 0, 1, 2 \dots.$$

These probabilities are independent of $[a_j^{(0)}]$, can be determined from the equation:

$$\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$$

$$\sum_j \boldsymbol{\pi}_j = 1$$

The expected number of transitions before the system returns to a state j for the first time is known as *the mean first return time or the mean recurrence time*, computed from the equation:

$$\mu_{jj} = \frac{1}{\pi_j},\ j = 0, 1, 2 \dots.$$

For the Game of craps, The mean first return times (rolls) are computed as

$$\begin{array}{c|ccccccccc} & S & W & L & P4 & P5 & P6 & P8 & P9 & P10 \\ a^{s.s.}_{start\to j} = & 0.000 & 0.4928 & 0.5071 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \end{array}$$

$$\mu_{start\to start} = \frac{1}{0.0} = \infty = \mu_{start\to P4} = \mu_{start\to P5} = \mu_{start\to P6} = \mu_{start\to P8} = \mu_{start\to P9} = \mu_{start\to P10}$$

$$\mu_{start\to Wt} = \frac{1}{0.4928} = 2.03 \text{ Rolls On Average}$$

win

$$\mu_{start\to L} = \frac{1}{0.5071} = 1.97 \text{ Rolls On Average}$$

Lose

If the player receive $50 if he wins and pay 45 for loose. Estimate the expected payoff

$$= \$50 \times \pi_{start\to W} - \$45 \times \pi_{start\to L}$$

$$= \$50 \times 0.4928 - \$45 \times 0.5071$$

$$= \$1.82$$

A simpler way to determine the mean first return time for all the states in an m-transition matrix, P, is to use the following

$$\|\mu_{ij}\| = (\mathbf{I} - \mathbf{N}_j)^{-1}\,\mathbf{1},\ j \neq i$$

- **I**: *(m-1) identity matrix*
- $\mathbf{N}_j$: *transition matrix P less its j^{th} row and j^{th} column of target state j*
- **1** *(m-1) column vector with all elements equal to 1*

Consider the passage from states S, L, P4, P5, P6, P8, P9, and P10, to W.

thus j = W, i = S, L, P4, P5, P6, P8, P9, and P10

$$
\begin{bmatrix} \mu_{S\to\pi} \\ \mu_{L\to\pi} \\ \mu_{P4\to\pi} \\ \mu_{P5\to\pi} \\ \mu_{P6\to\pi} \\ \mu_{P8\to\pi} \\ \mu_{9\to\pi} \\ \mu_{P10\to\pi} \end{bmatrix}
=
\left[
\begin{bmatrix}
1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1
\end{bmatrix}
-
\begin{array}{c}
\\ S \\ L \\ P4 \\ P5 \\ P6 \\ P8 \\ P9 \\ P10
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} S & L & P4 & P5 & P6 & P8 & P9 & P10 \end{array} \\
\begin{bmatrix}
0 & 0.1111 & 0.0833 & 0.1111 & 0.1389 & 0.1389 & 0.1111 & 0.0833\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0.1667 & 0.75 & 0 & 0 & 0 & 0 & 0\\
0 & 0.1667 & 0 & 0.7222 & 0 & 0 & 0 & 0\\
0 & 0.1667 & 0 & 0 & 0.694 & 0 & 0 & 0\\
0 & 0.1667 & 0 & 0 & 0 & 0.6944 & 0 & 0\\
0 & 0.1667 & 0 & 0 & 0 & 0 & 0.7222 & 0\\
0 & 0.1667 & 0 & 0 & 0 & 0 & 0 & 0.750
\end{bmatrix}
\end{array}
\right]^{-1}
\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix}
$$

$$
\begin{bmatrix} \mu_{S\to\pi} \\ \mu_{L\to\pi} \\ \mu_{P4\to\pi} \\ \mu_{P5\to\pi} \\ \mu_{P6\to\pi} \\ \mu_{P8\to\pi} \\ \mu_{9\to\pi} \\ \mu_{P10\to\pi} \end{bmatrix}
=
\left[
\begin{array}{c}
\\ S \\ L \\ P4 \\ P5 \\ P6 \\ P8 \\ P9 \\ P10
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} S & L & P4 & P5 & P6 & P8 & P9 & P10 \end{array} \\
\begin{bmatrix}
1 & -0.1111 & -0.0833 & -0.1111 & -0.1389 & -0.1389 & -0.1111 & -0.0833\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -0.1667 & 0.25 & 0 & 0 & 0 & 0 & 0\\
0 & -0.1667 & 0 & 0.2778 & 0 & 0 & 0 & 0\\
0 & -0.1667 & 0 & 0 & 0.306 & 0 & 0 & 0\\
0 & -0.1667 & 0 & 0 & 0 & 0.3056 & 0 & 0\\
0 & -0.1667 & 0 & 0 & 0 & 0 & 0.2778 & 0\\
0 & -0.1667 & 0 & 0 & 0 & 0 & 0 & 0.25
\end{bmatrix}
\end{array}
\right]^{-1}
\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix}
$$

Operations Research - II

A simpler way to determine the mean first return time for all the states in an m-transition matrix, P, is to use the following

$$\left\|\mu_{ij}\right\| = \left(\mathbf{I} - \mathbf{N}_j\right)^{-1}\mathbf{1},\ j \neq i$$

- **I**: *(m-1) identity matrix*
- $\mathbf{N}_j$: *transition matrix P less its j^{th} row and j^{th} column of target state j*
- **1** *(m-1) column vector with all elements equal to 1*

Consider the passage from states S, P4, P5, P6, P8, P9, and P10, to W and L.

thus j = W, i = S, L, P4, P5, P6, P8, P9, and P10

$$
\begin{bmatrix} \mu_{S\to W} & \mu_{S\to L} \\ \mu_{P4\to W} & \mu_{P4\to L} \\ \mu_{P5\to W} & \mu_{P5\to L} \\ \mu_{P6\to W} & \mu_{P6\to L} \\ \mu_{P8\to W} & \mu_{P8\to L} \\ \mu_{9\to W} & \mu_{9\to L} \\ \mu_{P10\to W} & \mu_{P10\to L} \end{bmatrix}
=
\left[
\begin{bmatrix} 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{bmatrix}
-
\begin{array}{c} \\ S \\ P4 \\ P5 \\ P6 \\ P8 \\ P9 \\ P10 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} S & P4 & P5 & P6 & P8 & P9 & P10 \end{array} \\
\begin{bmatrix} 0 & 0.0833 & 0.1111 & 0.1389 & 0.1389 & 0.1111 & 0.0833 \\ 0 & 0.75 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7222 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6944 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6944 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.7222 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.750 \end{bmatrix}
\end{array}
\right]^{-1}
\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
$$

$$
\begin{bmatrix} \mu_{S\to W} & \mu_{S\to L} \\ \mu_{P4\to W} & \mu_{P4\to L} \\ \mu_{P5\to W} & \mu_{P5\to L} \\ \mu_{P6\to W} & \mu_{P6\to L} \\ \mu_{P8\to W} & \mu_{P8\to L} \\ \mu_{9\to W} & \mu_{9\to L} \\ \mu_{P10\to W} & \mu_{P10\to L} \end{bmatrix}
=
\begin{array}{c} \\ S \\ P4 \\ P5 \\ P6 \\ P8 \\ P9 \\ P10 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} S & P4 & P5 & P6 & P8 & P9 & P10 \end{array} \\
\begin{bmatrix} 1 & 0.3332 & 0.3999 & 0.4545 & 0.4545 & 0.3999 & 0.3332 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3.5997 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.2723 & 0 & & 0 \\ 0 & 0 & 0 & 0 & 3.2723 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.5997 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}
\end{array}
\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}
=
\begin{bmatrix} 3.3752 & 3.3752 \\ 4 & 4 \\ 3.5997 & 3.5997 \\ 3.2723 & 3.2723 \\ 3.2723 & 3.2723 \\ 3.5997 & 3.5997 \\ 4 & 4 \end{bmatrix}
$$

It takes 3.38 rolls on average to pass from S to W, and 3.38 rolls to pass from S to L
It takes 4 rolls on average to pass from P4 to W, and 4 rolls to pass from P4 to L
And so on

A simpler way to determine the mean first return time for all the states in an m-transition matrix, P, is to use the following

$$\left\| \mu_{ij} \right\| = \left(I - N_j \right)^{-1} 1, \; j \neq i$$

- **I**: *(m-1) identity matrix*
- N_j: *transition matrix P less its j^{th} row and j^{th} column of target state j*
- **1** *(m-1) column vector with all elements equal to 1*

Consider the passage from states P4, P5, P6, P8, P9, and P10, to W and L.

