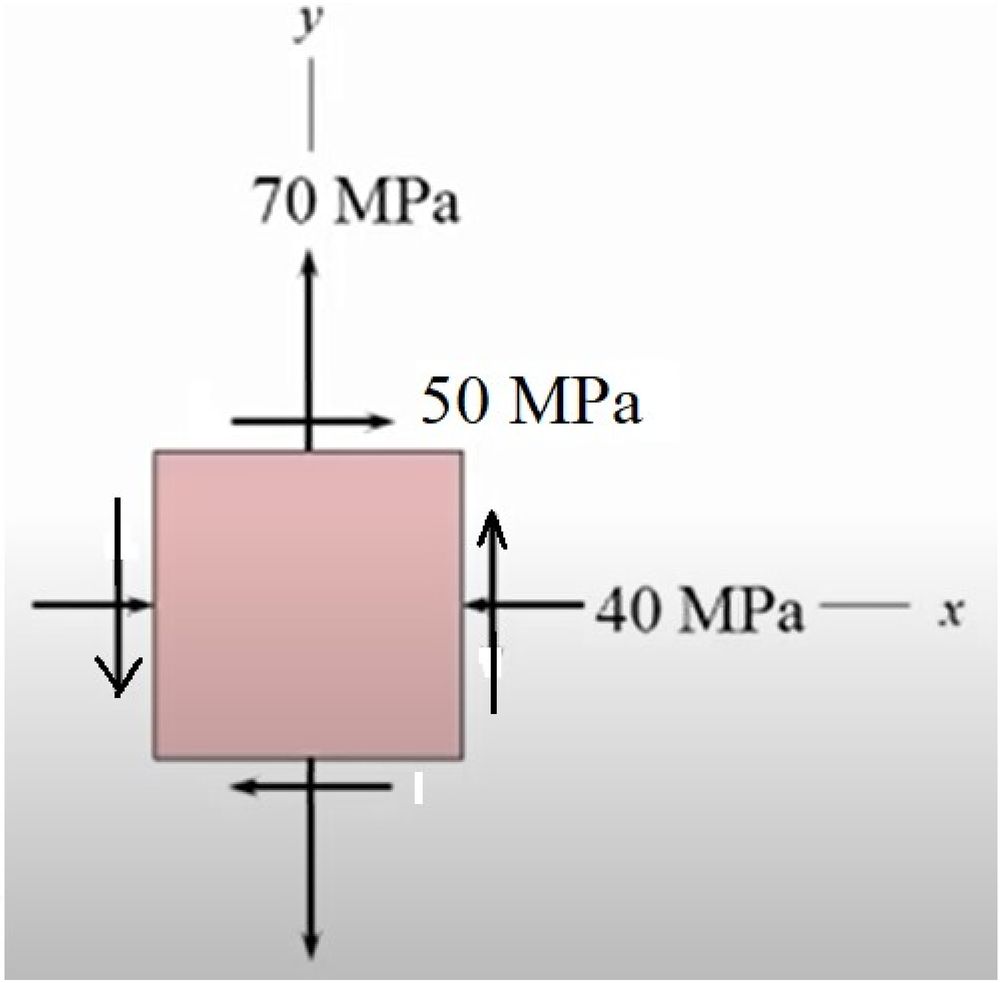
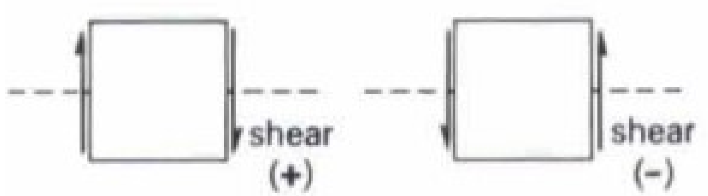
Example Determine the principal stresses and the maximum in-plane shear stress of the element.



$$\sigma_x = -40 \text{ MPa}$$

$$\sigma_y = 70 \text{ MPa}$$

$$\tau_{xy} = -50 \text{ MPa}$$



Principal stresses:

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= 89.3 \text{ MPa or } -59.3 \text{ MPa}$$

