Ordinary Differential Equations . ODEs

Definition: A differential equation (D.E.) is an equation with a function and one or more of its derivatives.

Example: ① $y' + y = 2x$

② $(x+1)\,dy - 2\,dx = 0$

The ordinary differential equation is an equation where the function depends on one independent variable.

Order: The highest derivative in the D.E. is called the order.

The degree of the D.E. is the exponent of the highest derivative

Example: ① $y'' + 2x y' = x^2$

order = 2

② $y''' + 2(y'')^2 = e^x$

order = 3

③ $x\,dx - y\,dy = 0$

order = 1

(III) Linearity: The ODE is linear if it is linear in y and its derivatives

Ex: ① $y''' - 2x y' = \sin x$

3rd order linear

② $y'' - (y')^2 = x$

Non linear second-order

③ $y^{(4)} - \sin y = x$

Non linear 4th order

④ $y'' + yy' = x$

Non linear 2^{nd} - order

Ex: show that $y(x) = e^{2x} - 1$ is a solution of

$$y'' - 4y = 4$$

The function y is a solution if it satisfies the D.E.

$$y = e^{2x} - 1$$

$$y' = 2e^{2x}$$

$$y'' = 4e^{2x}$$

By substitution in the D.E we obtain:

$$4e^{2x} - 4(e^{2x} - 1) \stackrel{?}{=} 4$$

$$4e^{2x} - 4e^{2x} + 4 \stackrel{?}{=} 4$$

$$\Longrightarrow \quad 4 = 4$$

$\therefore y = e^{2x} - 1$ is a solution of $y'' - 4y = 4$

Separable Differential Equations

Def: A separable ODE is any first-order D.E. which can be written as:

$$f(x)\,dx = g(y)\,dy$$

Example: Which of the following ODEs is sep.

① $x\cos^2 y\,dx + e^x\,dy = 0$

$$x\cos^2 y\,dx = -e^x\,dy$$

$$\frac{x}{e^x}\,dx = -\frac{1}{\cos^2 y}\,dy$$

$$\Rightarrow \quad x e^{-x}\,dx = -\sec^2 y\,dy$$

$\therefore$ separable

② $2xy\,dx + (x^2+1)\,dy = 0$

③ $(x+y^2)\,dx + x^2 e^{y}\,dy = 0$

④ $(xy+x)\,dx = (x^2+1)\,y\,dy$

$x(y+1)\,dx = (x^2+1)\,y\,dy$

$\Rightarrow \dfrac{x}{x^2+1}\,dx = \dfrac{y}{y+1}\,dy$

is separable

Example: Solve the following ODEs

① $y' = y^3 \cos^2 x$

$\dfrac{dy}{dx} = y^3 \cos^2 x$

$\Rightarrow \dfrac{dy}{y^3} = \cos^2 x \, dx$

$\Rightarrow \int y^{-3}\,dy = \frac{1}{2}\int (1+\cos 2x)\,dx$

$= \dfrac{y^{-2}}{-2} + c_1 = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + c_2$

$= \dfrac{-1}{2y^2} = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + C$

② $e^{2x-y}\,dx = e^{x+y}\,dy$

$$e^{2x}\,e^{-y}\,dx = e^{x}\,e^{y}\,dy$$

$$\frac{e^{2x}}{e^{x}}\,dx = \frac{e^{y}}{e^{-y}}\,dy$$

$$e^{x}\,dx = e^{2y}\,dy$$

$$\Longrightarrow \quad e^{x} = \frac{1}{2}\,e^{2y} + C$$

③ $\dfrac{dy}{dx} = \dfrac{xy\sin x}{y+1}$, $y(0)=1$

$$\frac{y+1}{y}\,dy = x\sin x\,dx$$

$$\int \left[1+\frac{1}{y}\right] dy = \int x\sin x\,dx$$

x		$\sin x$
	$+$	
1		$-\cos x$
	$-$	
0		$-\sin x$

$$\Rightarrow \quad y + \ln|y| = -x\cos x + \sin x + C$$

$$y(0)=1 \Longrightarrow 1 + \ln 1 = c \Longrightarrow \boxed{c=1}$$

$$\therefore \quad y + \ln|y| = -x\cos x + \sin x + 1$$

④ $(x^2y+y)\,dy + (y+1)\,dx = 0$

⑤ $\dfrac{dy}{dx} = x+1+xy^2+y^2$

$dy = \left[x+1+xy^2+y^2\right]dx$

$\Rightarrow dy = \left[x+1+y^2(x+1)\right]dx$

$\Rightarrow dy = (x+1)(1+y^2)\,dx$

$\Rightarrow \dfrac{dy}{1+y^2} = (x+1)\,dx$

$\Rightarrow \tan^{-1}y = \frac{1}{2}x^2+x+C$

First-Order Linear D.E.

(i) If $p(x)$, $f(x)$ are continuous functions of x, then

$$y' + p(x)\, y = f(x)$$

is a first-order linear D.E.

(ii) How can we solve such an equation

ⓐ The D.E. should be written as:

$$y' + p(x)\, y = f(x) \quad \text{----} \; (*)$$

ⓑ Find the integrating factor $e^{\int p(x)\,dx}$

ⓒ Multiply $(*)$ by $e^{\int p(x)\,dx}$

ⓓ simplify and integrate

Example: Solve the following ODEs

① $x\dfrac{dy}{dx}+3y=\dfrac{\sin x}{x^2}$

$$y'+\frac{3}{x}y=\frac{\sin x}{x^3} \quad \text{------} \quad (*)$$

(ii) $e^{\int p(x)dx}=e^{\int \frac{3}{x}dx}=e^{3\int \frac{1}{x}dx}=e^{3\ln x}$

$$=e^{\ln x^3}=x^3$$

(iii) Multiply $(*)$ by x^3

$$x^3y'+3x^2y=\sin x$$

$$(x^3y)'=\sin x$$

$$x^3y=-\cos x+C$$

② $y' + \frac{4x}{1+x^2} y = \frac{1}{(1+x^2)^3}$ ----- ⊛

(i) $e^{\int p(x)dx} = e^{\int \frac{4x}{1+x^2}dx} = e^{2\int \frac{2x}{1+x^2}dx}$

$= e^{2\ln(1+x^2)} = e^{\ln(1+x^2)^2}$

$= (1+x^2)^2$

(ii) Multiply ⊛ by $(1+x^2)^2$

$(1+x^2)^2 y' + 4x(1+x^2) y = \frac{1}{1+x^2}$

$\Rightarrow \left((1+x^2)^2 y\right)' = \frac{1}{1+x^2}$

$(1+x^2)^2 y = \tan^{-1}x + c$

Remark: As a method of solution, one can consider the following form

$x' + p(y)\,x = f(y) \quad , \quad x' = \frac{dx}{dy}$

③ $\dfrac{dy}{dx} = \dfrac{y}{4y^2 - 2x}$

$$\frac{dx}{dy} = \frac{4y^2 - 2x}{y}$$

$$x' = 4y - \frac{2}{y}x$$

$$\Longrightarrow \quad x' + \frac{2}{y}x = 4y \quad \text{-----} \ (*)$$

(ii) $e^{\int p(y)\,dy} = e^{\int \frac{2}{y}\,dx} = e^{2\ln y} = y^2$

(iii) Multiply $(*)$ by y^2

$$\Longrightarrow \quad y^2 x' + 2yx = 4y^3$$

$$\Longrightarrow \quad (y^2 x)' = 4y^3$$

$$\Longrightarrow \quad y^2 x = y^4 + c$$

Bernoulli D.E.

① A Bernoulli D.E. is any D.E. of the form:

$$y' + p(x)\,y = f(x)\,y^n, \quad n \in \mathbb{R} - \{0,1\}$$

where $p(x)$ and $f(x)$ are continuous

Remark: If $u(x) = (y(x))^n$, then

$$u'(x) = n\,(y(x))^{n-1}\,y'(x)$$

i.e., $u' = n\,y^{n-1}\,y'$

So, if $u = y^{1-n}$, then

$$u' = (1-n)\,y^{-n}\,y'$$

② Now, let us consider the D.E. ⊛

$$y' + p(x)\,y = f(x)\,y^n \;\cdots\cdots ⊛ \quad n \neq 0,1$$

This D.E. is nonlinear

To solve equation ⊛, we assume

$$u = y^{1-n}$$

$$\Rightarrow \quad u' = (1-n)\, y^{-n}\, y'$$

Multiply ⊛ by $(1-n)\, y^{-n}$:

$$(1-n)\, y^{-n}\, y' + (1-n)\, y^{-n}\, P(x)\, y = (1-n)\, y^{-n}\, f(x)\, y^{n}$$

$$(1-n)\, y^{-n}\, y' + (1-n)\, p(x)\, y^{1-n} = (1-n)\, f(x)$$

$$\Rightarrow \quad u' + (1-n)\, p(x)\, u = (1-n)\, f(x) \quad \text{----} \; (**)$$

which in linear !