nos

thus j = W, L, i = P4, P5, P6, P8, P9, and P10

Markov Chains

Chapter 17: First Passage Time

$$
\begin{bmatrix}
\mu_{P4\to W} & \mu_{P4\to L}\\
\mu_{P5\to W} & \mu_{P5\to L}\\
\mu_{P6\to W} & \mu_{P6\to L}\\
\mu_{P8\to W} & \mu_{P8\to L}\\
\mu_{P9\to W} & \mu_{P9\to L}\\
\mu_{P10\to W} & \mu_{P10\to L}
\end{bmatrix}
=
\left[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
-
\begin{array}{c}
\\ P4 \\ P5 \\ P6 \\ P8 \\ P9 \\ P10
\end{array}
\begin{array}{c}
\begin{array}{cccccc} P4 & P5 & P6 & P8 & P9 & P10 \end{array}\\
\begin{bmatrix}
0.75 & 0 & 0 & 0 & 0 & 0\\
0 & 0.7222 & 0 & 0 & 0 & 0\\
0 & 0 & 0.694 & 0 & 0 & 0\\
0 & 0 & 0 & 0.6944 & 0 & 0\\
0 & 0 & 0 & 0 & 0.7222 & 0\\
0 & 0 & 0 & 0 & 0 & 0.750
\end{bmatrix}
\end{array}
\right]^{-1}
\begin{bmatrix}
1\\ 1\\ 1\\ 1\\ 1\\ 1
\end{bmatrix}
$$

$$
\begin{bmatrix}
\mu_{P4\to W} & \mu_{P4\to L}\\
\mu_{P5\to W} & \mu_{P5\to L}\\
\mu_{P6\to W} & \mu_{P6\to L}\\
\mu_{P8\to W} & \mu_{P8\to L}\\
\mu_{P9\to W} & \mu_{P9\to L}\\
\mu_{P10\to W} & \mu_{P10\to L}
\end{bmatrix}
=
\begin{bmatrix}
0.25 & 0 & 0 & 0 & 0 & 0\\
0 & 0.2778 & 0 & 0 & 0 & 0\\
0 & 0 & 0.3056 & 0 & 0 & 0\\
0 & 0 & 0 & 0.3056 & 0 & 0\\
0 & 0 & 0 & 0 & 0.2778 & 0\\
0 & 0 & 0 & 0 & 0 & 0.25
\end{bmatrix}^{-1}
\begin{bmatrix}
1 & 1\\
1 & 1\\
1 & 1\\
1 & 1\\
1 & 1\\
1 & 1
\end{bmatrix}
$$

Operations Research - II

Markov Chains
Chapter 17: First Passage Time

$$\begin{bmatrix} \mu_{P4\to W} & \mu_{P4\to L} \\ \mu_{P5\to W} & \mu_{P5\to L} \\ \mu_{P6\to W} & \mu_{P6\to L} \\ \mu_{P8\to W} & \mu_{P8\to L} \\ \mu_{P9\to W} & \mu_{P9\to L} \\ \mu_{P10\to W} & \mu_{P10\to L} \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3.5997 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3.2723 & 0 & 0 & 0\ 4 \\ 0 & 0 & 0 & 3.2723 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3.5997 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \mu_{P4\to W} & \mu_{P4\to L} \\ \mu_{P5\to W} & \mu_{P5\to L} \\ \mu_{P6\to W} & \mu_{P6\to L} \\ \mu_{P8\to W} & \mu_{P8\to L} \\ \mu_{P9\to W} & \mu_{P9\to L} \\ \mu_{P10\to W} & \mu_{P10\to L} \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 3.5997 & 3.5997 \\ 3.2723 & 3.2723 \\ 3.2723 & 3.2723 \\ 3.5997 & 3.5997 \\ 4 & 4 \end{bmatrix}$$

Markov Chains

Chapter 17: First Passage Probability

To determine the probability that the system will pass to absorbing state j if it begins in transient state i (First passage probability, (fp_{ij}). The following equation is used

$$fp_{ij} = (I - N_j)^{-1}\, T,$$

- **I**: *(m-1) identity matrix*
- **N_j**: *transition matrix P less its j^{th} row and j^{th} column of targeted absorbing state j*
- **T** *The transition probabilities matrix from the transient state i to the desired absorbing state j*

الجامعة
الاردنية

The Game of Craps
Find the Absorbing State Probabilities ?

$$P = \begin{array}{c} \\ S \\ W \\ L \\ P4 \\ P5 \\ P6 \\ P8 \\ P9 \\ P10 \end{array}\begin{array}{c} \begin{array}{ccccccccc} S & P4 & P5 & P6 & P8 & P9 & P10 & W & L \end{array} \\ \begin{bmatrix} 0 & 0.083 & 0.111 & 0.139 & 0.139 & 0.111 & 0.083 & 0.222 & 0.111 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0.75 & 0 & 0 & 0 & 0 & 0 & 0.083 & 0.167 \\ 0 & 0 & 0.722 & 0 & 0 & 0 & 0 & 0.111 & 0.167 \\ 0 & 0 & 0 & 0.694 & 0 & 0 & 0 & 0.139 & 0.167 \\ 0 & 0 & 0 & 0 & 0.694 & 0 & 0 & 0.139 & 0.167 \\ 0 & 0 & 0 & 0 & 0 & 0.722 & 0 & 0.111 & 0.167 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.750 & 0.083 & 0.167 \end{bmatrix} \end{array}$$

$$fp = [I - N]^{-1} T$$

$$fd = \begin{array}{c} \begin{array}{cc} \text{Win} & \text{Lose} \end{array} \\ \begin{bmatrix} 0.493 & 0.507 \\ 0.333 & 0.667 \\ 0.400 & 0.600 \\ 0.455 & 0.545 \\ 0.455 & 0.545 \\ 0.400 & 0.600 \\ 0.333 & 0.667 \end{bmatrix} \end{array} \begin{array}{c} \\ \text{Start} \\ P4 \\ P5 \\ P6 \\ P8 \\ P9 \\ P10 \end{array}$$

- the probability that The system will pass to absorbing state j if it begins in transient state i.

The Game of Craps
Find the Absorbing State Probabilities ?

P =

	S	P4	P5	P6	P8	P9	P10	W	L
S	0	0.083	0.111	0.139	0.139	0.111	0.083	0.222	0.111
W	0	0	0	0	0	0	0	1	0
L	0	0	0	0	0	0	0	0	1
P4	0	0.75	0	0	0	0	0	0.083	0.167
P5	0	0	0.722	0	0	0	0	0.111	0.167
P6	0	0	0	0.694	0	0	0	0.139	0.167
P8	0	0	0	0	0.694	0	0	0.139	0.167
P9	0	0	0	0	0	0.722	0	0.111	0.167
P10	0	0	0	0	0	0	0.750	0.083	0.167

P =

	S	P4	P5	P6	P8	P9	P10	W	L
S	0	0.083	0.111	0.139	0.139	0.111	0.083	0.222	0.111
P4	0	0.75	0	0	0	0	0	0.083	0.167
P5	0	0	0.722	0	0	0	0	0.111	0.167
P6	0	0	0	0.694	0	0	0	0.139	0.167
P8	0	0	0	0	0.694	0	0	0.139	0.167
P9	0	0	0	0	0	0.722	0	0.111	0.167
P10	0	0	0	0	0	0	0.750	0.083	0.167
W	0	0	0	0	0	0	0	1	0
L	0	0	0	0	0	0	0	0	1

$$P = \begin{bmatrix} N & T \\ 0 & I \end{bmatrix}$$

$$P = \begin{bmatrix} \text{Transient} & \text{Absorbing} \\ \text{Zeros} & \text{Identity} \end{bmatrix}$$

$$fp = [I - N]^{-1} T$$

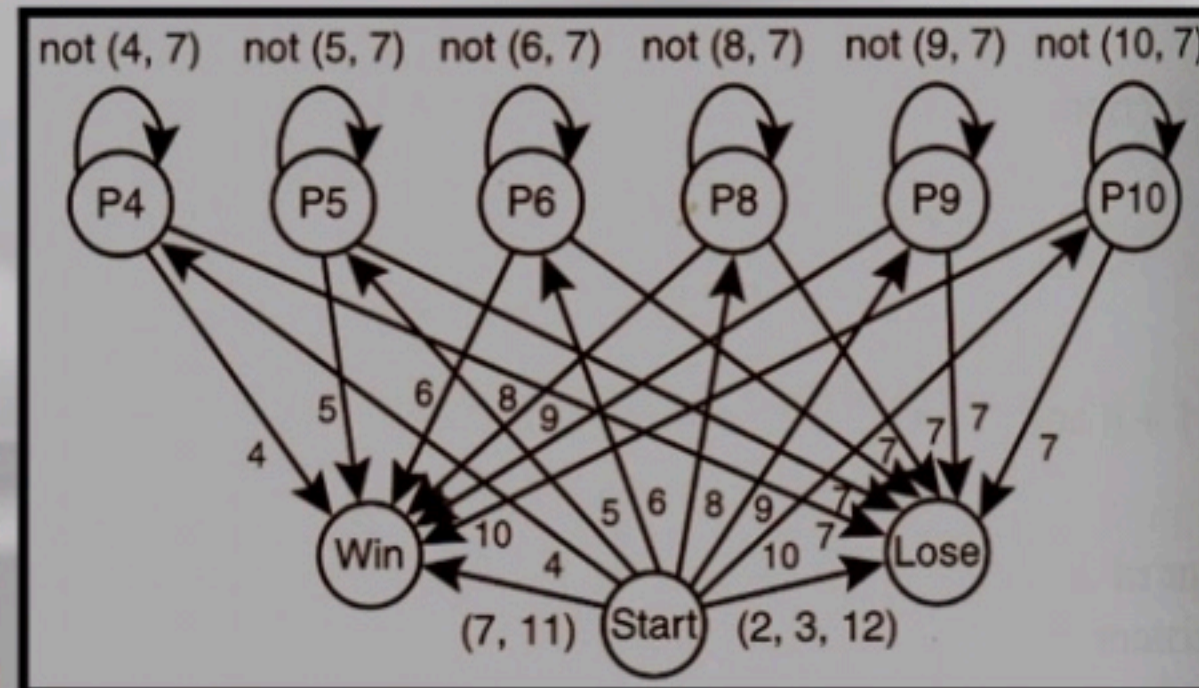
$$fd = \begin{array}{c} \begin{array}{cc} \text{Win} & \text{Lose} \end{array} \\ \begin{bmatrix} 0.493 & 0.507 \\ 0.333 & 0.667 \\ 0.400 & 0.600 \\ 0.455 & 0.545 \\ 0.455 & 0.545 \\ 0.400 & 0.600 \\ 0.333 & 0.667 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Start} \\ \text{P4} \\ \text{P5} \\ \text{P6} \\ \text{P8} \\ \text{P9} \\ \text{P10} \end{array}$$

- the probability that The system will pass to absorbing state j if it begins in transient state i.