$$\tan 2\theta_p = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}$$

$$\theta_{p_1} = \frac{1}{2}\tan^{-1} 0.909 = 21.1°$$

Positive sign of the angle means clockwise rotation of the element

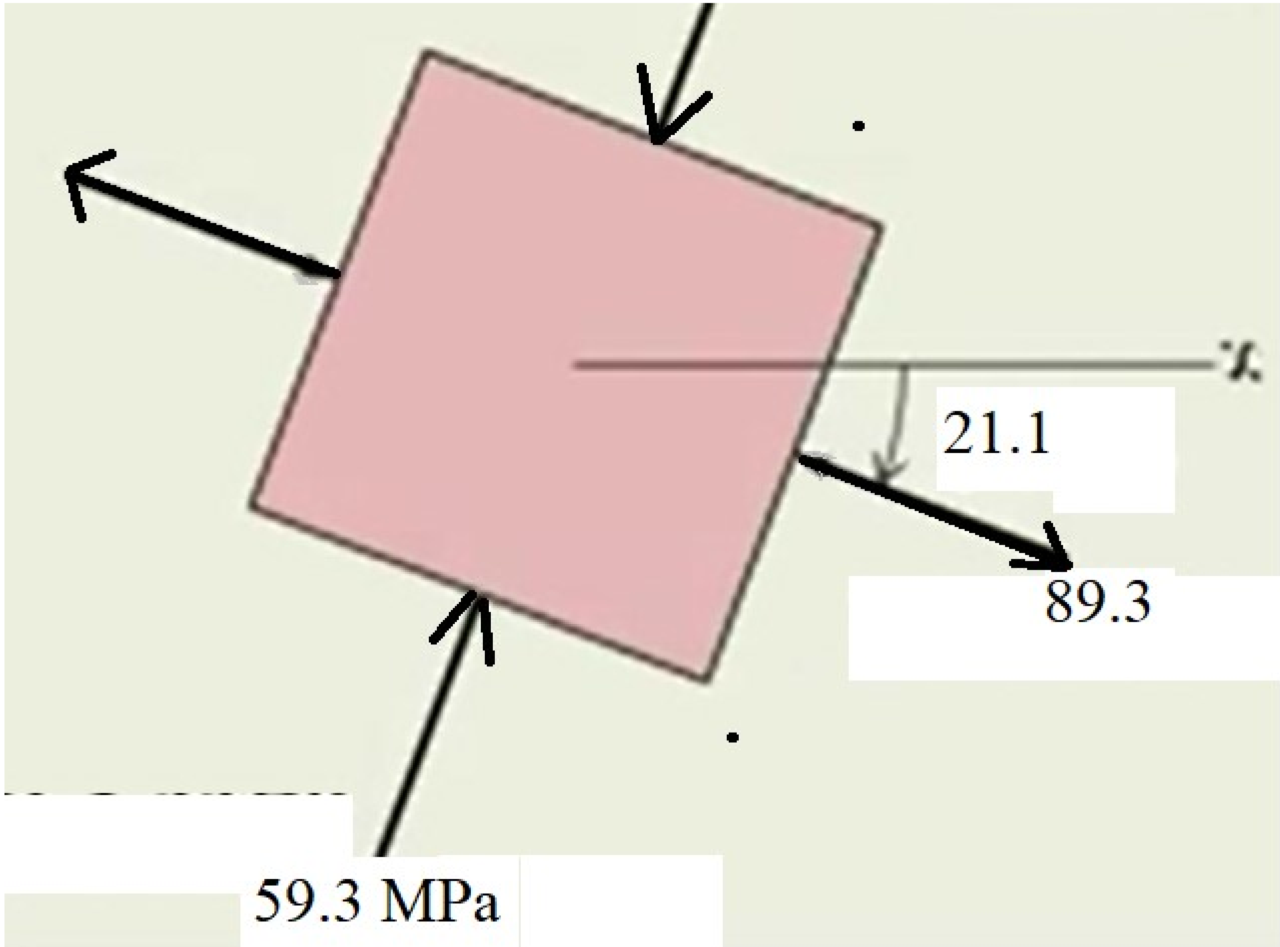
$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\phi + \tau_{xy} \sin 2\phi$$

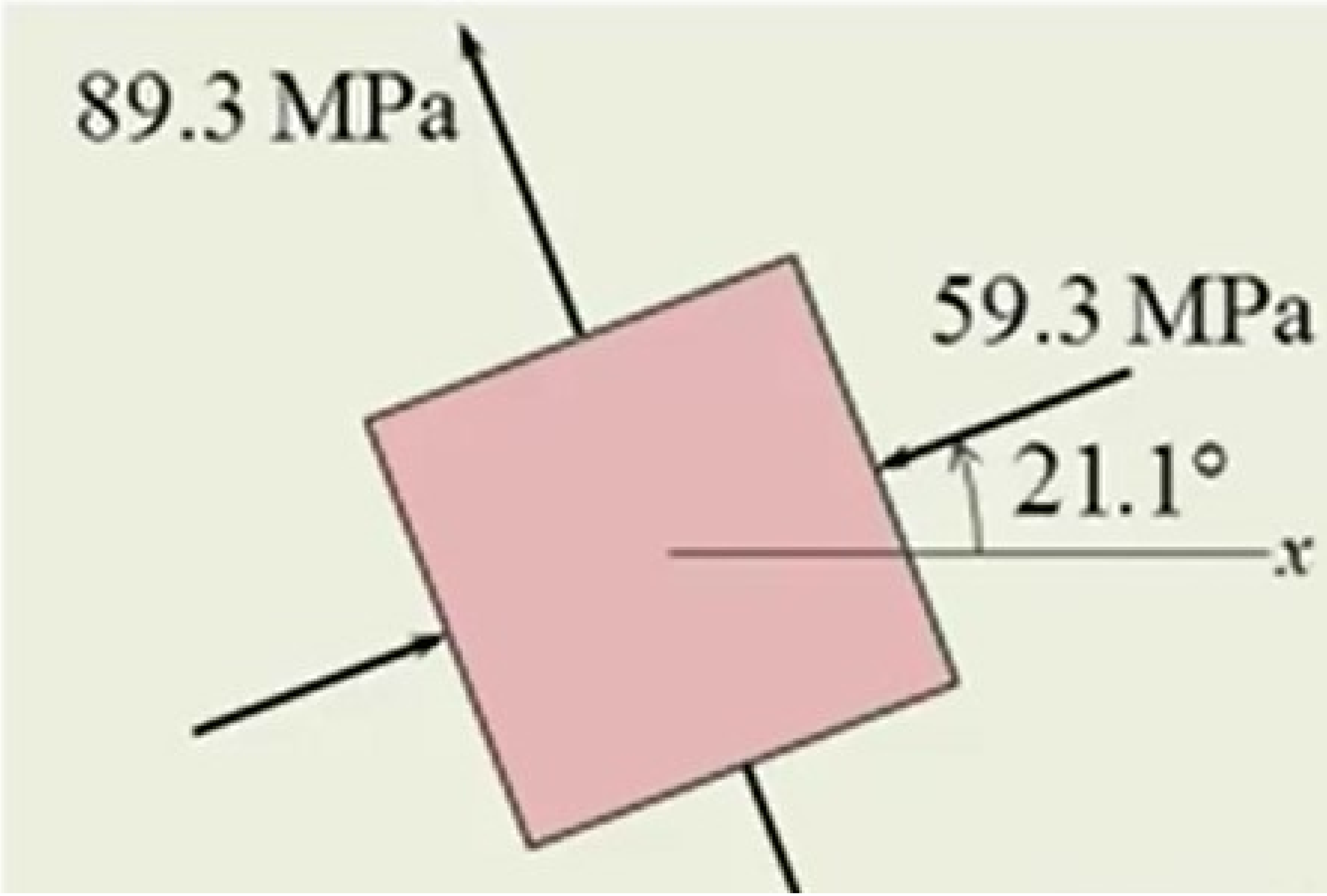
$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\phi - \tau_{xy} \sin 2\phi$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\phi + \tau_{xy} \cos 2\phi$$