Example : Solve the following ODEs :

① $x^2 y' + 2xy = y^3$

$$y' + \frac{2}{x} y = \frac{1}{x^2} y^3 \quad \text{----} \quad (*)$$

Let $u = y^{1-3}$, i.e., $u = y^{-2}$

$$\Longrightarrow \quad u' + (1-n) p(x)\, u = (1-n) f(x)$$

$$u' - 2 \cdot \frac{2}{x} u = -2 \cdot \frac{1}{x^2}$$

$$\therefore \quad u' - \frac{4}{x} u = \frac{-2}{x^2} \quad \text{---} \quad (**)$$

(ii) $e^{\int p(x)\,dx} = e^{\int \frac{-4}{x} dx} = e^{-4 \int \frac{1}{x} dx} = e^{-4 \ln x}$

$$= x^{-4}$$

(iii) Multiply (**) by x^{-4}

$$\Rightarrow \quad x^{-4} u' - 4 x^{-5} u = -2 x^{-6}$$

$$\Longrightarrow \quad (x^{-4} u)' = -2 x^{-6}$$

$$\therefore \quad x^{-4} u = \frac{-2}{-5} x^{-5} + c$$

$$x^{-4} y^{-2} = \frac{2}{5} x^{-5} + c$$

or $$\frac{1}{x^4 y^2} = \frac{2}{5} \frac{1}{x^5} + c$$

② $$y' + x^2 y = \frac{e^{-x^3} \sinh x}{3 y^2}$$

$$y' + x^2 y = \frac{1}{3} e^{-x^3} \sinh x \; y^{-2} \quad \text{---} \ (*)$$

Let $u = y^{1-(-2)} \Rightarrow u = y^3$

$$\therefore \quad u' + (1-n) P(x) \, u = (1-n) f(x)$$

$$u' + 3 \cdot x^2 u = 3 \cdot \frac{1}{3} e^{-x^3} \sinh x$$

$$\therefore \quad u' + 3x^2 u = e^{-x^3} \sinh x \quad \ldots\ldots (**)$$

$$e^{\int P(x)\,dx} = e^{\int 3x^2\,dx} = e^{x^3}$$

Multiply $(**)$ by e^{x^3}

$\therefore \quad x' + \frac{1}{y} x = y^2 x^2 \quad \text{----} \circledast$

Let $u = x^{1-2} \Rightarrow u = x^{-1}$

Equation $\circledast$ is reduced to:

$$u' + (1-n) P(y) u = (1-n) f(y)$$

$$u' - \frac{1}{y} u = -y^2 \quad \text{-----} \quad (\ast\ast)$$

$$e^{\int P(y)\,dy} = e^{-\int \frac{1}{y} dy} = e^{-\ln y} = e^{\ln y^{-1}} = \frac{1}{y}$$

Multiply $(\ast\ast)$ by $\frac{1}{y}$

$$\frac{1}{y} u' - \frac{1}{y^2} u = -y$$

$$\left(\frac{1}{y} u\right)' = -y$$

$$\frac{1}{y} u = -\frac{y^2}{2} + C$$

$$\frac{1}{y} \cdot \frac{1}{x} = -\frac{1}{2} y^2 + C$$

$$e^{x^3}u' + 3x^2 e^{x^3} u = e^{x^3} e^{-x^3} \sinh x$$

$$\therefore \left(e^{x^3} u\right)' = \sinh x$$

$$e^{x^3} u = \cosh x + c$$

$$\therefore e^{x^3} y^3 = \cosh x + c$$

Remark: Similarly, one can solve:

$$x' + p(y)\, x = f(y)\, x^n \quad , n \neq 0,1$$

Let $u = x^{1-n}$. Then the D.E. can be reduced to

$$u' + (1-n)\, p(y)\, u = (1-n)\, f(y)$$

③ $$\frac{dy}{dx} = \frac{y}{x^2 y^3 - x}$$

$$\frac{dx}{dy} = \frac{x^2 y^3 - x}{y}$$

$$\frac{dx}{dy} = x^2 y^2 - \frac{x}{y}$$

Remark: Consider the D.E.

$$\frac{dy}{dx} = \frac{x}{y - yx^2}$$

One can try $\frac{dx}{dy} = \frac{y - yx^2}{x}$

$$\Rightarrow \frac{dx}{dy} = \frac{y}{x} - yx$$

i.e., $x' + yx = yx^{-1}$

<u>or</u> $\frac{dy}{dx} = \frac{x}{y - yx^2}$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{y(1-x^2)}$$

$$\Rightarrow y\,dy = \frac{x}{1-x^2}\,dx$$

$$\Rightarrow \frac{1}{2}y^2 = \frac{-1}{2}\ln|1-x^2| + c$$

Homogeneous D.E.

Def: The D.E $y' = f(x,y)$ is called homogeneous if it can be written as

$$y' = g\left(\frac{y}{x}\right)$$

Ex: Which of the following ODEs is homogeneous

① $\frac{dy}{dx} = \frac{y-x}{x}$

$\frac{dy}{dx} = \left(\frac{y}{x}\right) - 1$ $\therefore$ homogeneous

② $\frac{dy}{dx} = \frac{2x^2 - y^2}{xy + x^2}$

$$\frac{dy}{dx} = \frac{\frac{2x^2}{x^2} - \frac{y^2}{x^2}}{\frac{xy}{x^2} + \frac{x^2}{x^2}}$$

$$\therefore \frac{dy}{dx} = \frac{2 - \left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right) + 1}$$

$\therefore$ homogeneous

③ $\dfrac{dy}{dx} = \dfrac{x - 2y + 1}{y - x}$

$$\frac{dy}{dx} = \frac{1 - 2\left(\frac{y}{x}\right) + \frac{1}{x}}{\left(\frac{y}{x}\right) - 1}$$

Not homogeneous

(iii) The homogeneous D.E. $y' = g\left(\frac{y}{x}\right)$ ..... ⊛

Can be solved by reducing it to a separable D.E. as follows:

Let $u = \frac{y}{x} \implies y = xu$

$$\implies y' = xu' + u$$

The D.E. ⊛ becomes:

$$xu' + u = g(u)$$

$$x\frac{du}{dx} = g(u) - u$$

$$\implies \frac{du}{g(u) - u} = \frac{dx}{x} \quad \text{Separable}$$

Ex: Solve the following ODEs:

(1) $$\frac{dy}{dx} = \frac{xy + y^2 + x^2}{x^2}$$

$$y' = \frac{xy}{x^2} + \frac{y^2}{x^2} + \frac{x^2}{x^2}$$

$$\therefore \quad y' = \frac{y}{x} + \left(\frac{y}{x}\right)^2 + 1 \quad \text{----} \quad (*)$$

let $u = \frac{y}{x} \Rightarrow y = xu$

$$\Rightarrow y' = xu' + u$$

Substitute in $(*)$

$$xu' + u = u + u^2 + 1$$

$$\Rightarrow x\,u' = u^2 + 1$$

$$\Rightarrow x\,\frac{du}{dx} = u^2 + 1$$

$$\Rightarrow \frac{du}{u^2+1} = \frac{dx}{x}$$

$$\Rightarrow \tan^{-1} u = \ln|x| + c$$

$$\Rightarrow u = \tan(\ln|x| + c) \Rightarrow \frac{y}{x} = \tan(\ln|x| + c)$$

$$|1-2u-u^2| = |x|^{-2} \cdot e^c$$

$|a| = |b|$
$a = \pm b$

$$1 - 2\frac{y}{x} - \frac{y^2}{x^2} = \pm e^c \, x^{-2}$$

call it $C \in \mathbb{R}$.

Multiply by x^2:

$$x^2 - 2xy - y^2 = C$$

③ $\dfrac{dy}{dx} = \dfrac{y}{x}\left(\ln y - \ln x + 1\right)$

$$y' = \frac{y}{x}\left(\ln\frac{y}{x} + 1\right) \quad \text{----} \ (*)$$

Let $u = \frac{y}{x} \Longrightarrow y = xu$

$$\Longrightarrow y' = xu' + u$$

From $(*)$ we have:

$$xu' + u = u(\ln u + 1)$$

$$xu' + u = u\ln u + u$$

$$xu' = u\ln u$$

② $(x+y)\,dy = (x-y)\,dx$

$$\frac{dy}{dx} = \frac{x-y}{x+y}$$

$$\therefore \frac{dy}{dx} = \frac{1-\frac{y}{x}}{1+\frac{y}{x}} \quad \cdots \circledast$$

Let $u = \frac{y}{x} \implies y = xu$

$$\implies y' = xu' + u$$

$$\implies xu' + u = \frac{1-u}{1+u}$$

$$\frac{1-u}{1+u} - u$$

$$\frac{1-u-u-u^2}{1+u}$$

$$\frac{1-2u-u^2}{1+u}$$

$$\implies xu' = \frac{1-u}{1+u} - u$$

$$\therefore x\frac{du}{dx} = \frac{1-2u-u^2}{1+u}$$

$$\therefore \frac{1+u}{1-2u-u^2}\,du = \frac{dx}{x}$$

$$\implies -\frac{1}{2}\ln|1-2u-u^2| = \ln|x| + c$$

$$\implies \ln|1-2u-u^2| = -2\ln|x| + c$$

$\boxed{\text{H.W}}$ Solve the following ODEs:

① $(xy+y^2)\,dx - x^2\,dy = 0$

② $$\frac{dy}{dx} = \frac{x\sec\left(\frac{y}{x}\right)+y}{x}$$

③ $$\frac{dy}{dx} = \frac{\sqrt{xy}+y}{x}, \quad x>0$$

$$x\frac{du}{dx} = u\ln u$$

$$\frac{du}{u\ln u} = \frac{dx}{x}$$

$$\frac{\frac{1}{u}}{\ln u}\,du = \frac{dx}{x}$$

$$\ln|\ln u| = \ln|x| + c$$

$$|\ln u| = e^{\ln|x|+c}$$

$$|\ln u| = e^{c}e^{\ln|x|} = e^{c}|x|$$

$$\ln u = \underbrace{\pm e^{c}}_{\text{call it } C\in\mathbb{R}}x$$

$$\therefore\ u = e^{cx}$$

$$\frac{y}{x} = e^{cx}$$

$$\Longrightarrow\ y = x e^{cx}$$

Remark: Note that $y = x$ is also a solution

$$\frac{dy}{dx} = f(ax+by+c) \quad \cdots \quad ⊗$$

Examples of $f(ax+by+c)$

(i) $f(x+y+2) = (x+y+2)^3 - 1$

(ii) $f(x+y) = \sin(x+y) - 1$

(iii) $f(x-y) = \dfrac{2x-2y+1}{x-y}$

$$= \frac{2(x-y)+1}{x-y}$$

To solve ⊗ we set: $u = ax+by+c$

$\Longrightarrow u' = a + by'$

$\Longrightarrow y' = \dfrac{u'-a}{b}$

By this substitution, D.E. ⊗ is reduced to separable.

Ex: Solve the following ODEs:

① $\frac{dy}{dx} = (x+y+2)^2 \quad \cdots \circledast$

Let $u = x+y+2$

$\Rightarrow u' = 1 + y' \Rightarrow y' = u' - 1$

Substitute in $\circledast$:

$u' - 1 = u^2$

$\Rightarrow u' = u^2 + 1 \Rightarrow \frac{du}{dx} = u^2 + 1$

$\Rightarrow \frac{du}{u^2+1} = dx \Rightarrow \tan^{-1} u = x + c$

$\Rightarrow u = \tan(x+c)$

$\Rightarrow x + y + 2 = \tan(x+c)$

② $\frac{dy}{dx} = \sqrt{x+y} - 1$

③ $\frac{dy}{dx} = \sin(x-y)$

(4) $y' = (2x+y)^2 - 2$ --- ⊛

Let $u = 2x+y$

$\Rightarrow u' = 2 + y'$

$\Rightarrow y' = u' - 2$

By substitution in ⊛ we obtain:

$u' - 2 = u^2 - 2$

$\frac{du}{dx} = u^2 \Longrightarrow \frac{du}{u^2} = dx$

$\Longrightarrow -\frac{1}{u} = x + c$

$\Rightarrow \frac{1}{u} = -x + c$

$\rightarrow u = \frac{1}{-x+c}$

$\therefore 2x + y = \frac{1}{-x+c}$

⑤ $y' = \frac{x+y+2}{x+y}$

$y' = \frac{x+y}{x+y} + \frac{2}{x+y}$

$y' = 1 + \frac{2}{x+y}$ ---- ⊛

Let $u = x+y \Longrightarrow u' = 1+y'$

$\Longrightarrow y' = u' - 1$

From ⊛ we obtain

$u' - 1 = 1 + \frac{2}{u}$

$u' = 2 + \frac{2}{u} \Longrightarrow u' = \frac{2u+2}{u}$

$\frac{du}{dx} = \frac{2(u+1)}{u} \Longrightarrow \frac{u}{u+1} du = 2\,dx$

$\left(1 - \frac{1}{u+1}\right) du = 2\,dx$

$u - \ln|u+1| = 2x + C$

$x+y - \ln|x+y+1| = 2x + C$

$\longrightarrow u = \frac{\quad}{-x+c}$

$\therefore \; 2x + y = \frac{1}{-x+c}$

⑤ $y' = \frac{x+y+2}{x+y}$

$y' = \frac{x+y}{x+y} + \frac{2}{x+y}$

$y' = 1 + \frac{2}{x+y} \quad ---- \; *$

let $u = x+y \Longrightarrow u' = 1 + y'$

$\longrightarrow y' = u' - 1$

From $*$ we obtain

$u' - 1 = 1 + \frac{2}{u}$

$u' = 2 + \frac{2}{u} \Longrightarrow u' = \frac{2u+2}{u}$

$\frac{du}{dx} = \frac{2(u+1)}{u} \Longrightarrow \frac{u}{u+1} du = 2\,dx$

$\left(1 - \frac{1}{u+1}\right) du = 2\,dx$

$u - \ln|u+1| = 2x + C$

$x+y - \ln|x+y+1| = 2x + C$

Revision

Ex: If $f(x,y) = x^3 + 2y^2$, find

① $\frac{\partial f}{\partial y}$ ② $\frac{\partial f}{\partial x}$

$\frac{\partial f}{\partial y} = 4y$ $\frac{\partial f}{\partial x} = 3x^2$

Ex: If $f(x,y) = x^2 + xy + \sin y$,

find $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial x}$

$\frac{\partial f}{\partial y} = x + \cos y$ $\frac{\partial f}{\partial x} = 2x + y$

Ex: If $f(x,y) = x\cos y - 2y$,

find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$

$\frac{\partial f}{\partial x} = \cos y$ $\frac{\partial f}{\partial y} = -x\sin y - 2$

Ex: If $\frac{\partial f}{\partial y} = x\cos y - 2y$, find f.

Integrate with respect to y, gives

$f(x,y) = x\sin y - y^2 + g(x)$

Exact D.E.

We will be interested in D.E of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

Ex: $(x-2y)\,dx + (y-2x)\,dy = 0$

Theorem: Let $M(x,y)$, $N(x,y)$, $\frac{\partial M}{\partial y}$, $\frac{\partial N}{\partial x}$ be continuous. Then

$$M\,dx + N\,dy = 0 \quad \cdots \circledast$$

is an exact D.E. iff

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The solution of $\circledast$ is $u(x,y) = c$

where $\frac{\partial u}{\partial x} = M$, $\frac{\partial u}{\partial y} = N$

$$\Rightarrow \quad u = \int M\,\partial x + g(y) \quad , \quad u = \int N\,\partial y + h(x)$$

Ex: Solve :

$$(2x\cos y + 3x^2 y)\,dx + (x^3 - x^2\sin y - y)\,dy = 0$$

$x^2\cos y + x^3 y$

$x^3 y + x^2\cos y - \frac{y^2}{2}$

$$\frac{\partial M}{\partial y} = -2x\sin y + 3x^2$$

$\therefore$ Exact

$$\frac{\partial N}{\partial x} = 3x^2 - 2x\sin y$$

The Solution is $u(x,y) = c$ where

$$\frac{\partial u}{\partial x} = M \,,\quad \frac{\partial u}{\partial y} = N$$

$$\Rightarrow \frac{\partial u}{\partial x} = 2x\cos y + 3x^2 y$$

$$\Rightarrow u(x,y) = x^2\cos y + x^3 y + f(y)$$

$$\frac{\partial u}{\partial y} = -x^2\sin y + x^3 + f'(y)$$

$$\Rightarrow x^3 - x^2\sin y - y = -x^2\sin y + x^3 + f'(y)$$

$$\Rightarrow -y = f'(y) \Longrightarrow f(y) = -\frac{y^2}{2} + c_1$$

$$x^2\cos y + x^3 y - \frac{y^2}{2}$$

$\therefore$ The Solution is: $u(x,y) = c$

i.e., $x^2\cos y + x^3 y - \frac{y^2}{2} = c$

Ex: Solve:

$$(e^x + y)\,dx + (x - \sin y)\,dy = 0$$

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = 1 \implies \text{Exact}$$

$\therefore$ The solution is $u(x,y) = c$ where

$$\frac{\partial u}{\partial x} = M, \quad \frac{\partial u}{\partial y} = N$$

$$\frac{\partial u}{\partial x} = e^x + y \implies u(x,y) = \int (e^x + y)\,\partial x$$

$$\implies u(x,y) = e^x + yx + f(y)$$

Differentiate w.r.to y

$$\frac{\partial u}{\partial y} = x + f'(y)$$

$$\implies x - \sin y = x + f'(y) \implies f'(y) = -\sin y$$

$$\implies f(y) = \cos y + c_1$$

$$\therefore \quad u(x,y) = e^x + yx + \cos y + c_1$$

The solution is $u(x,y) = c$

$$e^x + yx + \cos y + c_1 = c$$

$$\implies e^x + yx + \cos y = c$$

H.W Solve : $\dfrac{dy}{dx} = -\dfrac{\sin y + y\cos x}{\sin x + x\cos y}$

or $(\sin y + y\cos x)\,dx + (\sin x + x\cos y)\,dy = 0$

Ex: If $(2x + 6x^2y^2)\,dx + (4x^n y - 12y^3)\,dy = 0$

is exact, find n.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$12x^2 y = 4n\,x^{n-1}\,y$$

$$3x^2 = n\,x^{n-1}$$

$$\Longrightarrow \boxed{n = 3}$$

Integrating Factor

suppose the D.E.