The Game of Craps
First passage Probabilities

- First passage distribution for 4 rolls of the dice, one from the start to the Win state, and one from the Start to the Lose state.

Pr{4|R1} * Pr{7|R2} + Pr{5|R1} * Pr{7|R2} +
Pr{6|R1} * Pr{7|R2} + Pr{8|R1} * Pr{7|R2} +
Pr{9|R1} * Pr{7|R2} + Pr{10|R1} * Pr{ 7|R2}
0.083 * 0.167 + 0.111 * 0.167 +
0.139 * 0.167 + 0.139 * 0.167 +
0.111 * 0.167 + 0.083 * 0.167 = 0.111

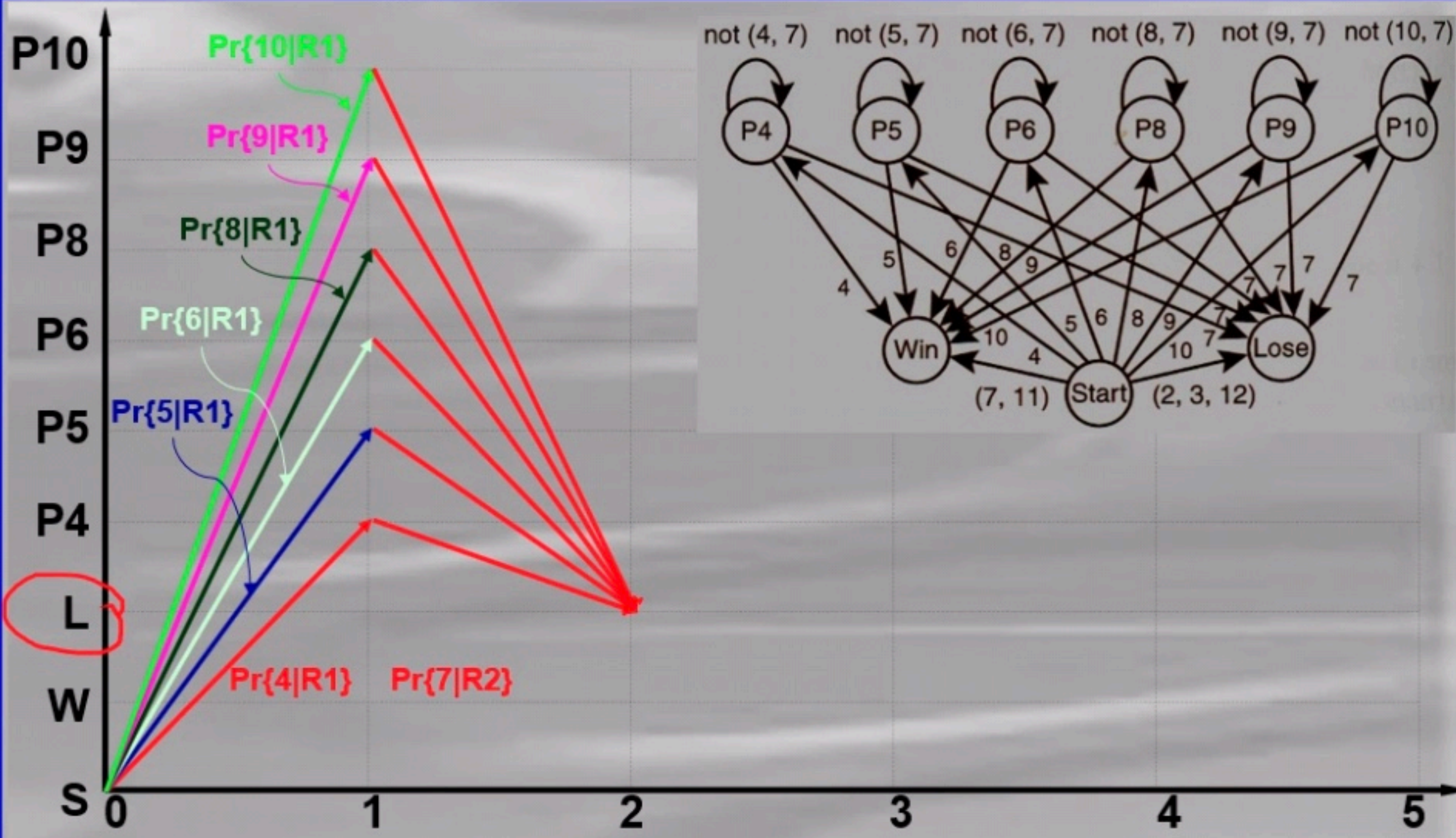


Roll (R)	Start - Win	Start - Lose	Sum	Cumulative	Remarks
1	Pr{7,11} 0.222	Pr {2,3,11} 0.111	0.333	0.333	
2	0.077	0.111	0.188	0.522	
3	0.055	0.080	0.135	0.656	
4	0.039	0.057	0.097	0.753	
5	0.028	0.041	0.069	0.822	

The probability that the first passage from start to Lose will take 2 roll is 0.111

Operations Research - II

The Game of Craps
First passage Probabilities

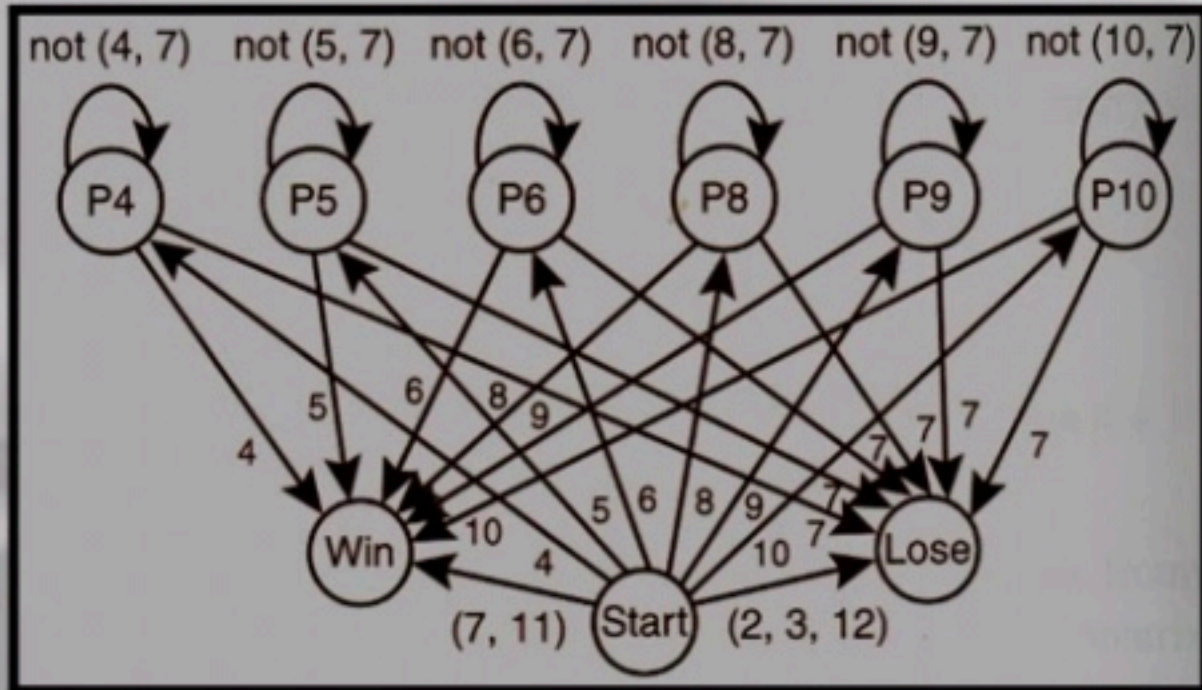


Possible Paths of a 2 step transition for the game of craps Roll (n)

The Game of Craps
First passage Probabilities

- First passage distribution for 5 rolls of the dice, one from the start to the Win state, and one from the Start to the Lose state.

Pr{4|R1} * Pr{4|R2} + Pr{5|R1} * Pr{5|R2} +
Pr{6|R1} * Pr{6|R2} + Pr{8|R1} *Pr{8|R2} +
Pr{9|R1} * Pr{9|R2} + Pr{10|R1} * Pr{10|R2}
0.083 * 0.083 + 0.111 * 0.111 +
0.139 * 0.139 + 0.139 * 0.139 +
0.111 * 0.111 + 0.083 * 0.083 = 0.077



Roll (R)	Start - Win	Start - Lose	Sum	Cumulative	Remarks
1	Pr{7,11} 0.222	Pr {2,3,11} 0.111	0.333	0.333	
2	0.077	0.111	0.188	0.522	
3	0.055	0.080	0.135	0.656	
4	0.039	0.057	0.097	0.753	
5	0.028	0.041	0.069	0.822	

The probability that the first passage from start to Win will take 2 roll is 0.077