$\sigma 2 = -59.3$

$\sigma 1 = 89.3$


x
21.1
89.3
59.3 MPa


89.3 MPa
59.3 MPa
21.1°
x

Maximum in-plane shear stress:

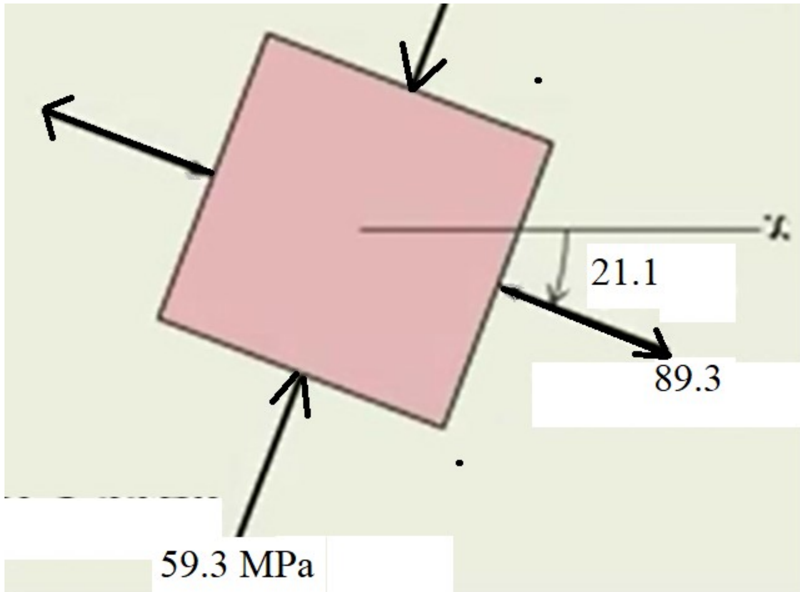
$$\left|\tau_{\substack{\max \\ \text{in-plane}}}\right| = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

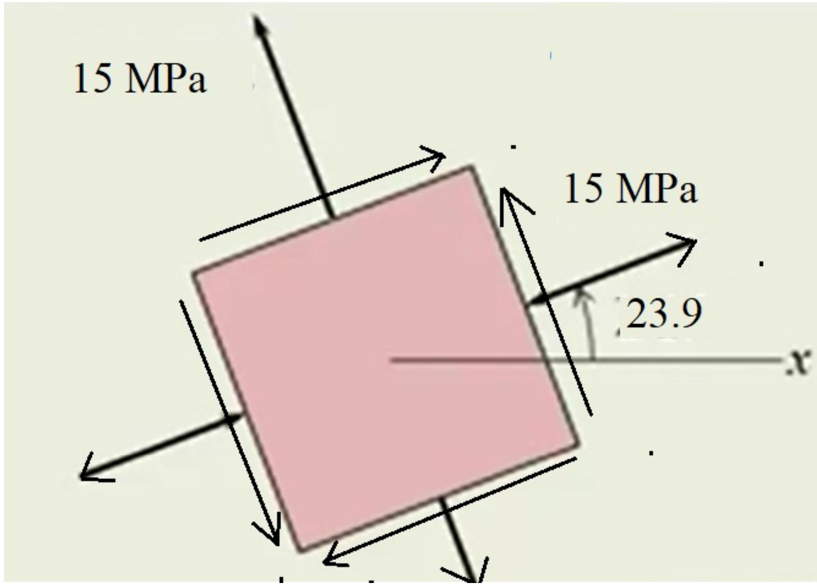
$$= 74.3 \text{ MPa}$$

$$\tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)/2}{\tau_{xy}}$$

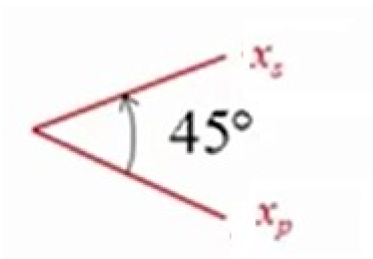
$$\theta_{s_1} = \frac{1}{2}\tan^{-1}(-1.1) = -23.9^\circ$$