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is not exact, i.e., $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

- If $\mu(x,y)$ is a function such that

$$\mu(x,y)\,M(x,y)\,dx + \mu(x,y)\,N(x,y)\,dy = 0$$

is exact, then $\mu(x,y)$ is called an integrating factor.

- We will be interested in finding μ as a function of x alone or as a function of y alone

- $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = R(x) \Longrightarrow \mu(x) = e^{\int R(x)\,dx}$$

- $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M} = R(y) \Longrightarrow \mu(y) = e^{\int R(y)\,dy}$$

② $(3x^2y + y^2)\,dx + (2x^3 + 3xy)\,dy = 0$

$$\frac{\partial M}{\partial y} = 3x^2 + 2y$$

$\therefore$ Not exact

$$\frac{\partial N}{\partial x} = 6x^2 + 3y$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -3x^2 - y = -(3x^2+y)$$

$$\Rightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M} = \frac{-(3x^2+y)}{-(3x^2y+y^2)}$$

$$= \frac{(3x^2+y)}{y(3x^2+y)} = \frac{1}{y} = R(y)$$

$$\Rightarrow \mu(y) = e^{\int R(y)\,dy} = e^{\int \frac{1}{y}\,dy} = e^{\ln y} = y$$

Multiply the D.E. by y :

$(3x^2y^2 + y^3)\,dx + (2x^3y + 3xy^2)\,dy = 0$

This D.E. is exact

$x^3y^2 + y^3x$

$x^3y^2 + xy^3$

The solution is :

$x^3y^2 + xy^3 = c$

Ex Solve the following ODEs :

① $(y^2 - 3xy - 2x^2)\,dx + (xy - x^2)\,dy = 0$

$$\frac{\partial M}{\partial y} = 2y - 3x$$

$\therefore$ not exact

$$\frac{\partial N}{\partial x} = y - 2x$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2y - 3x - y + 2x$$

$$= y - x$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{y-x}{xy - x^2} = \frac{y-x}{x(y-x)} = \frac{1}{x} = R(x)$$

$$\mu(x) = e^{\int R(x)\,dx} = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$$

Multiply the D.E. by x :

$$(xy^2 - 3x^2y - 2x^3)\,dx + (x^2y - x^3)\,dy = 0$$

This D.E is exact and its solution is given by

$$\frac{1}{2}x^2y^2 - x^3y - \frac{1}{2}x^4 = c$$

suppose $\mu(x)$ is an integrating factor of

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$\Rightarrow$ $\mu(x) M(x,y)\,dx + \mu(x) N(x,y)\,dy = 0$ is <u>exact</u>

$$\frac{\partial}{\partial y}\left[\mu(x) M(x,y)\right] = \frac{\partial}{\partial x}\left[\mu(x) N(x,y)\right]$$

$$\mu(x) \frac{\partial M}{\partial y} = \mu(x) \frac{\partial N}{\partial x} + N \mu'(x)$$

$$\mu(x) \frac{\partial M}{\partial y} - \mu(x) \frac{\partial N}{\partial x} = N \mu'(x)$$

$$\mu(x) \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = N \mu'(x)$$

$$\frac{\mu'(x)}{\mu(x)} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

$$\ln|\mu(x)| = \int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\,dx$$

$$\Rightarrow \mu(x) = e^{\int R(x)\,dx} \quad \text{where } R(x) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

$\boxed{\text{H.W}}$: Find the integrating factor:

① $(3x^2y + 2xy + y^3)\,dx + (x^2+y^2)\,dy = 0$

$$\mu = e^{3x}$$

② $y\,dx + (2xy - e^{-2y})\,dy = 0$

$$\mu = \frac{e^{2y}}{y}$$

③ $y\,dx + (2x - ye^{y})\,dy = 0$

$$\mu = y$$

④ $2xy\,dx + (y^2 - x^2)\,dy = 0$

$$\mu = \frac{1}{y^2}$$

$\boxed{\text{H.W}}$ show that $\mu_{(x,y)} = xy^2$ is an integrating factor of:

$$(2y - 6x)\,dx + (3x - 4x^2y^{-1})\,dy = 0$$

Second-order ODEs

Consider the D.E.:

$$y'' + p(x)\, y' + q(x)\, y = f(x) \quad \text{---- (1)}$$

where $p(x)$, $q(x)$ and $f(x)$ are continuous functions on some open interval I.

- If $f(x) = 0$, the D.E. (1) is called homogeneous, otherwise it is called nonhomogeneous

Theorem: Consider the homogeneous D.E.

$$y'' + p(x)\, y' + q(x)\, y = 0 \quad \text{--- (2)}$$

If y_1, y_2 are two solutions of (2) on I, then $y = c_1 y_1 + c_2 y_2$, c_1, c_2 constants is also a solution on I.

Def: Two functions $f_1(x)$, $f_2(x)$ are said to be linearly dependent on I if there exists constants c_1, c_2, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) = 0 \quad \forall x \in I.$$

Otherwise, they are called independent, i.e.,

$$c_1 f_1(x) + c_2 f_2(x) = 0 \implies c_1 = c_2 = 0$$

NB: f_1, f_2 are linearly dependent $\implies f_1 = c f_2$

otherwise, they are linearly independent

Ex: $f(x) = \sin x$, $g(x) = \cos x$, $x \in (0, \frac{\pi}{4})$

$$\frac{f(x)}{g(x)} = \tan x \quad \forall x \in I \implies \text{independent}$$

Ex: $f(x) = x^2$, $g(x) = 3x^2$

$$\frac{f(x)}{g(x)} = \frac{x^2}{3x^2} = \frac{1}{3} \implies \text{Dependent}$$

Def: Let y_1, y_2 be two independent solutions of ②. Then $y = c_1 y_1 + c_2 y_2$, c_1, c_2: constants is called the general solution of ②.

Theorem: (Abel's Theorem)

If y_1, y_2 are solutions of

$$y'' + p(x) y' + q(x) y = 0$$

where $p(x), q(x)$ are continuous on some open interval I, then

$$W(y_1, y_2)(x) = c\, e^{-\int p(x)\,dx}$$

This means that:

$W(y_1, y_2)(x)$ either is zero for all $x \in I$ or else is never zero in I

Def: suppose f, g are differentiable functions. Then $W(f,g)$ is called the Wronskian of f and g and it is defined by

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = f g' - g f'$$

If f, g and h are differentiable functions up to order 2, then

$$W(f,g,h) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix}$$

Theorem: Let y_1, y_2 be two solutions of the homogeneous D.E.

$$y'' + p(x)\, y' + q(x)\, y = 0 \quad \text{--- (2)}$$

on I. Then $\{y_1, y_2\}$ is linearly independent iff $W(y_1, y_2) \neq 0 \ \forall x \in I$, and $\{y_1, y_2\}$ is called a fundamental set of solutions.

Ex: If y_1, y_2 are linearly independent solutions of: $t y'' + 2y' + t e^t y = 0$ and $W(y_1, y_2)(2) = 3$, find $W(y_1, y_2)(5)$.

Sol:

$$y'' + \frac{2}{t} y' + e^t y = 0$$

$$e^{-\int p(x)\,dx} = e^{-\int \frac{2}{t}\,dt} = e^{-2\ln t} = e^{\ln t^{-2}} = t^{-2}$$

$$= \frac{1}{t^2}$$

$$\therefore W(y_1, y_2) = c\, e^{-\int p(x)\,dx}$$

$$\Longrightarrow W(y_1, y_2)(x) = \frac{c}{t^2}$$

$$W(y_1, y_2)(2) = 3$$

$$\frac{c}{4} = 3 \Longrightarrow c = 12$$

$$\therefore W(y_1, y_2)(x) = \frac{12}{t^2}$$

$$\therefore W(y_1, y_2)(5) = \frac{12}{25}$$

Ex: If $W(f,g) = 3e^{4t}$ and $f(t) = e^{2t}$ find $g(t)$

Sol:

$$W(f,g) = 3e^{4t}$$

$$f g' - g f' = 3e^{4t}$$

$$e^{2t} g' - 2e^{2t} g = 3e^{4t}$$

$$g' - 2g = 3e^{2t} \quad \text{----} \ \circledast$$

$$e^{\int p(t)\,dt} = e^{\int -2\,dt} = e^{-2t}$$

Multiply $\circledast$ by e^{-2t}

$$\therefore e^{-2t} g' - 2e^{-2t} g = 3$$

$$\therefore \left(e^{-2t} g\right)' = 3$$

$$\Rightarrow e^{-2t} g = 3t + c$$

$$\therefore g(t) = \frac{3t + c}{e^{-2t}}$$

Ex: If $W(f,g)=2$, find $W(f,f+2g)$

Sol:

$$W(f,f+2g) = f\,(f+2g)' - (f+2g)\cdot f'$$

$$= f\,(f'+2g') - ff' - 2gf'$$

$$= ff' + 2fg' - \cancel{ff'} - 2gf'$$

$$= 2\,(fg' - gf')$$

$$= 2\,W(f,g)$$

$$= 2\,(2)$$

$$= 4$$

$$F(x, y', y'') = 0 \quad : \quad y\text{-missing}$$

Let $u = y'$, $u' = y''$

$$\Rightarrow F(x, u, u') = 0$$

Ex: Solve: $y'' + \frac{2}{x} y' = \frac{1}{x^2}$

Sol: y- missed

Let $u = y'$, $u' = y''$

$$\Rightarrow u' + \frac{2}{x} u = \frac{1}{x^2} \quad \text{----} \quad (*)$$

$$e^{\int p(x)\,dx} = e^{\int \frac{2}{x} dx} = e^{2\ln x} = e^{\ln x^2} = x^2$$

Multiply $(*)$ by x^2 :

$$\Rightarrow x^2 u' + 2x u = 1$$

$$\Rightarrow (x^2 u)' = 1$$

$$\Rightarrow x^2 u = x + c_1$$

$$\Rightarrow u = \frac{x + c_1}{x^2}$$

$F(x, y', y'') = 0$: y-missing

Let $u = y'$, $u' = y''$

$\Longrightarrow F(x, u, u') = 0$

Ex: Solve: $y'' + \frac{2}{x} y' = \frac{1}{x^2}$

Sol: y- missed

Let $u = y'$, $u' = y''$

$\Longrightarrow u' + \frac{2}{x} u = \frac{1}{x^2}$ ---- ⊛

$e^{\int p(x)\,dx} = e^{\int \frac{2}{x} dx} = e^{2\ln x} = e^{\ln x^2} = x^2$

Multiply ⊛ by x^2:

$\Longrightarrow x^2 u' + 2x u = 1$

$\Longrightarrow (x^2 u)' = 1$

$\Longrightarrow x^2 u = x + c_1$

$\Longrightarrow u = \dfrac{x + c_1}{x^2}$

$$\therefore \quad \frac{dy}{dx} = \frac{x + c_1}{x^2}$$

$$\Longrightarrow \quad dy = \frac{x + c_1}{x^2}\, dx$$

$$\Longrightarrow \quad dy = \left[\frac{1}{x} + \frac{c_1}{x^2}\right] dx$$

$$\Longrightarrow \quad y = \ln|x| - \frac{c_1}{x} + c_2$$

H.W: Solve the following ODEs:

1. $x y'' + y' = 1$

2. $x^2 y'' + (y')^2 = 0$

3. $y'' + (y')^2 + 1 = 0$

$F(y, y', y'') = 0$ $\quad$ x-missed

$u = y'$, $u' = y''$

$\Longrightarrow \quad F(y, u, u') = 0$

$u' = \frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} \Longrightarrow u' = u \frac{du}{dy}$

$\therefore \ F(y, u, u') = 0$ where $u' = u \frac{du}{dy}$.

Ex: Solve: $y y'' + (y')^2 = 0$

x-missed

Let $u = y'$, $u' = y''$

$\Longrightarrow \quad y u' + u^2 = 0$

$\Longrightarrow \quad y u \frac{du}{dy} + u^2 = 0$

$\Longrightarrow \quad y u \frac{du}{dy} = -u^2$

$\Longrightarrow \quad \frac{du}{u} = -\frac{dy}{y}$

$\Longrightarrow \quad \ln|u| = -\ln|y| + c_1$

$$\therefore \quad \frac{dy}{dx} = c_1 (y+1)$$

$$\Longrightarrow \quad \frac{dy}{y+1} = c_1 \, dx$$

$$\Longrightarrow \quad \ln |y+1| = c_1 x + c_2 .$$

$$\Longrightarrow \quad y+1 = e^{c_1 x + c_2}$$

$$y+1 = c_2 e^{c_1 x}$$

$$\therefore \quad y = c_2 e^{c_1 x} - 1$$

H.W: Solve:

1. $y'' + 2y(y')^3 = 0$

2. $y'' = \dfrac{-1}{2y^2}$, $\quad y(0) = 1$, $y'(0) = -1$

$\therefore \ln|u| + \ln|y| = c_1$

$\Rightarrow \ln|uy| = c_1$

$\Rightarrow uy = e^{c_1}$ or $uy = c_1$

$\therefore y \cdot \frac{dy}{dx} = c_1 \Rightarrow y\,dy = c_1\,dx$

$\Rightarrow \frac{1}{2}y^2 = c_1 x + c_2$

Ex: Solve : $(y+1)y'' = y'^2$

Let $u = y'$, $u' = y''$

$\Rightarrow (y+1)u' = u^2$ where $u' = u\frac{du}{dy}$

$\Rightarrow (y+1)u\frac{du}{dy} = u^2$

$\Rightarrow \frac{du}{u} = \frac{dy}{y+1}$

$\Rightarrow \ln|u| = \ln|y+1| + c_1$

$\Rightarrow u = (y+1)\cdot e^{c_1}$

or $u = c_1(y+1)$

Reduction of order

Consider

$$y'' + p(x)\,y' + q(x)\,y = 0 \quad \text{---} \; ②$$

Given y_1 a solution of ②, then we can find a 2^{nd} linearly independent solution y_2 using the formula:

$$y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx$$

Ex: Given $y_1 = x$ a solution of

$$(x^2 - x)\,y'' - x\,y' + y = 0$$

find a 2^{nd} linearly independent solution y_2.

Sol:

$$y'' - \frac{x}{x^2 - x}\,y' + \frac{1}{x^2 - x}\,y = 0$$

$$\therefore\; y'' - \frac{1}{x-1}\,y' + \frac{1}{x^2 - x}\,y = 0$$

$$e^{-\int p(x)dx} = e^{-\int \frac{-1}{x-1} dx} = e^{\int \frac{1}{x-1} dx}$$

$$= e^{\ln|x-1|} = |x-1|$$

$$\therefore y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} dx$$

$$= x \int \frac{x-1}{x^2} dx$$

$$= x \int \left[\frac{1}{x} - \frac{1}{x^2} \right] dx$$

$$= x \left(\ln|x| + \frac{1}{x} \right)$$

$$\therefore y_2 = x \ln|x| + 1$$

Ex: Consider $x y'' + 2y' + xy = 0$,

Given $y_1 = \frac{\cos x}{x}$ a solution, find

a 2^{nd} - linearly independent solution.

<u>Sol:</u>

$$y'' + \frac{2}{x} y' + y = 0$$

$$e^{-\int p(x)dx} = e^{-\int \frac{2}{x}dx} = e^{-2\ln x} = x^{-2}$$

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2}\, dx$$

$$= \frac{\cos x}{x} \int \frac{x^{-2}}{\frac{\cos^2 x}{x^2}}$$

$$= \frac{\cos x}{x} \int \frac{x^{-2} \cdot x^2}{\cos^2 x}\, dx$$

$$= \frac{\cos x}{x} \int \sec^2 x\, dx$$

$$= \frac{\cos x}{x} \cdot \tan x$$

$$= \frac{\cos x}{x} \cdot \frac{\sin x}{\cos x} = \frac{\sin x}{x}$$

H.W: Consider:

$$(1-x^2)y'' - 2xy' + 2y = 0, \quad y_1 = x$$

Find a 2^{nd}-independent solution y_2.

Hint: $\frac{1}{x^2(1-x^2)} = \frac{1}{x^2} + \frac{1/2}{x+1} - \frac{1/2}{x-1}$

Homogeneous Linear ODEs with Constant Coefficients

$$y'' + ay' + by = 0 \quad \text{----} \quad (*)$$

Ex: Solve: $y' + by = 0$

$$y' = -by \Rightarrow \frac{y'}{y} = -b$$

$$\Rightarrow \ln|y| = -bx + c$$

$$\Rightarrow y = e^{-bx+c} \Rightarrow y = c\,e^{-bx}$$

Now, to solve $(*)$, we search for a solution of the form $y = e^{rx}$

Since $y' = re^{rx}$, $y'' = r^2 e^{rx}$

$$\Rightarrow r^2 e^{rx} + are^{rx} + be^{rx} = 0$$

$$\Rightarrow e^{rx}\left[r^2 + ar + b\right] = 0$$

$$\therefore \quad r^2 + ar + b = 0$$

This equation is called characteristic equation or auxiliary equation.