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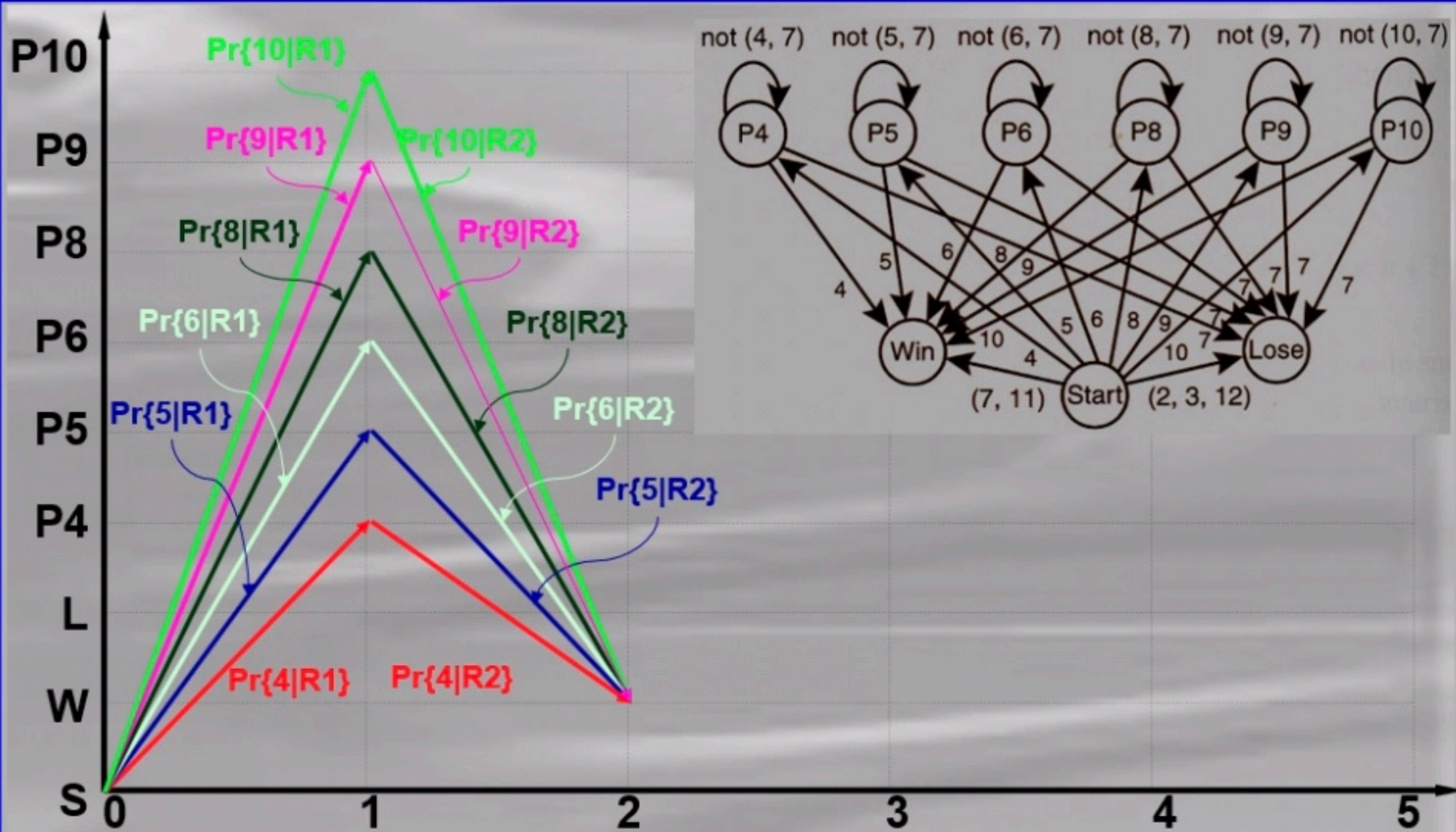
The Game of Craps
First passage Probabilities



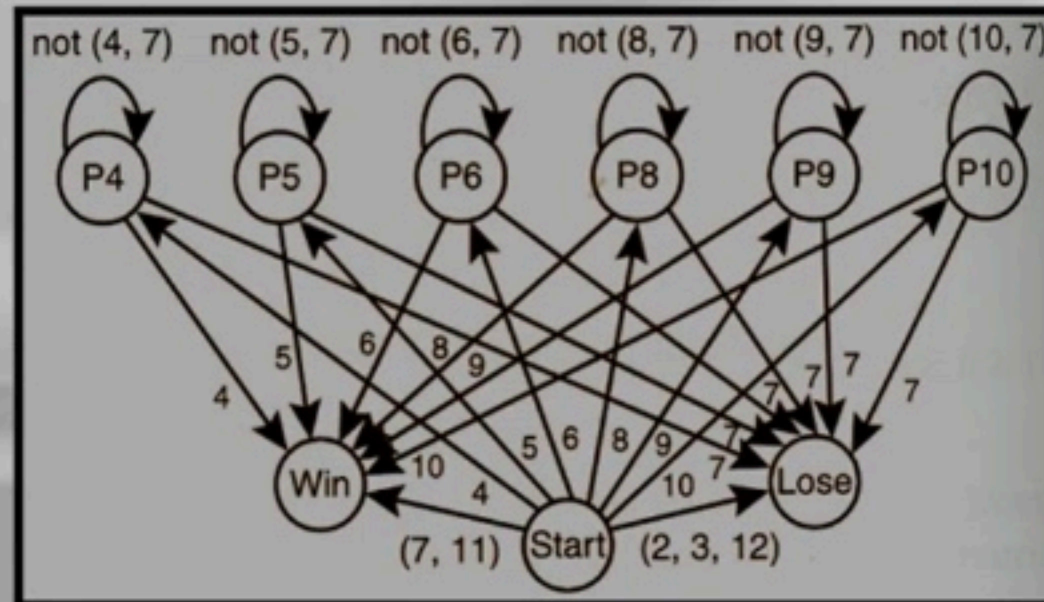
Possible Paths of a 2 step transition for the game of craps Roll (n)

الجامعة الأردنية

The Game of Craps
First passage Probabilities

كل اشي عدا 4,7 probability

Pr{4|R1} * (1-Pr{4}- Pr{7})^3*Pr{4|R5} +
Pr{5|R1} * (1-Pr{5}- Pr{7})^3*Pr{5|R5} +
Pr{6|R1} * (1-Pr{6}- Pr{7})^3*Pr{6|R5} +
Pr{8|R1} * (1-Pr{8}- Pr{7})^3*Pr{8|R5} +
Pr{9|R1} * (1-Pr{9}- Pr{7})^3*Pr{9|R5} +
Pr{10|R1} * (1-Pr{10}- Pr{7})^3*Pr{10|R5}
0.083 * (1- 0.083 – 0.167) ^3* 0.083 +
0.111 * (1- 0.111 – 0.167) ^3* 0.111 +
0.139 * (1- 0.139 – 0.167) ^3* 0.139 +
0.139 * (1- 0.139 – 0.167) ^3* 0.139 +
0.111 * (1- 0.111 – 0.167) ^3* 0.111 +
0.083 * (1- 0.083 – 0.167) ^3* 0.083 = 0.028

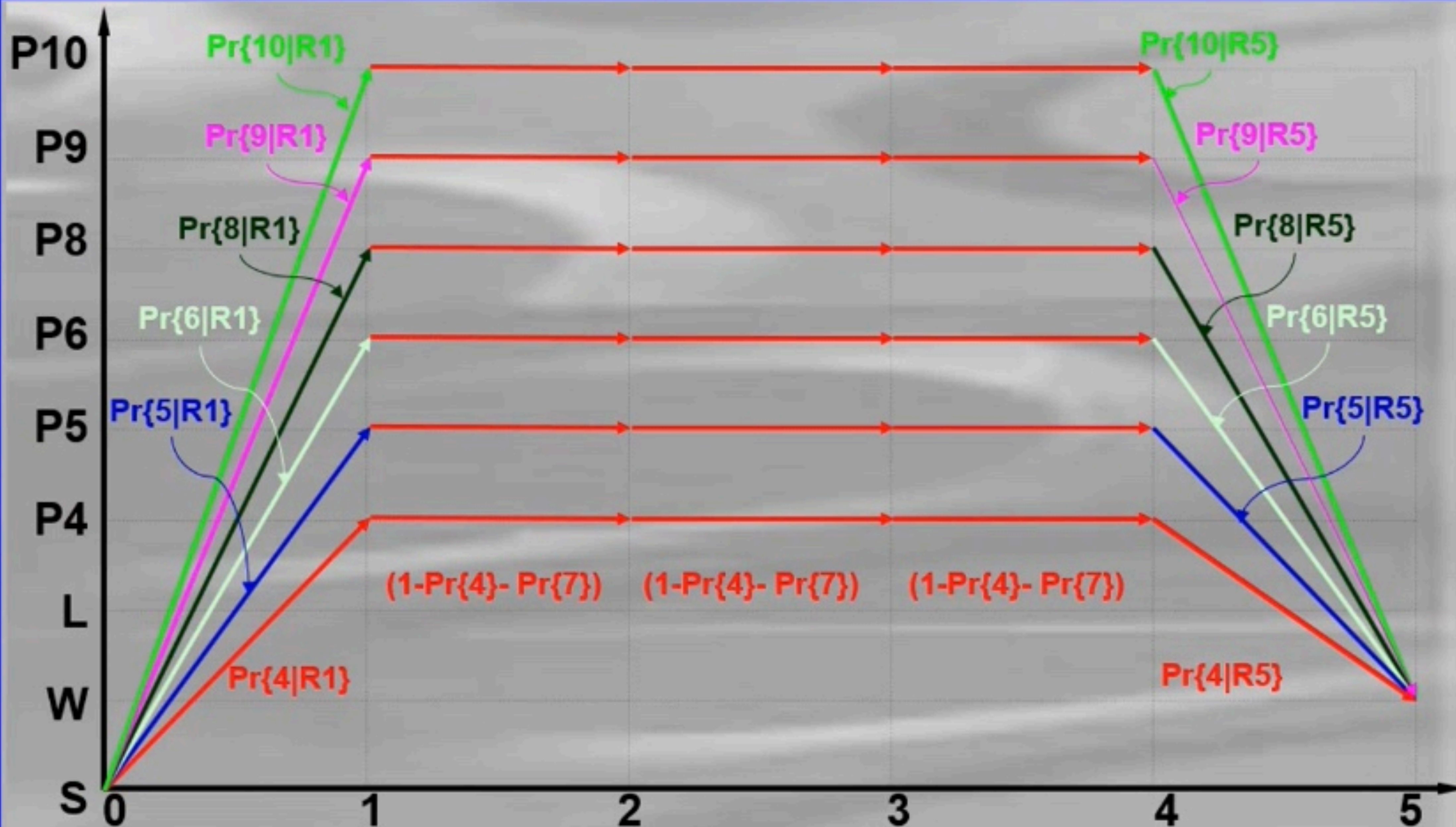


Roll (R)	Start - Win	Start - Lose	Sum	Cumulative	Remarks
1	Pr{7,11} 0.222	Pr {2,3,11} 0.111	0.333	0.333	
2	0.077	0.111	0.188	0.522	
3	0.055	0.080	0.135	0.656	
4	0.039	0.057	0.097	0.753	
5	0.028	0.041	0.069	0.822	