$$= 74.3 \text{ MPa}$$





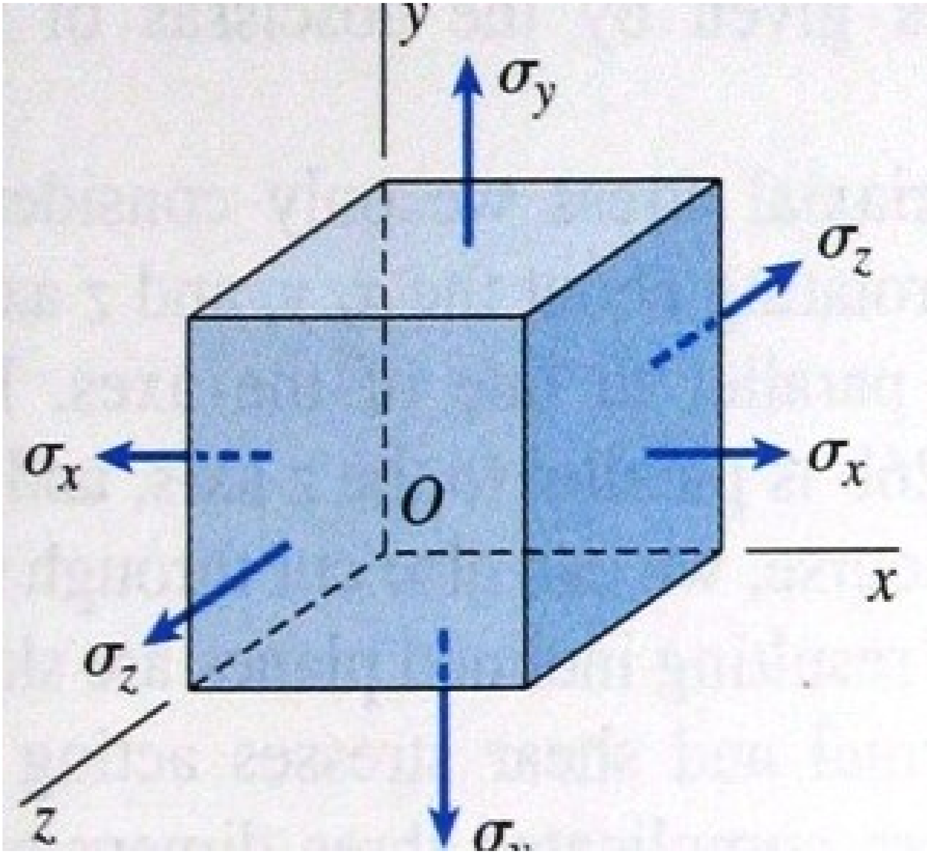
Principal stresses



Maximum in-plane shear stress

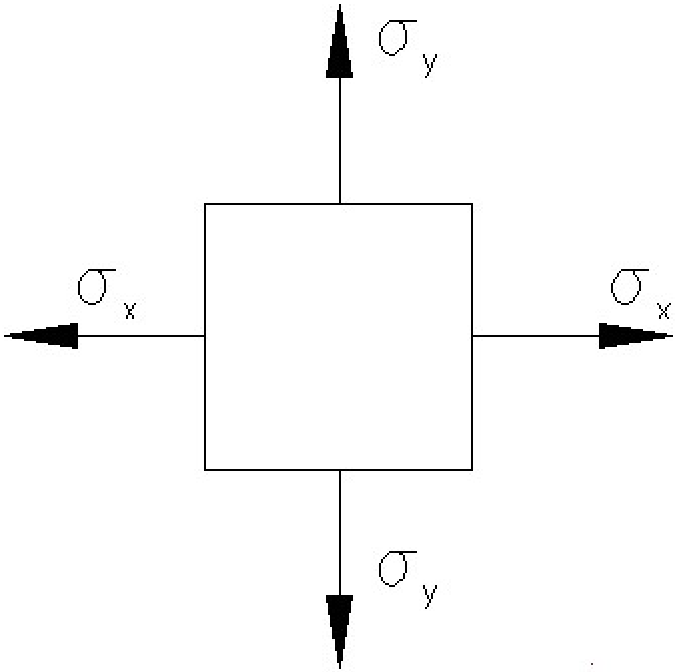
3-7 General Three Dimensional Stress (Triaxial Stresses)

Triaxial Stress
(no shear stress)

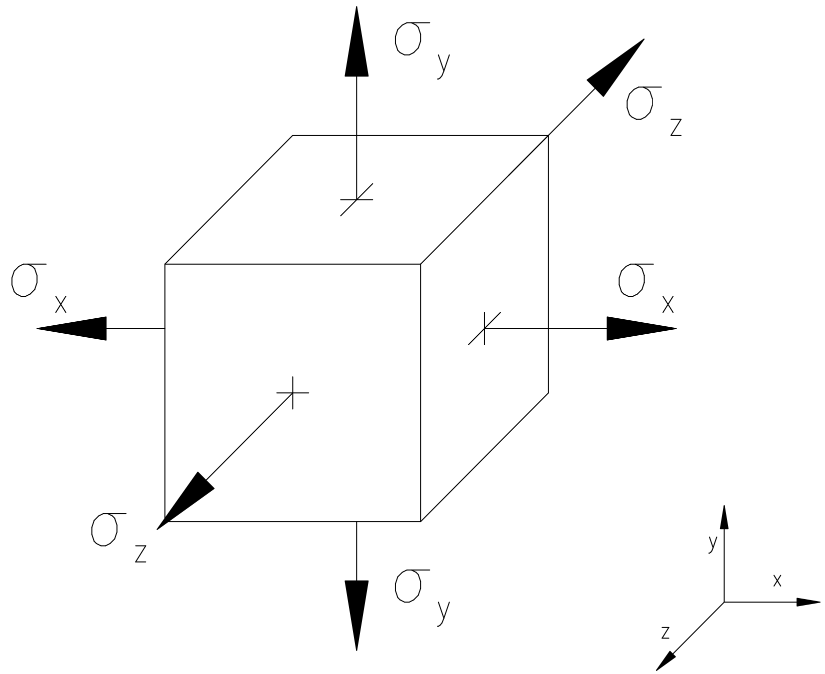


stress states:

- **Biaxial** stresses are stresses σ applied along **two** axes of the element and **$\tau = 0$.**



- **Triaxial** stresses are stresses σ applied along **three** axes of the element and **$\tau = 0$.**



3D Stress – Principal Stresses

The three principal stresses are obtained as the three real roots of the following equation:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

where

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

$$I_2 = \sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z - \tau_{xy}^2 - \tau_{xz}^2 - \tau_{yz}^2$$

$$I_3 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{xz}\tau_{yz} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2$$

I_1, I_2, and I_3 are known as stress invariants as they do not change in value when the axes are rotated to new positions.