(iii) Distinct Real Roots

Ex: Solve $y'' + y' - 2y = 0$

$r^2 + r - 2 = 0 \Longrightarrow (r+2)(r-1) = 0$

$\Longrightarrow r = -2, 1$

$\therefore y_1 = e^{-2x}, \quad y_2 = e^{x}$

$\therefore$ The general solution is:

$y(x) = c_1 y_1 + c_2 y_2$

$y = c_1 e^{-2x} + c_2 e^{x}$

(iii) $\{ e^{-2x}, e^{x} \}$ is called a fundamental set of solutions or a Basis of solutions

Ex: Solve: $2y'' + 3y' = 0$

$2r^2 + 3r = 0 \Longrightarrow r(2r+3) = 0$

$\Longrightarrow r = 0, -\frac{3}{2}$

$\therefore y_1 = e^{0x} = 1, \quad y_2 = e^{-\frac{3}{2}x}$

Ex: Find a 2^{nd}-order linear homogeneous D.E. whose solution is: $y = c_1 e^{2x} + c_2 e^{3x}$

Sol: $r_1 = 2$, $r_2 = 3$

$\Rightarrow \quad (r-2)(r-3) = 0$

$\therefore \quad r^2 - 5r + 6 = 0$

$\therefore \quad y'' - 5y' + 6y = 0$

Real Double Roots

If the characteristic equation $r^2 + ar + b = 0$ has equal roots $r_1 = r_2 = r$, then

$y_1 = e^{rx}$, $\quad y_2 = x e^{rx}$

$\therefore$ The general solution is:

$$y = c_1 + c_2 e^{-\frac{3}{2}x}$$

Ex: Solve: $y'' + 4y' + 2y = 0$

$$r^2 + 4r + 2 = 0$$

$$b^2 - 4ac = 16 - 4(1)(2) = 8$$

$$\therefore r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-4 \pm \sqrt{8}}{2} = \frac{-4 \pm 2\sqrt{2}}{2}$$

$$\therefore r = -2 \pm \sqrt{2}$$

$$\therefore y_1 = e^{(-2-\sqrt{2})x} \quad , \quad y_2 = e^{(-2+\sqrt{2})x}$$

$\therefore$ The general solution is:

$$y = c_1 y_1 + c_2 y_2$$

$$\therefore y = c_1 e^{(-2-\sqrt{2})x} + c_2 e^{(-2+\sqrt{2})x}$$

Ex: Find a 2^{nd}-order Linear homogeneous D.E. whose general solution is: $y=(c_1+c_2x)e^{-2x}$

Sol: $y = c_1 e^{-2x} + c_2 x e^{-2x}$

$\therefore r_1=-2, \quad r_2=-2$

$\Longrightarrow (r+2)(r+2)=0$

$\therefore \quad r^2+4r+4=0$

$\therefore \quad y''+4y'+4y=0$

Ex: Solve the IVP:

$y''-9y=0, \quad y(0)=-2, \quad y'(0)=-12$

Sol: $r^2-9=0 \Longrightarrow r^2=9 \Longrightarrow r=3,-3$

$\therefore \quad y_1=e^{-3x}, \quad y_2=e^{3x}$

$\therefore y=c_1e^{-3x}+c_2e^{3x}$

$y'(x)=-3c_1e^{-3x}+3c_2e^{3x}$

Ex: Solve: $y'' - 6y' + 9y = 0$

$r^2 - 6r + 9 = 0$

$\Rightarrow (r-3)(r-3) = 0 \Rightarrow r = 3, 3$

$\therefore y_1 = e^{3x}, \quad y_2 = xe^{3x}$

$\therefore$ The general solution is:

$y = c_1 e^{3x} + c_2 x e^{3x}$

$= e^{3x}(c_1 + c_2 x)$

Ex: Solve $y'' + 4\pi y' + 4\pi^2 y = 0$

$r^2 + 4\pi r + 4\pi^2 = 0$

$\Rightarrow (r + 2\pi)(r + 2\pi) = 0$

$\Rightarrow r = -2\pi, -2\pi$

$\Rightarrow y_1 = e^{-2\pi x}, \quad y_2 = xe^{-2\pi x}$

$\therefore y = c_1 e^{-2\pi x} + c_2 x e^{-2\pi x}$

$$y(0) = -2 \implies c_1 + c_2 = -2 \quad \text{---} ①$$

$$y'(0) = -12 \implies -3c_1 + 3c_2 = -12$$

$$\therefore \; -c_1 + c_2 = -4 \quad \text{---} ②$$

$$\implies 2c_2 = -6 \implies c_2 = -3$$

$$c_1 = -2 - c_2 = -2 + 3 = 1$$

$$\therefore \; y(x) = e^{-3x} - 3e^{3x}$$

Revision : Complex Numbers

- $\sqrt{-1} = i \quad , \quad i^2 = -1$
- $\sqrt{4} = 2 \quad , \sqrt{-4} = 2i$
- $\sqrt{-7} = \sqrt{7}\, i$

- Complex Number:

The complex number is any number of the form $a + bi$, $a, b \in \mathbb{R}$.

Ex: The numbers:

$2 + 3i \quad , \quad 2 \quad , \quad 5i$

are all complex numbers

Ex: Solve $r^2 + 2r + 2 = 0$

$a = 1 \quad b = 2 \quad c = 2$

$b^2 - 4ac = 4 - 4(1)(2) = -4$

$$\because\ r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2}$$

$$= -1 \pm i$$

Complex Roots:

$$r = \lambda \pm \mu i$$

$$y_1 = e^{(\lambda+\mu i)x}, \quad y_2 = e^{(\lambda-\mu i)x}$$

Remark: From the previous two complex solutions, we can obtain the fellow two real solutions.

$$y_1 = e^{\lambda x} \cos \mu x$$

$$y_2 = e^{\lambda x} \sin(\mu x)$$

Ex: Solve: $y'' + 2y' + 5y = 0$

Sol, $r^2 + 2r + 5 = 0$

$a=1 \quad b=2 \quad c=5$

$b^2 - 4ac = 4 - 4(1)(5) = -16$

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2}$$

$$\therefore r = \underbrace{-1}_{\lambda} \pm \underbrace{2}_{\mu} i$$

$y_1 = e^{\lambda x} \cos \mu x$

$\therefore y_1 = e^{-x} \cos 2x$

$\therefore y_2 = e^{-x} \sin 2x$

$\Longrightarrow$ The general solution is given by:

$$y = c_1 e^{-x} \cos 2x + c_2 e^{-x} \sin 2x$$

$$= e^{-x} [c_1 \cos 2x + c_2 \sin 2x]$$

Ex: Solve: $y'' + 16y = 0$

$$r^2 + 16 = 0 \Longrightarrow r^2 = -16 \Longrightarrow r = \pm\sqrt{-16}$$

$$\therefore r = \pm 4i$$

$$\therefore \lambda = 0, \quad \mu = 4$$

$$y_1 = e^{\lambda x} \cos \mu x = \cos 4x$$

$$\therefore y_1 = \cos 4x$$

$$y_2 = \sin 4x$$

$\therefore$ The general solution is given by:

$$y = c_1 \cos 4x + c_2 \sin 4x$$

Remark: Let r_1, r_2 be two roots of

$$(r - r_1)(r - r_2) = 0.$$

Then $$r^2 - (r_1 + r_2) r + r_1 r_2 = 0$$

Ex: Find a 2^{nd} order linear homogeneous D.E. whose solution is:

$$y = e^{2x}\left[C_1 \cos 3x + C_2 \sin 3x\right]$$

Sol, $y = C_1 e^{2x} \cos 3x + C_2 e^{2x} \sin 3x$

$\lambda = 2, \quad \mu = 3$

The roots of the char. eqn are:

$$\lambda \pm \mu i = 2 \pm 3i$$

$\therefore \quad r_1 \longrightarrow 2 + 3i$

$r_2 \longrightarrow 2 - 3i$

$r_1 + r_2 = 4$

$r_1 \cdot r_2 = (2+3i)(2-3i) = 4-(3i)^2$

$= 4 - (3)^2(i)^2 = 4+9 = 13$

$r^2 - (r_1+r_2)r + r_1 r_2 = 0$

$r^2 - 4r + 13 = 0$

$y'' - 4y' + 13y = 0$

Euler Formula: $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$$y_1 = e^{(\lambda+\mu i)x} = e^{\lambda x} e^{i\mu x}$$

$$= e^{\lambda x}[\cos\mu x + i\sin\mu x]$$

$$\therefore y_1 = e^{\lambda x}\cos\mu x + i e^{\lambda x}\sin\mu x \quad \text{-----} (1)$$

$$y_2 = e^{(\lambda-\mu i)x} = e^{\lambda x} e^{-i\mu x}$$

$$= e^{\lambda x}[\cos\mu x - i\sin\mu x]$$

$$\therefore y_2 = e^{\lambda x}\cos\mu x - i e^{\lambda x}\sin\mu x \quad \text{----} (2)$$

$$\therefore y_1 + y_2 = 2 e^{\lambda x}\cos\mu x$$

$$\Longrightarrow \frac{1}{2}(y_1+y_2) = e^{\lambda x}\cos\mu x$$

Similarly, $\frac{1}{2i}(y_1 - y_2) = e^{\lambda x}\sin\mu x$

H.W

① If $\{e^{x}, e^{2x}\}$ is a basis of $y'' + ay' + by = 0$, find a, b.

② If $\{e^{(-1+i)x}, e^{(-1-i)x}\}$ is a basis of $y'' + ay' + by = 0$, find a, b.