The probability that the first passage from start to Win will take 5 roll is 0.028

Operations Research - II

The Game of Craps
First passage Probabilities



Possible Paths of a 5 step transition for the game of craps Roll (n)

Operations Research - II

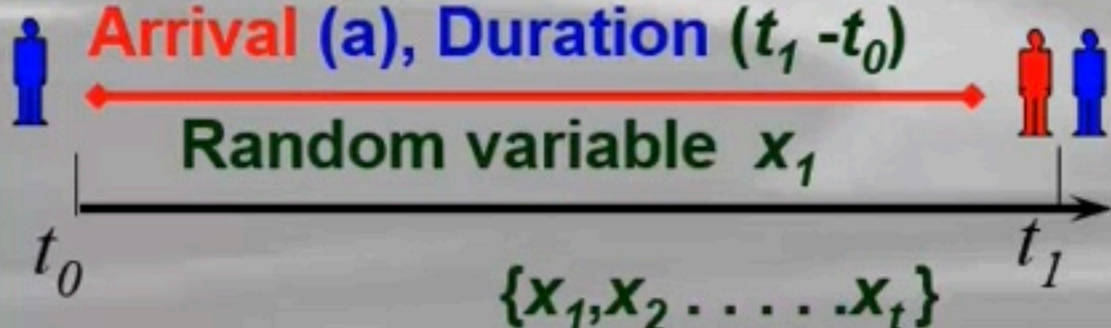
Continuous Time Markov Chains *CTMCs*
ATM Machine

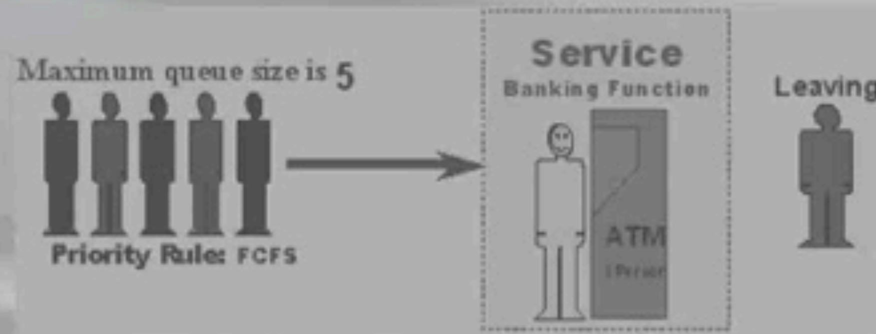
To illustrate the elements of the stochastic process model, we use the example of a single Automated Teller Machine (ATM) located in foyer of a bank. The ATM performs banking operations for people arriving for service. The machine is used by only one person at a time, and that person said to be *in service*. Others arriving when the machine is busy must wait in a single queue, and these people are said to be *in the queue*. Following the rule of *first-come-first-served*, a person in the queue will eventually enter service and will ultimately leave the system. The number in the *system* is the total of the number in service plus the number in the queue. The foyer is limited in size so that it can hold only five people. Since the weather is generally bad in this part of the country, when the foyer is full, arriving people do not enter. We have gathered statistics on ATM usage that show the time between arrivals averages 30 seconds (or 0.5 minutes). The time for service averages 24 seconds (or 0.4 minutes). Although the ATM has sufficient capacity to meet all demand, we frequently observe queues at the machine and occasionally customers are lost.

We want to perform an analysis to determine statistical measures that describe the number of people in the system, the waiting time for customers, the efficiency of the ATM machine, and the number of customers not served because there is no room in the foyer. We intend to use these statistics to guide managers in design questions such as whether another ATM should be installed, or whether the size of the foyer should be expanded.

Continuous Time Markov Chains *CTMCs*
The Stochastic Process

- **Definition: a stochastic process**
  is a collection of random variables $\{x_t\}$, where t is a time index that takes values from a given set T





- **Definition: Markovian Property**
  Given that the current state is known, the conditional probability of the next state depends only on the current state and not on the past .

- **Definition: A discrete-time Markov chain (Markov chain)**
  is a stochastic process with the following characteristics
  1. A discrete state space S = { 0,1,2,3,4, ...}
  2. Markovian property
  3. The one-step transition probabilities, p_{ij} , from time n to time n+1 remain constant overtime (termed stationary transition probabilities)

Operations Research - II

- **Definition: A Continuous-Time Markov Chain (CTMC)**

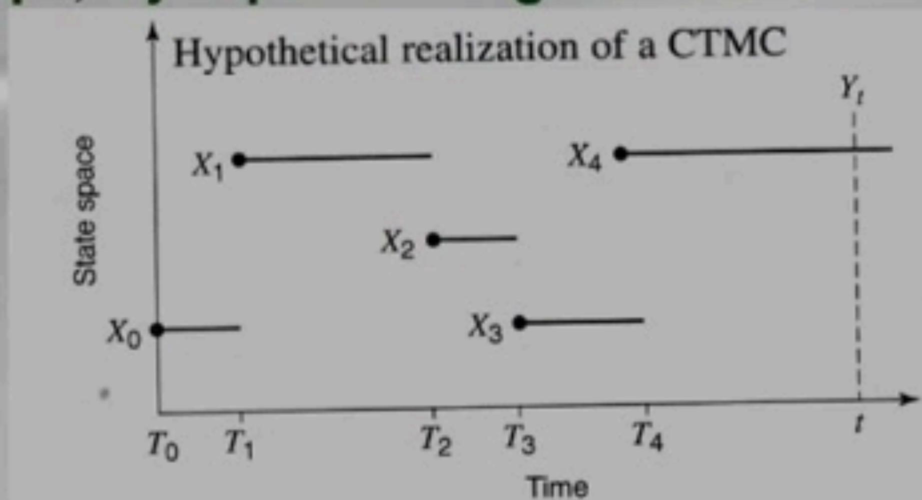
The process $Y = \{Y_t : t \geq o\}$ With state Space S is a CTMC if the following condition holds for all $j \in S, and\ t, s \geq o$

$$Pr\{Y_{t+s} = j \mid Y_u,\ 0 \leq u \leq s\} = Pr\{Y_{t+s} = j \mid Y_s\}$$

In addition the chain is said to have a stationary transitions if

$$Pr\{Y_{t+s} = j \mid Y_s = i\} = Pr\{Y_t = j \mid Y_0 = i\}$$

2. In CTMC, Markovian property must hold for all future times instead of just for one step
3. The process remains in each state for an exponential distributed length of time and then, when jumps, it jumps as though it were DTMC.



4. When a CTMC is generalized by allowing the state to be continuous, we have what is referred to as a Markov Process

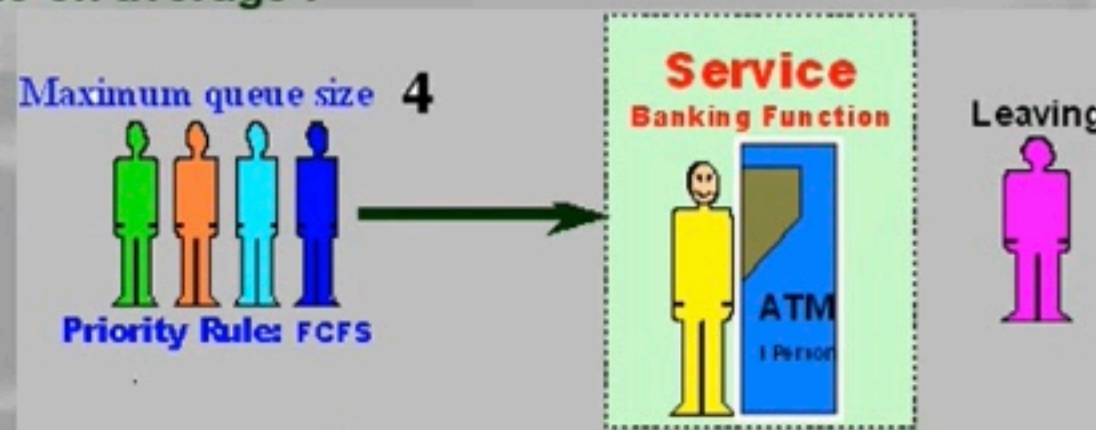
Continuous Time Markov Chains CTMCs
ATM Example