$$\sigma_{ij} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -2 \\ 0 & -2 & 1 \end{bmatrix}$$

$$I_1 = 2 + 3 + 1 = 6$$

$$I_2 = 6,$$

$$I_3 = -3$$

The principal shear stresses are given by

$$\tau_{1/2} = \frac{\sigma_1 - \sigma_2}{2}$$

$$\tau_{2/3} = \frac{\sigma_2 - \sigma_3}{2}$$

$$\tau_{1/3} = \frac{\sigma_1 - \sigma_3}{2}$$

OCTAHEDRAL SHEAR STRESS CRITERION

On the octahedral plane, the octahedral normal stress

$$\sigma_h = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

octahedral shear stress

$$\tau_{oct} = \tfrac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

The octahedral stress criterion in terms of the yield strength:

$$\tau_{oct} = \frac{\sqrt{2}}{3} Sy$$

we expect to observe yielding in a material under 3-D loading when

$$Sy = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

Example

Consider the stress state

$$[\sigma_{ij}] = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -6 & -12 \\ 0 & -12 & 1 \end{bmatrix}$$

The principal stresses $\sigma_1 = 10, \sigma_2 = 5, \sigma_3 = -15$

the maximum shear stress is

$$\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_3) = \frac{25}{2}$$

Example: triaxial stress state, not plane stress

- Determine the maximum principal stresses and the maximum shear stress for the following triaxial stress state.
- Will the material yield

$$\sigma = \begin{bmatrix} 20 & 40 & -30 \\ 40 & 30 & 25 \\ -30 & 25 & -10 \end{bmatrix} \text{MPa}$$

$$\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{zx} & \tau_{yz} & \sigma_z \end{bmatrix} = \begin{bmatrix} 20 & 40 & -30 \\ 40 & 30 & 25 \\ -30 & 25 & -10 \end{bmatrix} \quad \text{MPa}$$

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

$I_1 = \sigma_x + \sigma_y + \sigma_z$ = 20 + 30 –10 = 40 MPa

$I_2 = \sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z - \tau_{xy}^2 - \tau_{xz}^2 - \tau_{yz}^2$ = -3025 MPa

$I_3 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{xz}\tau_{yz} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2$

= -89500 MPa

$$\sigma^3 - 40\sigma^2 - 3025\sigma - 89500 = 0$$

$$\sigma_3 = 65.3MPa$$

$$\sigma_2 = 26.5MPa$$

$$\sigma_1 = -51.8MPa$$

$$\tau_{\max} = 1/2(65.3 + 51.8)$$

$$= 58.5MPa$$

5-3 Failure Theories

5-4 Maximum Shear Stress Theory for Ductile Materials (Tresca)
5-5 Distorsion Energy Theory for Ductile Materials (Von Misses)
5-14 Important Design Equations

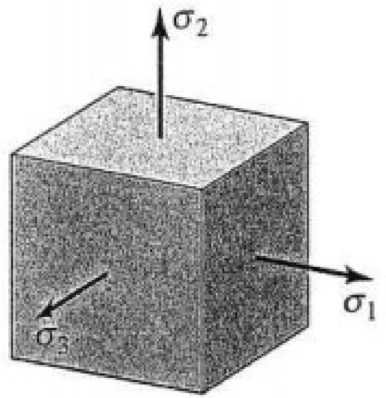
5-4. Maximum Shear Stress Theory for Ductile Materials Tresca

$$\tau_{max} = \frac{1}{2}\left|\sigma_{max} - \sigma_{min}\right|$$

According to Tresca yielding occurs when:

$$\tau_{max} = \frac{1}{2}\left|\sigma_{max} - \sigma_{min}\right| = \frac{S_y}{2}$$

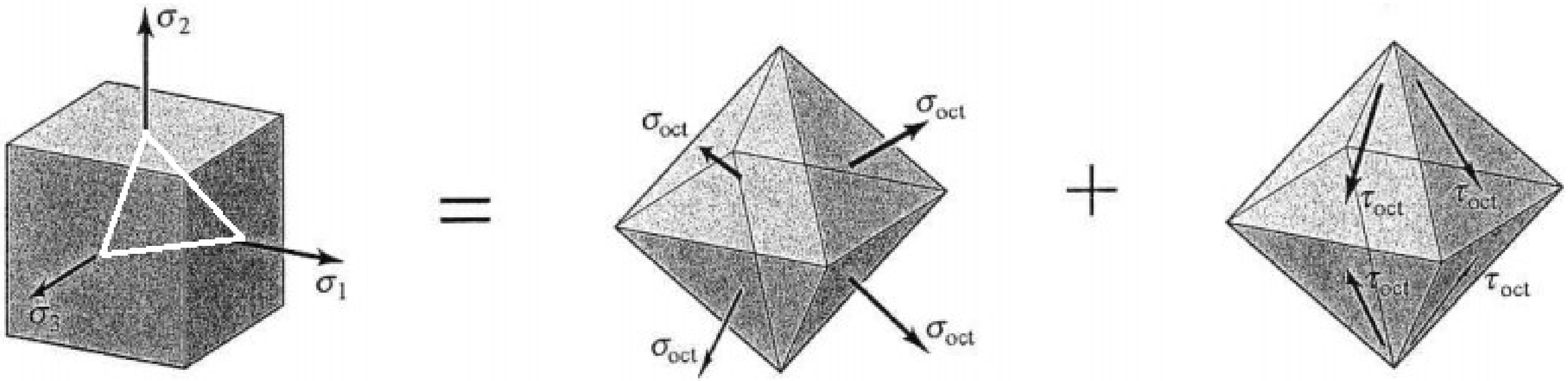
5-5 Distorsion Energy Theory for Ductile Materials (Von Misses)

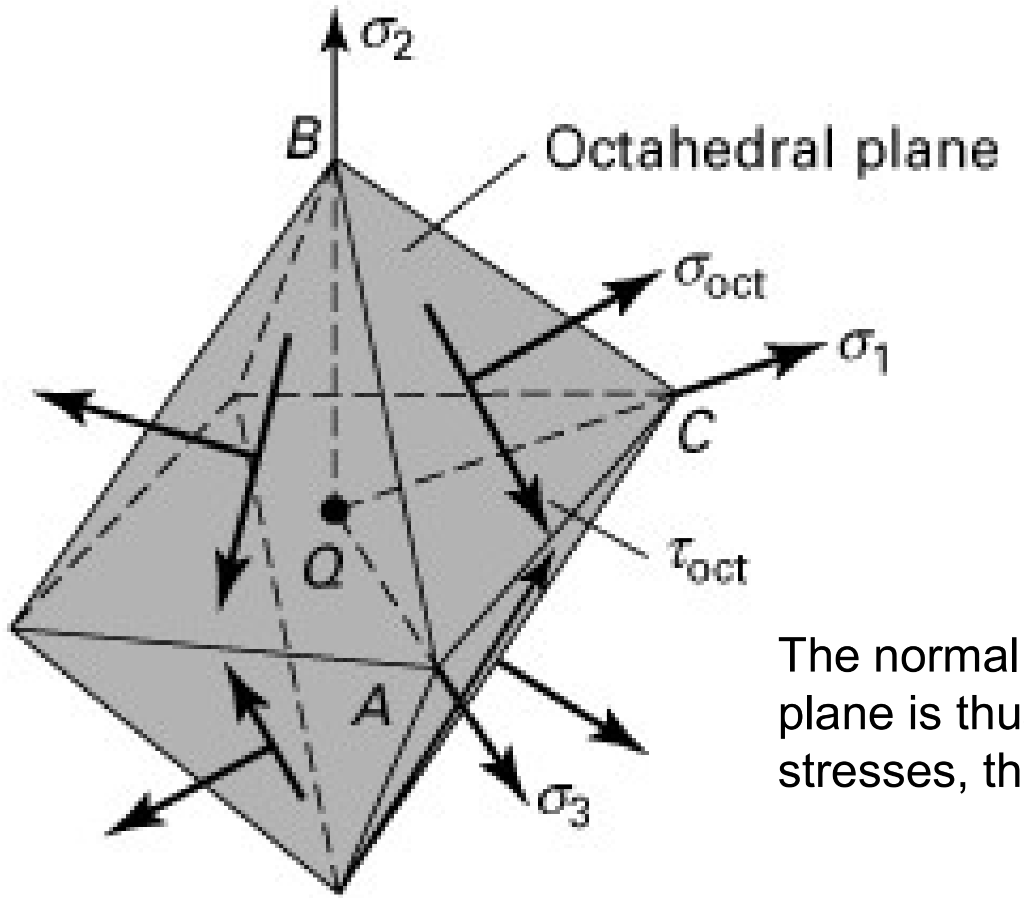


$$
= \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}
$$

(a)

There exist an Octahedral plane: same intercept with all three principle axes





The normal stress acting on an octahedral plane is thus the average of the principal stresses, the *mean stress*

That the normal and shear stresses are the same for the eight planes is a powerful tool for failure analysis of ductile materials

On the octahedral plane, the octahedral normal stress

$$\sigma_{oct} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

Hydrostatic stress

octahedral shear stress in erms of principle stesses

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

octahedral shear stress in terms of the stress compenents in the x , y and z

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}$$

According to Von Misses yielding occurs when:

$$\tau_{Oct} = \frac{\sqrt{2}}{3}\sigma_y$$

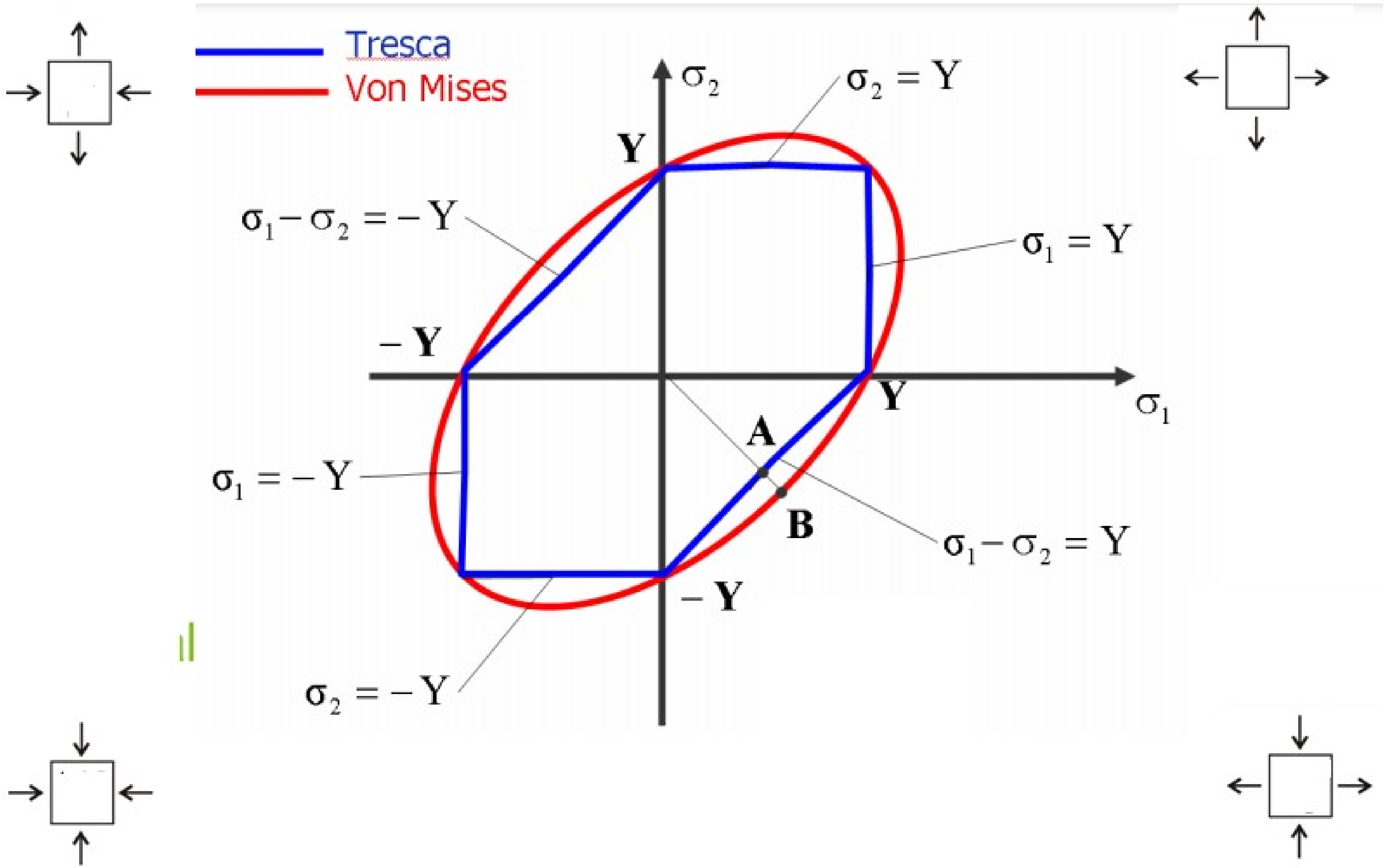
based on principal stress

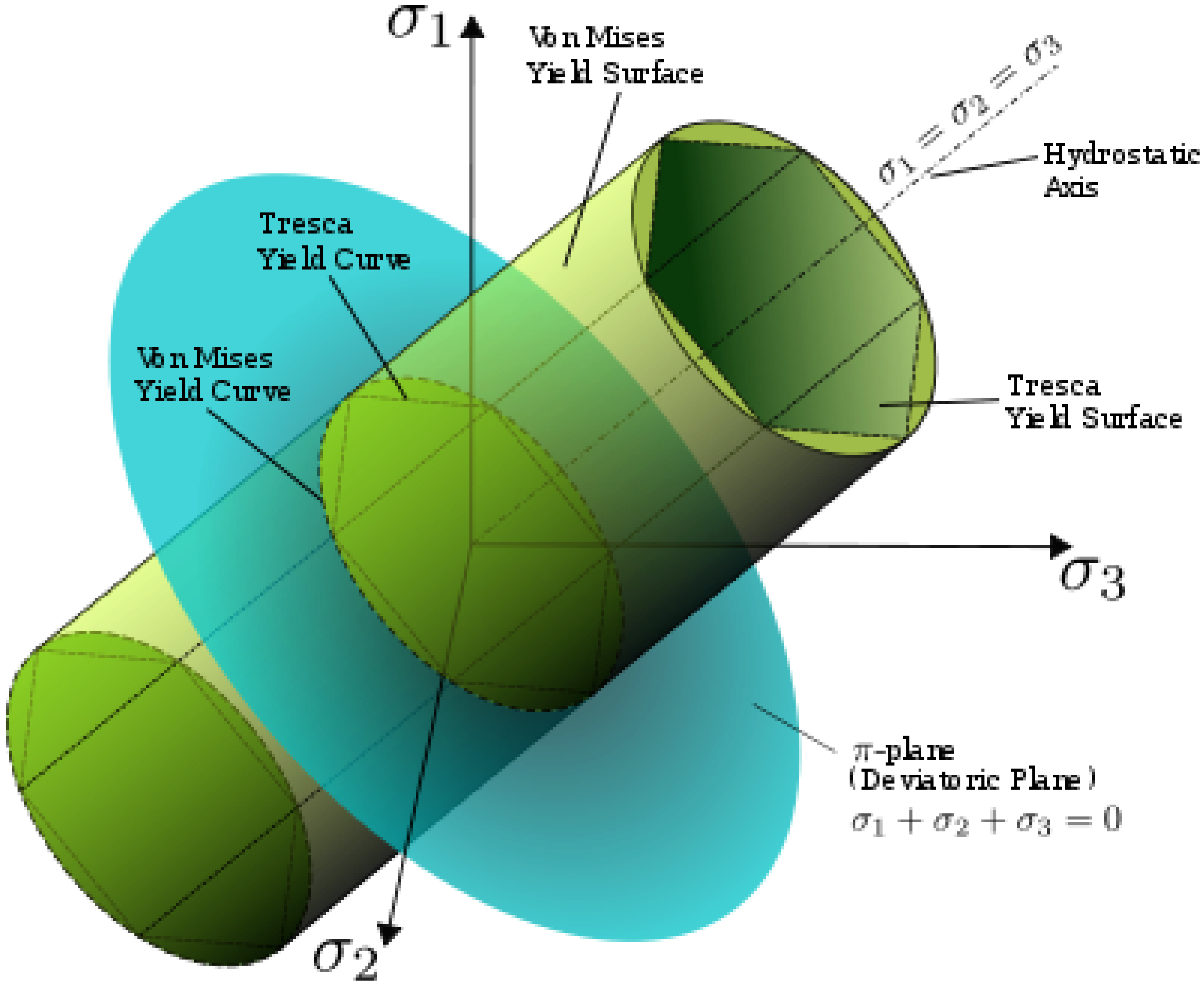
$$\frac{1}{3}\sqrt{(\sigma_1-\sigma_3)^2+(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2} = \frac{\sqrt{2}}{3}\sigma_y$$

based on x, y and z stresses

$$\frac{1}{3}\sqrt{(\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2+6(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2)} = \frac{\sqrt{2}}{3}\sigma_y$$