↳ always available, always changing

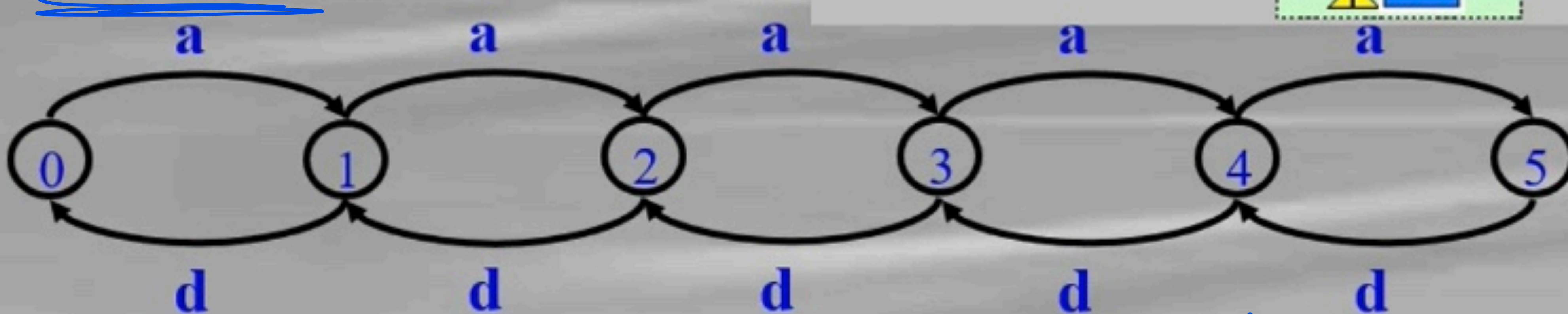
- A single Automated Teller Machine (ATM) located in the foyer of a bank. Only one person can use the machine at a time, so a queue forms when two or more customers are present. Following the rule of first come first served (FCFS), a customer in the queue will receive service and leave. The foyer is limited in size and can hold only four people arriving customers balk when the foyer is full. Statistics on the usage of this ATM indicate that the average time between arrivals is 30 second (0.5 Minute) whereas the time for service average 24 second (0.4 Minute).

W e want to perform an analysis to determine on average :

the number of people in the system
The waiting time for customers
The efficiency of the ATM
Number of customers not served.
not another ATM should be installed
the size of the foyer should be expanded.



- State transition network



- State is the number of customer in system S = {0, 1, 2, 3, 4, 5} → state-space
- State increases and decreases by an arrival (a) or a departure (d) → operations which change
- When system is full the states reaches 5 and no other arrivals occur until there is a departure

الجامعة الأردنية

Continuous Time Markov Chains CTMCs
Rate Network: ATM Example

Arrival activity (a)

Duration

Customer arrives in queue

Arrival of next Customer

Mean time between arrivals $= \frac{1}{\lambda} = \frac{1}{2}$

Arrival rate $= \lambda = 2$

Event: arrival (a)

- **Service activity**

Service activity (d)

Duration

Customer appears in front of ATM

Customer Departs

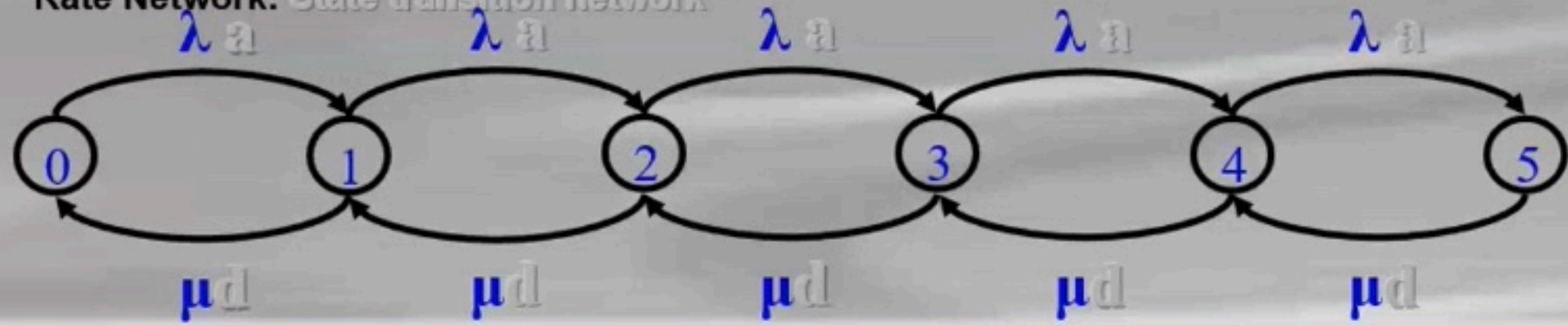
Event: service completion

Departure (d)

Average Time of service
($1/\mu = t_d = 0.4$ Minute)
Random variable, Poisson process

Mean time for service activity $= \frac{1}{\mu} = 0.4$, Service rate $= \mu = 2.5$

- **Rate Network:** State transition network



Operations Research - II

Continuous Time Markov Chains CTMCs

Rate Matrix R Transition P: ATM Example

- Rate matrix contains the transition rate from state i to state j

General Rate Matrix

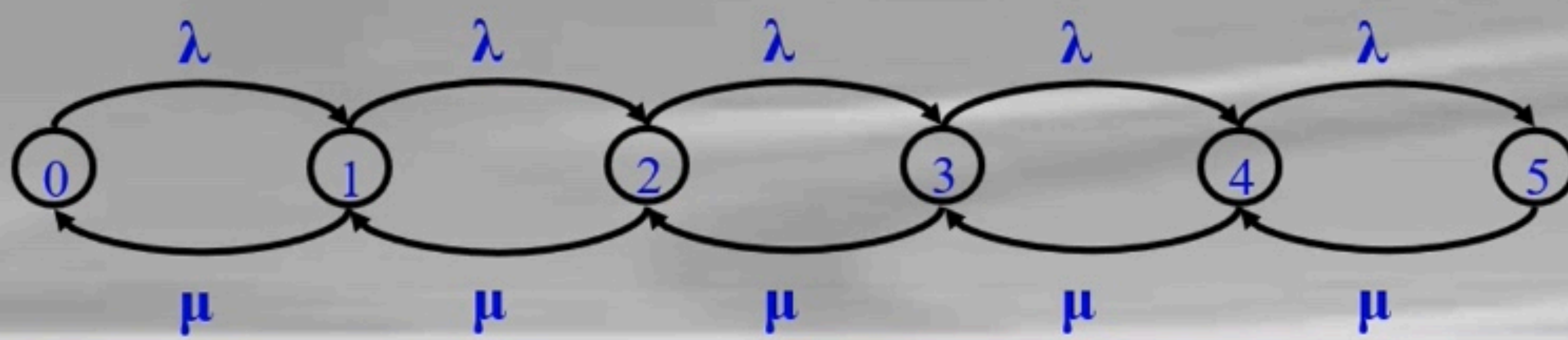
System State index

$$
R = \begin{array}{c} \\ 0 \\ 1 \\ \\ m-1 \end{array}
\begin{array}{c}
\begin{array}{ccccc} 0 & 1 & 2 & \dots & m-1 \end{array} \\
\begin{bmatrix}
0 & r_{01} & r_{02} & \dots & r_{0,m-1} \\
r_{10} & r_{11} & r_{12} & \dots & r_{1,m-1} \\
. & . & . & \dots & . \\
r_{m-1,0} & r_{m-1,1} & r_{m-1,2} & \dots & 0
\end{bmatrix}
\end{array}
$$

Rate matrix for the ATM example

$$
R = \begin{array}{c|cccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
0 & 0 & \lambda & 0 & 0 & 0 & 0 \\
1 & \mu & 0 & \lambda & 0 & 0 & 0 \\
2 & 0 & \mu & 0 & \lambda & 0 & 0 \\
3 & 0 & 0 & \mu & 0 & \lambda & 0 \\
4 & 0 & 0 & 0 & \mu & 0 & \lambda \\
5 & 0 & 0 & 0 & 0 & \mu & 0
\end{array}
=
\begin{array}{c|cccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
0 & 0 & 2 & 0 & 0 & 0 & 0 \\
1 & 2.5 & 0 & 2 & 0 & 0 & 0 \\
2 & 0 & 2.5 & 0 & 2 & 0 & 0 \\
3 & 0 & 0 & 2.5 & 0 & 2 & 0 \\
4 & 0 & 0 & 0 & 2.5 & 0 & 2 \\
5 & 0 & 0 & 0 & 0 & 2.5 & 0
\end{array}
$$

max num. of people in system



Operations Research - II

Continuous Time Markov Chains CTMCs
Transient Solutions: ATM Example

General Rate Matrix

System State index

$$\alpha_i = \sum_{j=0}^{m-1} r_{ij}$$

$$R = \begin{array}{c|c} & \begin{array}{ccccc} 0 & 1 & 2 & \dots & m-1 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \\ m-1 \end{array} & \begin{bmatrix} 0 & r_{01} & r_{02} & \dots & r_{0,m-1} \\ r_{10} & r_{11} & r_{12} & \dots & r_{1,m-1} \\ . & . & . & \dots & . \\ r_{m-1,0} & r_{m-1,1} & r_{m-1,2} & \dots & 0 \end{bmatrix} \end{array}$$

$$P = \begin{bmatrix} 1-\Delta\alpha_0 & \Delta r_{01} & \Delta r_{02} & \dots & \Delta r_{0,m-1} \\ \Delta r_{12} & 1-\Delta\alpha_1 & \Delta r_{12} & \dots & \Delta r_{1,m-1} \\ . & . & . & \dots & . \\ \Delta r_{m-1,0} & \Delta r_{m-1,1} & \Delta r_{m-1,2} & \dots & 1-\Delta\alpha_{M-1} \end{bmatrix}$$