Graphical comparison between Tresca and VonMises





σ_1
Von Mises
Yield Surface
$\sigma_1 = \sigma_2 = \sigma_3$
Hydrostatic
Axis
Tresca
Yield Curve
Von Mises
Yield Curve
Tresca
Yield Surface
σ_3
π-plane
(Deviatoric Plane)
$\sigma_1 + \sigma_2 + \sigma_3 = 0$
σ_2

5-14 Important Design Equations

Factor of safety according to Tresca

$$n = \frac{S_y}{2\tau_{max}}$$

Factor of safety according to Von Misses

$$n = \frac{\sigma_{yield}}{\frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}}$$

$$n = \frac{\sigma_{yield}}{\frac{1}{\sqrt{2}}\sqrt{(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}}$$

3-8 Elastic Strain in Three dimensions

- Consider the case of trial stress

$$trial-stress\,\sigma_x, \quad \sigma_y, \sigma_z$$

- Corresponding normal strains,

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu(\sigma_y + \sigma_z)}{E}$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \frac{\nu(\sigma_x + \sigma_z)}{E}$$

$$\varepsilon_y = \frac{\sigma_z}{E} - \frac{\nu(\sigma_x + \sigma_y)}{E}$$

$$\tau = G\gamma$$

$$\tau = 2G(1+\nu)$$

3. Von Mises Criterion:

$$\sigma_3 = 65.3 MPa$$
$$\sigma_2 = 26.5 MPa$$
$$\sigma_1 = -51.8 MPa$$

$$S_y = 300$$

$$\mathbf{Sy} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

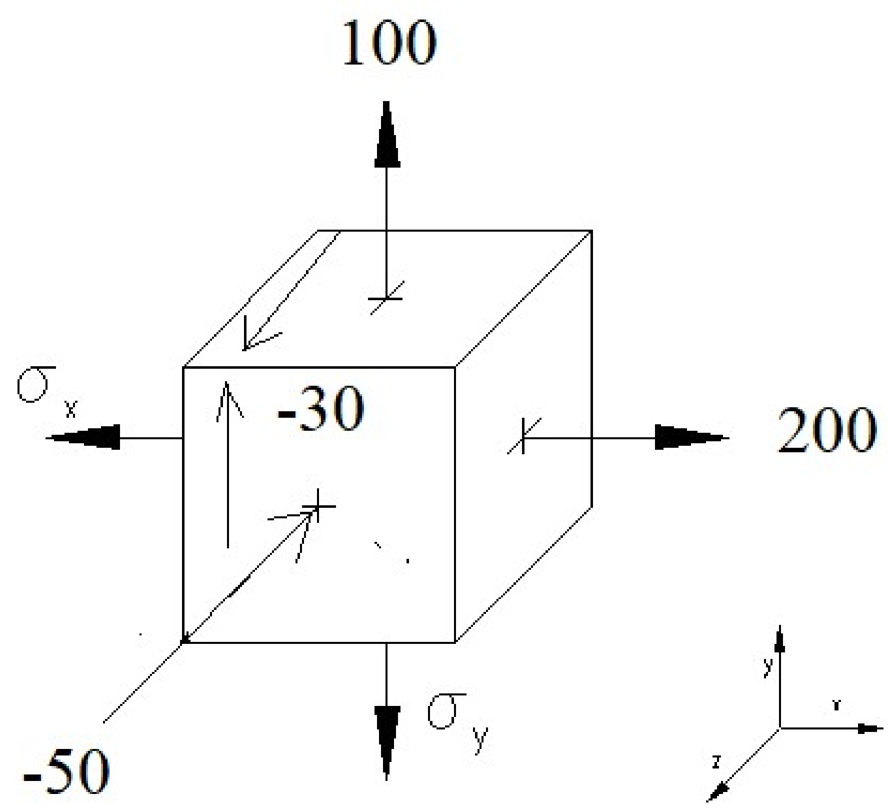
$$S_Y \overset{?}{=} \frac{1}{\sqrt{2}}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2}$$

$$S_Y \overset{?}{=} \frac{1}{\sqrt{2}}\left[(-51.8 - 26.5)^2 + (26.5 - 65.3)^2 + (65.3 - (-51.8))^2\right]^{1/2}$$

=103.31<300 Mpa
It will not yield

Ex: a uint volume in a structural member gives the state of stress as shown below. If the part is made from a steel with a yielding strength = 500 MPa, check yielding according to Tresca and von Mises criteria. What is its safety factor for each criterion?

$$\sigma = \begin{bmatrix} 200 & 0 & 0 \\ 0 & 100 & -30 \\ 0 & -30 & -50 \end{bmatrix} \textbf{ MPa}$$



100
σ_x
-30
200
σ_y
-50
y
x
z

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

$$I_2 = \sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z - \tau_{xy}^2 - \tau_{xz}^2 - \tau_{yz}^2$$

$$I_3 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{xz}\tau_{yz} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2$$

$$I_1 = \sigma_x + \sigma_y + \sigma_z = 200 + 100 - 50 = 250 \text{ MPa}$$

$$I_2 = \sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z - \tau_{xy}^2 - \tau_{xz}^2 - \tau_{yz}^2$$

$$= 200(100) + 200(-50) + 100(-50) - 0 - 0 - (30)^2 = 4100$$

$$I_3 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{xz}\tau_{yz} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2$$

$$= 200(-50)100 + 2(30)(0)(0) - (200)30^2 - 0 - 0 = -1180000$$

$$\sigma^3 - 250\sigma^2 + 4100\sigma + 1180000 = 0$$

The principal stresses are:

$$\sigma_1 = 200; \quad \sigma_2 = 105.77; \quad \sigma_3 = -55.77;$$

$$\tau_{max} = \frac{1}{2}\left|\sigma_{max} - \sigma_{min}\right| = (200+55.77)/2 = 127.885$$

Tresca

Factor of safety FS=250/127.89=1.95

$$\frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}=224$$

Factor of safety FS=500/224=2.2