- **For the ATM example, the Markov chain transition matrix is**

$$P = \begin{array}{c|c} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 1-\Delta\lambda & \Delta\lambda & 0 & 0 & 0 & 0 \\ \Delta\mu & 1-\Delta(\lambda+\mu) & \Delta\lambda & 0 & 0 & 0 \\ 0 & \Delta\mu & 1-\Delta(\lambda+\mu) & \Delta\lambda & 0 & 0 \\ 0 & 0 & \Delta\mu & 1-\Delta(\lambda+\mu) & \Delta\lambda & 0 \\ 0 & 0 & 0 & \Delta\mu & 1-\Delta(\lambda+\mu) & \Delta\lambda \\ 0 & 0 & 0 & 0 & \Delta\mu & 1-\Delta\mu \end{bmatrix} \end{array}, \quad \text{For } \lambda = 2, \text{and } \mu = 2.5$$

- **Where P is a state transition matrix determined from the rate matrix R**

Continuous Time Markov Chains CTMCs
Transient Solutions: ATM Example

- For the ATM example, the Markov chain transition matrix is

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1-2\Delta & 2\Delta & 0 & 0 & 0 & 0 \\ 2.5\Delta & 1-4.5\Delta & 2\Delta & 0 & 0 & 0 \\ 0 & 2.5\Delta & 1-4.5\Delta & 2\Delta & 0 & 0 \\ 0 & 0 & 2.5\Delta & 1-4.5\Delta & 2\Delta & 0 \\ 0 & 0 & 0 & 2.5\Delta & 1-4.5\Delta & 2\Delta \\ 0 & 0 & 0 & 0 & 2.5\Delta & 1-2.5\Delta \end{bmatrix} \end{array}$$

- To approximate transient probabilities at any time (t) an increment Δ should be used. (CTMC at any time (t) is divided into DTMC with (n) period each period is Δ) where (t) = (n).(Δ)
- let Δ = 0.05 for (t = 1) (n = 1/0.05) = 20 steps

20 steps

t_0 t_1 t_2

40 steps

- Applying

$$q(t+1) = q(t)\,P$$

$$q(n.\Delta + \Delta) = q(n.\Delta)\,P \quad \text{for all periods } (n = 0, 2, \ldots t/\Delta)$$

$\frac{1}{\text{num of steps}}$

$$q(0.\Delta + \Delta) = q(0.\Delta)\,P$$

$$q(1.\Delta + \Delta) = q(1.\Delta)\,P$$

$$q(2.\Delta + \Delta) = q(2.\Delta)\,P$$

$$q(19.\Delta + \Delta) = q(19.\Delta)\,P$$

Operations Research - II

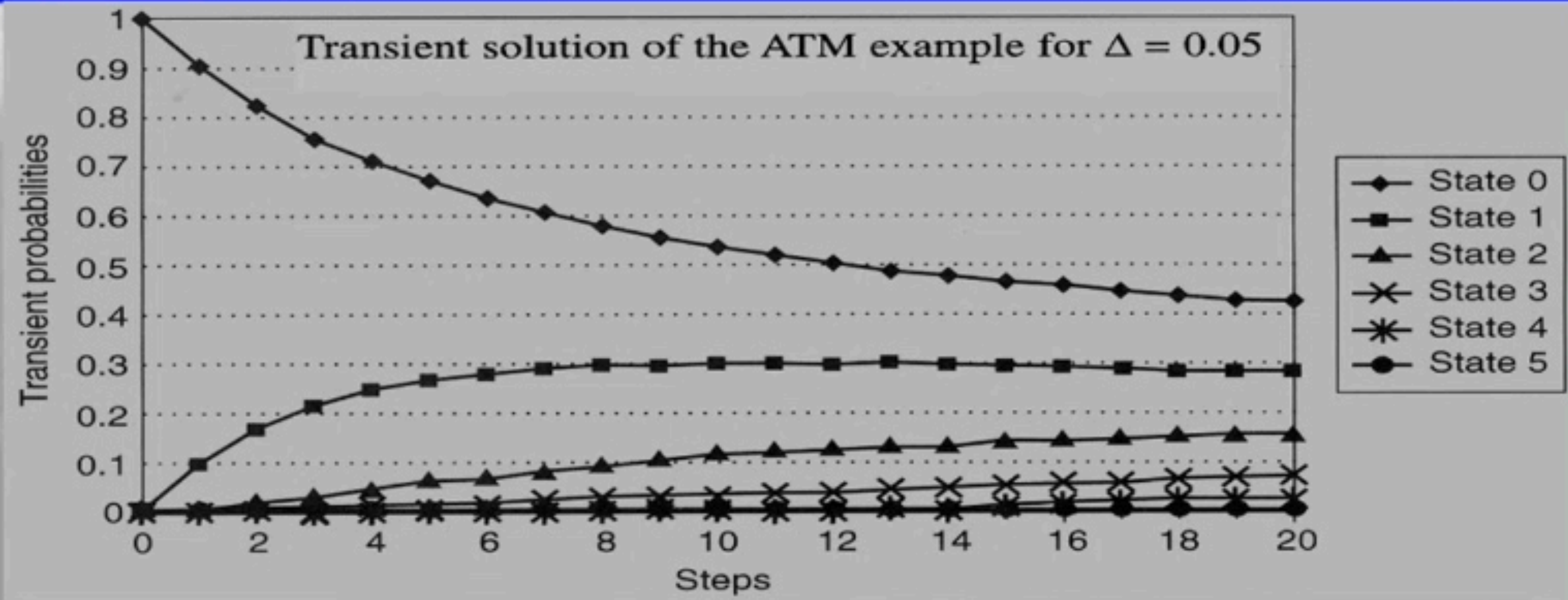
Continuous Time Markov Chains CTMCs
Transient Solutions: ATM Example

- $\Delta = 0.05$ for $(t = 1)$ $(n = 1/0.05) = 20$ steps

$$P = P^{n=1} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0.9 & 0.1 & 0 & 0 & 0 & 0 \\ 0.125 & 0.775 & 0.1 & 0 & 0 & 0 \\ 0 & 0.125 & 0.775 & 0.1 & 0 & 0 \\ 0 & 0 & 0.125 & 0.775 & 0.1 & 0 \\ 0 & 0 & 0 & 0.125 & 0.775 & 0.1 \\ 0 & 0 & 0 & 0 & 0.125 & 0.875 \end{bmatrix} \end{array} \begin{array}{l} \\ = 0 \text{ State vector at } \Delta, t = 1 \\ = 1 \text{ State vector at } \Delta, t = 1 \\ = 2 \text{ State vector at } \Delta, t = 1 \\ = 3 \text{ State vector at } \Delta, t = 1 \\ = 4 \text{ State vector at } \Delta, t = 1 \\ = 5 \text{ State vector at } \Delta, t = 1 \end{array}$$

Step Δ	Min.	Current	0	1	2	3	4	5	
0	0	0	1.000	0.000	0.000	0.000	0.000	0.000	1.000
1	0.05	0	0.900	0.100	0.000	0.000	0.000	0.000	1.000
2	0.1	0	0.823	0.168	0.010	0.000	0.000	0.000	1.000
3	0.15	0	0.761	0.213	0.025	0.001	0.000	0.000	1.000
4	0.2	0	0.712	0.244	0.040	0.003	0.000	0.000	1.000
5	0.25	0	0.671	0.266	0.056	0.007	0.000	0.000	1.000
6	0.3	0	0.637	0.280	0.071	0.011	0.001	0.000	1.000
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
201	10.05	0	0.271	0.217	0.173	0.139	0.111	0.089	1.000

Continuous Time Markov Chains CTMCs Transient Solutions: ATM Example



- **Transient solution:** Is the probability distribution of the number of customers in the system as a function of the time since opening at 8:00 AM
- **Transient period:** When the state of the system is influenced considerably by the initial conditions – customers queued up for service.
- **Steady state or equilibrium:** When the probability distribution becomes less dependent on the initial conditions, so state probabilities approach constant values. It does so only in time limit as time goes to infinity.
- **Probability State Vector:** Transition Matrix for a DTMCs consist of m State vectors at any period

Continuous Time Markov Chains CTMCs
Transient Solutions: ATM Example

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$$P = \begin{array}{c c} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 1-2\Delta & 2\Delta & 0 & 0 & 0 & 0 \\ 2.5\Delta & 1-4.5\Delta & 2\Delta & 0 & 0 & 0 \\ 0 & 2.5\Delta & 1-4.5\Delta & 2\Delta & 0 & 0 \\ 0 & 0 & 2.5\Delta & 1-4.5\Delta & 2\Delta & 0 \\ 0 & 0 & 0 & 2.5\Delta & 1-4.5\Delta & 2\Delta \\ 0 & 0 & 0 & 0 & 2.5\Delta & 1-2.5\Delta \end{bmatrix} \end{array}$$

trying

▪ $\Delta = 0.025$ for ($t = 1$) ($n = 1/0.025$) = 40 steps → increasing steps, more accuracy

$$P = \begin{array}{c c} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 1-2\Delta & 2\Delta & 0 & 0 & 0 & 0 \\ 2.5\Delta & 1-4.5\Delta & 2\Delta & 0 & 0 & 0 \\ 0 & 2.5\Delta & 1-4.5\Delta & 2\Delta & 0 & 0 \\ 0 & 0 & 2.5\Delta & 1-4.5\Delta & 2\Delta & 0 \\ 0 & 0 & 0 & 2.5\Delta & 1-4.5\Delta & 2\Delta \\ 0 & 0 & 0 & 0 & 2.5\Delta & 1-2.5\Delta \end{bmatrix} \end{array} = \begin{array}{c c} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0.95 & 0.05 & 0 & 0 & 0 & 0 \\ 0.0625 & 0.875 & 0.05 & 0 & 0 & 0 \\ 0 & 0.0625 & 0.875 & 0.05 & 0 & 0 \\ 0 & 0 & 0.0625 & 0.875 & 0.05 & 0 \\ 0 & 0 & 0 & 0.0625 & 0.875 & 0.05 \\ 0 & 0 & 0 & 0 & 0.0625 & 0.9375 \end{bmatrix} \end{array}$$

Continuous Time Markov Chains CTMCs
Transient Solutions: ATM Example

min

Step ~~#~~	~~Min.~~	Current	0	1	2	3	4	5	
0	0	0	1.000	0.000	0.000	0.000	0.000	0.000	1.000
1	0.025	0	0.950	0.050	0.000	0.000	0.000	0.000	1.000
2	0.05	0	0.906	0.092	0.003	0.000	0.000	0.000	1.000
3	0.075	0	0.866	0.127	0.007	0.000	0.000	0.000	1.000
4	0.1	0	0.831	0.156	0.012	0.000	0.000	0.000	1.000
404	10.1	0	0.272	0.217	0.173	0.139	0.111	0.089	1.000
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
405	**10.125**	**0**	**0.271**	**0.217**	**0.173**	**0.139**	**0.111**	**0.089**	**1.000**

الجامعة الأردنية

Continuous Time Markov Chains CTMCs
Transient Solutions: ATM Example

Transient Computations for the ATM Example with $\Delta = 0.025$

Steps, n	Time (min)	q_0	q_1	q_2	q_3	q_4	q_5
0	0	1	0	0	0	0	0
40	1	0.435	0.291	0.160	0.073	0.029	0.011
80	2	0.348	0.258	0.175	0.110	0.066	0.042
120	3	0.311	0.239	0.175	0.124	0.087	0.063
160	4	0.292	0.228	0.175	0.131	0.098	0.075
200	5	0.282	0.223	0.174	0.135	0.104	0.082
240	6	0.277	0.220	0.174	0.137	0.107	0.085
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
Steady state $(\pi^P =)$	∞	$(\pi_0^P = 0.271)$	$(\pi_1^P = 0.217)$	$(\pi_2^P = 0.173)$	$(\pi_3^P = 0.139)$	$(\pi_4^P = 0.111)$	$(\pi_5^P = 0.089)$

- The ATM is idle 0.271 of the time
- The efficiency of the machine is 1-0.271 = 0.729
- The proportion of customers obtaining immediate service is 0.271
- The proportion of customers who arrive and find the system full is 0.089
- The proportion of customers who wait is: 1- 0.271 – 0.089 = 0.640
- The expected average number in the system is $\sum_{i=0}^{5} i.\pi_i^P$ = 1.868 customers
- Throughput rate (average customers passing through system) = $\lambda(1-\pi_5^P)$ = 1.822 Cust. /Min.
- Balking rate (average customers lost to the system) = $\lambda\pi_5^P$ = 0.178 Cust. /Min.
- The average time in system = average number/ throughput rate = 1.868/1.822 = 1.025 Min

Operations Research - II

Transient Computations for the ATM Example with $\Delta = 0.025$

Steps, n	Time (min)	q_0	q_1	q_2	q_3	q_4	q_5
0	0	1	0	0	0	0	0
40	1	0.435	0.291	0.160	0.073	0.029	0.011
80	2	0.348	0.258	0.175	0.110	0.066	0.042
120	3	0.311	0.239	0.175	0.124	0.087	0.063
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200	5	0.282	0.223	0.174	0.135	0.104	0.082
240	6	0.277	0.220	0.174	0.137	0.107	0.085
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
Steady state $(\pi^P =)$	∞	$(\pi_0^P = 0.271)$	$(\pi_1^P = 0.217)$	$(\pi_0^P = 0.173)$	$(\pi_1^P = 0.139)$	$(\pi_0^P = 0.111)$	$(\pi_1^P = 0.089)$

The ATM is idle 0.271 of the time:

- The efficiency of the machine is 1-0.271 = 0.729
- The proportion of customers obtaining immediate service is 0.271
- The proportion of customers who arrive and find the system full is 0.089
- The proportion of customers who wait is: 1- 0.271 – 0.089 = 0.640
- The expected average number in the system is $\sum_{i=0}^{5} \mathbf{i}.\pi_i^P$ = 1.868 customers
- Throughput rate (average customers passing through system) = $\lambda(1-\pi_5^P)$ = 1.822 Cust. /Min.
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- The average time in system = average number/ throughput rate = 1.868/1.822 = 1.025 Min

π: Steady-state probability vector

Limiting state vector

$$\boldsymbol{\pi_j = lim_{n\to\infty} p_{ij}^n, i, j = 0,1,\ldots,m\text{-}1}$$

$$\pi = \pi P,$$

$$\sum_{j\in S} \pi_j = 1 \tag{1}$$

$$\pi(P - I) = 0$$

$$\pi = \begin{bmatrix} \pi_0 & \pi_1 & . & . & \pi_{m-1} \end{bmatrix}$$

$$P = \begin{bmatrix} p_{00} & p_{01} & \cdot & \cdot & p_{0,m-1} \\ p_{10} & p_{11} & \cdot & \cdot & p_{1,m-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ p_{m-1,0} & p_{m-1,1} & \cdot & \cdot & p_{m-1,m-1} \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & . & . & 0 \\ 0 & 1 & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & 1 \end{bmatrix}$$

$$(P-I)=\begin{bmatrix} ((p_{00})-1) & p_{01} & \cdot & \cdot & p_{0,m-1} \\ p_{10} & ((p_{11})-1) & \cdot & \cdot & p_{1,m-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ p_{m-1,0} & p_{m-1,1} & \cdot & \cdot & ((p_{m-1,m-1})-1) \end{bmatrix}$$

$$\begin{bmatrix}\pi_0 & \pi_1 & \cdot & \cdot & \pi_{m-1}\end{bmatrix}\begin{bmatrix} ((p_{00})-1) & p_{01} & \cdot & \cdot & p_{0,m-1} \\ p_{10} & ((p_{11})-1) & \cdot & \cdot & p_{1,m-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ p_{m-1,0} & p_{m-1,1} & \cdot & \cdot & ((p_{m-1,m-1})-1) \end{bmatrix}=\begin{bmatrix}0 & 0 & \cdot & \cdot & 0\end{bmatrix} \quad (2)$$

$$\begin{array}{llll} ((p_{00})-1)\pi_0 + p_{10}\pi_1 & + \cdot\ \cdot\ \cdot\ \cdot + p_{m-1,0}(\pi_{m-1}) & & = 0 \\ p_{01}\pi_0 \quad + ((p_{11})-1)\pi_1 & + \cdot\ \cdot\ \cdot\ \cdot + p_{m-1,1}(\pi_{m-1}) & & = 0 \\ \cdot \qquad \cdot & \cdot & & \\ \cdot \qquad \cdot & \cdot & & \\ \cdot \qquad \cdot & \cdot & & \\ p_{0,m-1}\pi_0 \quad + p_{1,m-1,}\pi_1 & + \cdot\ \cdot\ \cdot\ \cdot + ((p_{m-1,m-1})-1)(\pi_{m-1}) & & = 0 \end{array}$$

We Have m equations in m unknowns, one of these equations is redundant and must be replaced by equation (1), the first column of matrix (2) is replaced by equation (1), we obtain

$$\begin{bmatrix}\pi_0 & \pi_1 & \cdot & \cdot & \pi_{m-1}\end{bmatrix}\begin{bmatrix} 1 & p_{01} & \cdot & \cdot & p_{0,m-1} \\ 1 & ((p_{11})-1) & \cdot & \cdot & p_{1,m-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & p_{m-1,1} & \cdot & \cdot & ((p_{m-1,m-1})-1) \end{bmatrix}=\begin{bmatrix}1 & 0 & \cdot & \cdot & 0\end{bmatrix}$$

$$A_a=\begin{bmatrix} 1 & p_{01} & \cdot & \cdot & p_{0,m-1} \\ 1 & ((p_{11})-1) & \cdot & \cdot & p_{1,m-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & p_{m-1,1} & \cdot & \cdot & ((p_{m-1,m-1})-1) \end{bmatrix}$$

$$e_1 = \begin{bmatrix} 1 & 0 & . & . & 0 \end{bmatrix}$$

$$\pi A_a = e_1$$

for the computer repair example we have

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.8 & 0.2 & 0.0 \\ 1.0 & 0.0 & 0.0 \end{bmatrix} \end{array} \quad , \quad I = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \quad , \quad P - I = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} -0.4 & 0.3 & 0.1 \\ 0.8 & -0.8 & 0.0 \\ 1.0 & 0.0 & -1.0 \end{bmatrix} \end{array}$$

$$A_a = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0.3 & 0.1 \\ 1 & -0.8 & 0.0 \\ 1 & 0.0 & -1.0 \end{bmatrix} \end{array}$$

$$\pi A_a = e_1$$

$$\begin{bmatrix} \pi_0 & \pi_1 & \pi_2 \end{bmatrix} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0.3 & 0.1 \\ 1 & -0.8 & 0.0 \\ 1 & 0.0 & -1.0 \end{bmatrix} \end{array} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \pi_0 + \pi_1 + \pi_2 &= 1 \\ 0.3\pi_0 - 0.8\pi_1 + \quad &= 0 \Rightarrow \pi_1 = 0.375\pi_0 \\ 0.1\pi_0 - \quad \pi_2 &= 0 \Rightarrow \pi_2 = 0.1\pi_0 \end{aligned}$$

$$\pi_0 + 0.375\pi_0 + 0.1\pi_0 = 1 \Rightarrow \pi_0 = 0.6780, \pi_1 = 0.25425, \pi_2 = 0.0678$$