Euler-Cauchy Equations

$$x^2 y'' + a x y' + b y = 0 \qquad x > 0 \quad \text{---} \circledast$$

Let us consider first $xy' + by = 0$

$$\Rightarrow xy' = -by \Rightarrow \frac{y'}{y} = \frac{-b}{x}$$

$$\ln|y| = -b \ln x + c$$

$$\Rightarrow \ln|y| = \ln x^{-b} + c$$

$$\Rightarrow |y| = e^{\ln x^{-b}} e^{c}$$

$$\Rightarrow y = c\, x^{-b}$$

So, we try a solution for $\circledast$ of the form

$$\boxed{y = x^r}$$

$$\therefore y' = r x^{r-1}, \qquad y'' = r(r-1) x^{r-2}$$

Substitute in $\circledast$:

$$x^2 \cdot r(r-1) x^{r-2} + a x \cdot r x^{r-1} + b x^r = 0$$

$$\Rightarrow \quad r(r-1)\,x^r + a r\,x^r + b\,x^r = 0$$

$$\therefore \quad r(r-1) + ar + b = 0$$

This 2nd-order algebraic equation has 3-Cases:

① Different Real Roots $r_1 \neq r_2$

$$y_1 = x^{r_1} , \quad y_2 = x^{r_2}$$

② Equal Roots $r_1 = r_2 = r$

$$y_1 = x^r , \quad y_2 = x^r \ln x$$

③ Complex Roots $\lambda \pm \mu i$

$$y_1 = x^{\lambda} \cos(\mu \ln x)$$

$$y_2 = x^{\lambda} \sin(\mu \ln x)$$

Example: Solve the following ODEs:

① $2x^2y'' + 3xy' - y = 0$

Sol

$$2r(r-1) + 3r - 1 = 0$$

$$\Longrightarrow 2r^2 - 2r + 3r - 1 = 0 \Longrightarrow 2r^2 + r - 1 = 0$$

$$\Longrightarrow (2r-1)(r+1) = 0$$

$$\Longrightarrow 2r-1 = 0, \quad r+1 = 0$$

$$\Longrightarrow r = \frac{1}{2}, \ -1$$

$$\therefore y_1 = x^{1/2} = \sqrt{x}$$

$$y_2 = x^{-1} = \frac{1}{x}$$

$\therefore$ The General solution is:

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1\sqrt{x} + \frac{c_2}{x} \qquad x > 0$$

(2) $x^2y'' - 5xy' + 9y = 0$

Sol:

$r(r-1) - 5r + 9 = 0$

$\Longrightarrow r^2 - 6r + 9 = 0 \Longrightarrow (r-3)(r-3) = 0$

$\therefore r = 3, 3$

$\therefore y_1 = x^3 , \quad y_2 = x^3 \ln x$

$\therefore$ The general solution is:

$y = c_1 x^3 + c_2 x^3 \ln x$

$= x^3 (c_1 + c_2 \ln x)$

(3) $x^2 y'' - 5xy' + 13y = 0$

$r(r-1) - 5r + 13 = 0$

$\Longrightarrow r^2 - 6r + 13 = 0$

$a = 1 \quad b = -6 \quad c = 13$

$b^2 - 4ac = 36 - 4(1)(13)$

$= 36 - 52$

$= -16$

$\Longrightarrow r^2+3r=0 \Longrightarrow r(r+3)=0$

$\therefore \quad r=0, -3$

$y_1 = x^0 = 1$

$y_2 = x^{-3} = \frac{1}{x^3}$

$\therefore$ The general solution is:

$y = c_1 + \frac{c_2}{x^3}$

Ex: If $\{x^2\cos(\ln x), x^2\sin(\ln x)\}$ is a fundamental set of solutions of

$x^2y''+axy'+by=0$, find a, b.

Sol: $\lambda=2, \mu=1$

$\therefore \quad \lambda \pm \mu i = 2 \pm i$

$r_1 \longleftrightarrow 2+i$

$r_2 \longleftrightarrow 2-i$

$\therefore r_1+r_2=4$

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{6 \pm \sqrt{-16}}{2}$$

$$\therefore r = \frac{6 \pm 4i}{2} = 3 \pm 2i$$

$\lambda = 3 \quad \mu = 2$

$$\therefore y_1 = x^{\lambda} \cos(\mu \ln x)$$

$$\therefore y_1 = x^3 \cos(2 \ln x)$$

$$y_2 = x^3 \sin(2 \ln x)$$

$\therefore$ The general solution is:

$$y = c_1 x^3 \cos(2\ln x) + c_2 x^3 \sin(2\ln x)$$

$$= x^3 \left[c_1 \cos(2\ln x) + c_2 \sin(2\ln x) \right]$$

(4) $x y'' + 4y' = 0 \quad , \quad x > 0$

Sol:
$$x^2 y'' + 4 x y' = 0$$

$$r(r-1) + 4r = 0$$

H.W

Solve the following ODEs:

① $4x^2y'' + y = 0$

② $4x^2y'' + 4xy' - y = 0$

③ $x^2y'' - 3xy' + 6y = 0$

④ $y'' = \frac{y}{x^2}$

$$r_1 r_2 = (2+i)(2-i)$$

$$= 4-(i)^2 = 4+1 = 5$$

The char. Eqn can be written as:

$$r^2 - (r_1+r_2)r + r_1 r_2 = 0$$

$$r^2 - 4r + 5 = 0$$

$$\therefore \quad r(r-1) - 3r + 5 = 0$$

$$x^2 y'' - 3xy' + 5y = 0$$

$$\therefore \quad a = -3 \quad , \quad b = 5$$

H.W: If $\{\cos(2\ln x), \sin(2\ln x)\}$ is a basis of solutions of:

$$a x^2 y'' + b x y' + c y = 0,$$

find a, b, and c.

Method of Undetermined Coefficients

The method of undetermined coefficients is used to find $y_p(x)$ for a linear D.E with Constant Coefficients:

$$a y'' + b y' + c y = r(x)$$

and $r(x)$ is one of the following

① polynomial

② e^{ax}

③ $\sin ax$ or $\cos ax$

④ product of 1, 2, 3

⑤ Linear Combination of 1, 2, 3, 4

$\therefore$ The General Solution of ⊛ is:

$$y(x) = C_1 \cos x + C_2 \sin x + 2x^2 + 2$$

$$y'(x) = -C_1 \sin x + C_2 \cos x + 4x$$

$$y(0) = 0 \Longrightarrow C_1 + 2 = 0 \Longrightarrow C_1 = -2$$

$$y'(0) = 1 \Longrightarrow C_2 = 1$$

$$\therefore y(x) = -2\cos x + \sin x + 2x^2 + 2$$

Ex: Solve: $y'' - 2y' = e^{3x}$ ------ ⊛

(I) $y'' - 2y' = 0$

$$r^2 - 2r = 0 \Longrightarrow r(r-2) = 0 \Longrightarrow r = 0, 2$$

$$y_1 = e^{0x} = 1 \qquad y_2 = e^{2x}$$

$$\therefore y_h(x) = C_1 + C_2 e^{2x}$$

(II) Let $y_p(x) = Ae^{3x}$ be a solution of ⊛

$$y_p' = 3Ae^{3x}$$

$$y_p'' = 9Ae^{3x}$$

Ex: Solve: $y'' + y = 2x^2 + 6$ $\quad y(0)=0,\ y'(0)=1$ ....(*)

(1) $y'' + y = 0$

$r^2 + 1 = 0 \Rightarrow r^2 = -1 \Rightarrow r = \pm i$

$\therefore y_1 = \cos x, \quad y_2 = \sin x$

$\therefore y_h(x) = c_1 \cos x + c_2 \sin x$

Let $y_p(x) = Ax^2 + Bx + C$ be a solution of (*)

$y_p' = 2Ax + B$

$y_p'' = 2A$

By substitution in (*) we obtain

$2A + Ax^2 + Bx + C = 2x^2 + 6$

$\Rightarrow Ax^2 + Bx + 2A + C = 2x^2 + 6$

$\Rightarrow A = 2 \qquad B = 0 \qquad 2A + C = 6$

$\Rightarrow 4 + C = 6$

$\Rightarrow C = 2$

$\therefore C = 2$

$\therefore y_p(x) = 2x^2 + 2$

By substitution in ⊛ we obtain:

$$9Ae^{3x} - 2(3Ae^{3x}) = e^{3x}$$

$$3Ae^{3x} = e^{3x} \Longrightarrow 3A = 1$$

$$\therefore A = \frac{1}{3}$$

$$\therefore y_p(x) = \frac{1}{3}e^{3x}$$

$\therefore$ The general solution of ⊛ is:

$$y = y_h + y_p$$

$$= c_1 + c_2 e^{2x} + \frac{1}{3}e^{3x}$$

Ex: Solve: $y'' + 2y' = 12\sin x$

(1) $y'' + 2y' = 0$

$$r^2 + 2r = 0 \Rightarrow r(r+2) = 0 \Rightarrow r = 0, -2$$

$$y_1 = e^{0x} = 1, \qquad y_2 = e^{-2x}$$

$$\therefore y_h(x) = c_1 + c_2 e^{-2x}$$

(iii) Let $y_{p}(x) = A\sin x + B\cos x$

$$y_p' = A\cos x - B\sin x$$

$$y_p'' = -A\sin x - B\cos x$$

By substitution in ⊛ we obtain:

$$-A\sin x - B\cos x + 2\left[A\cos x - B\sin x\right] = 12\sin x$$

$$(-A - 2B)\sin x + (2A - B)\cos x = 12\sin x$$

$$\Longrightarrow \quad -A - 2B = 12 \quad \text{----} ①$$

$$2A - B = 0 \quad \text{---} ②$$

$$2① + ② \Longrightarrow -5B = 24 \Longrightarrow B = \frac{-24}{5}$$

from ② $\quad A = \dfrac{B}{2} = \dfrac{-12}{5}$

$$\therefore\ y_p(x) = -\frac{12}{5}\sin x - \frac{24}{5}\cos x$$

$\therefore$ The General solution of ⊛ is:

$$y(x) = C_1 + C_2 e^{-2x} - \frac{12}{5}\sin x - \frac{24}{5}\cos x$$

Ex: Solve: $y'' - 2y' = 2e^{2x}$ ---- ⊛

(1) $y'' - 2y' = 0$

$r^2 - 2r = 0 \Longrightarrow r(r-2) = 0$

$\Longrightarrow r = 0, 2$

$y_1 = e^{0x} = 1, \quad y_2 = e^{2x}$

$\therefore y_h(x) = c_1 + c_2 e^{2x}$

(2) Let $y_p(x) = A e^{2x}$

$y_p' = 2Ae^{2x}$

$y_p'' = 4Ae^{2x}$

Substitute in ⊛ we obtain:

$4Ae^{2x} - 2(2Ae^{2x}) = 2e^{2x}$

$\therefore 0 = 2e^{2x}$ Contradiction!

So, what is the problem!!

Try $y_{p(x)} = Ax e^{2x}$

$$y_p' = Ax(2e^{2x}) + Ae^{2x}$$

$$\therefore\ y_p' = (2Ax + A)e^{2x}$$

$$y_p'' = (2Ax + A)(2e^{2x}) + 2Ae^{2x}$$

$$= 4Axe^{2x} + 4Ae^{2x}$$

Substitute in ⊗:

$$4Axe^{2x} + 4Ae^{2x} - 2(2Ax + A)e^{2x} = 2e^{2x}$$

$$4Axe^{2x} + 4Ae^{2x} - 4Axe^{2x} - 2Ae^{2x} = 2e^{2x}$$

$$\Rightarrow\ 2Ae^{2x} = 2e^{2x}$$

$$\therefore\ 2A = 2 \Longrightarrow A = 1$$

$$\therefore\ y_{p(x)} = xe^{2x}$$

$\therefore$ The general solution of ⊗ is:

$$y = c_1 + c_2e^{2x} + xe^{2x}$$

Ex: Give the form of $y_p(x)$ if the method of undetermined coefficients is to be used:

① $y'' - 5y' + 6y = x^2 + 2x + 4 + 2e^{-x}$

(H) $y'' - 5y' + 6y = 0$

$r^2 - 5r + 6 = 0 \Rightarrow (r-2)(r-3) = 0$

$\Rightarrow r = 2, 3$

$\therefore y_1 = e^{2x}, \quad y_2 = e^{3x}$

Try $y_p(x) = Ax^2 + Bx + C + De^{-x}$

② $y'' + 4y = 2x\sin x + 4\cos 2x$

(H) $y'' + 4y = 0$

$r^2 + 4 = 0 \Rightarrow r^2 = -4 \Rightarrow r = \pm 2i$

$y_1 = \cos 2x \qquad y_2 = \sin 2x$

$y_p(x) = (Ax + B)\sin x + (Cx + D)\cos x + xE\cos 2x + xF\sin 2x$

⑤ $y'' + 2y' = 3x e^{-2x} + x^2 + 4$

$y'' + 2y' = 0$

$r^2 + 2r = 0 \Longrightarrow r(r+2) = 0 \Longrightarrow r = 0, -2$

$\therefore y_1 = 1 \quad , \quad y_2 = e^{-2x}$

$y_{p(x)} = x(Ax + B)e^{-2x} + x(Cx^2 + Dx + E)$

⑥ $y'' - 4y' + 4y = x e^{2x} + 3e^{-2x} + 5$

$y'' - 4y' + 4y = 0$

$r^2 - 4r + 4 = 0 \Longrightarrow (r-2)(r-2) = 0$

$\therefore r = 2, 2$

$\therefore y_1 = e^{2x} \quad , \quad y_2 = x e^{2x}$

$y_{p(x)} = x^2(Ax + B)e^{2x} + Ce^{-2x} + D$

③ $y'' + 2y' + 2y = e^{x}\cos x + 2x\,e^{-x}\sin x$

(h) $y'' + 2y' + 2y = 0$

$r^2 + 2r + 2 = 0$

$a = 1 \quad b = 2 \quad c = 2$

$b^2 - 4ac = 4 - 4(1)(2) = -4$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2}$$

$$= -1 \pm i$$

$y_1 = e^{-x}\cos x, \quad y_2 = e^{-x}\sin x$

$$\therefore\ y_{p(x)} = A e^{x}\cos x + B e^{x}\sin x + x(A_2 x + A_1)\,e^{-x}\sin x + x(B_2 x + B_1)\,e^{-x}\cos x$$

⑦ $y'' + 4y = \cos^2 x$

⑧ $y'' + 4y = \sin x \cos x$

⑨ $y'' - 4y = e^{2(x+1)}$

Ex: Solve: $y'' + y = \sec x$

(I) $y'' + y = 0$

$r^2 + 1 = 0 \Rightarrow r^2 = -1 \Rightarrow r = \pm i$

$y_1 = \cos x, \quad y_2 = \sin x$

$\therefore \; y_h(x) = c_1 \cos x + c_2 \sin x$

(II) $W(\cos x, \sin x) = \cos x . \cos x - \sin x \, (-\sin x)$

$= \cos^2 x + \sin^2 x$

$= 1$

(III) $r(x) = \sec x$

$\therefore \; y_p(x) = -y_1 \int \frac{y_2}{W} r + y_2 \int \frac{y_1}{W} r$

$= -\cos x \int \sin x \sec x \, dx + \sin x \int \cos x . \sec x \, dx$

$= -\cos x \int \frac{\sin x .}{\cos x} dx + \sin x \int \cos x . \frac{1}{\cos x} dx$

$\therefore \; y_p(x) = \cos x \ln|\cos x| + x \sin x$

Variation of Parameters

Consider:

$$y'' + p(x)\,y' + q(x)\,y = r(x) \quad , \quad r(x) \neq 0 \quad --- \ (*)$$

where $p(x)$, $q(x)$ and $r(x)$ are continuous functions on some open interval I.

① Let y_1, y_2 be independent solutions of

$$y'' + p(x)\,y' + q(x)\,y = 0 \quad --- \ (2)$$

i.e., $y_h(x) = c_1 y_1 + c_2 y_2$.

② We can find $y_p(x)$ for ① as follows:

$$y_p(x) = -y_1 \int \frac{y_2}{W}\, r + y_2 \int \frac{y_1}{W}\, r$$

$\therefore$ The General Solution is :

$$y = y_h + y_p$$

$$= C_1 \cos x + C_2 \sin x + \cos x \ln|\cos x| + x \sin x$$

<u>Ex</u>: Solve: $x^2 y'' - 2x y' - 4y = 12x^{-3}$ ---- (1)

(i) $x^2 y'' - 2x y' - 4y = 0$

$$r(r-1) - 2r - 4 = 0$$

$$r^2 - 3r - 4 = 0 \Longrightarrow (r+1)(r-4) = 0$$

$$\Longrightarrow r = -1, 4$$

$\therefore$ $y_1 = x^{-1}$, $y_2 = x^4$

$\therefore$ $y_h(x) = \dfrac{c_1}{x} + c_2 x^4$

(ii) $W(\frac{1}{x}, x^4) = \frac{1}{x}(4x^3) - x^4 \cdot (\frac{-1}{x^2})$

$$= 4x^2 + x^2 = 5x^2$$

(iii) $r(x) = \dfrac{12x^{-3}}{x^2} = 12x^{-5}$

VII $y_p(x) = -y_1 \int \frac{y_2}{W} r + y_2 \int \frac{y_1}{W} r$

$$= -\frac{1}{x} \int \frac{x^4}{5x^2} \cdot 12x^{-5}\, dx + x^4 \int \frac{\frac{1}{x}}{5x^2} \cdot 12x^{-5}\, dx$$

$$= \frac{-12}{5} \cdot \frac{1}{x} \int x^{-3}\, dx + \frac{12}{5} x^4 \int x^{-8}\, dx$$

$$= \frac{-12}{5x} \cdot \frac{x^{-2}}{-2} + \frac{12}{5} x^4 \cdot \frac{x^{-7}}{-7}$$

$$= \frac{12}{10} x^{-3} - \frac{12}{35} x^{-3}$$

$$= \frac{6}{7} x^{-3}$$

VIII The General Solution is:

$$y = y_h + y_p$$

$$= \frac{c_1}{x} + c_2 x^4 + \frac{6}{7} x^{-3}$$

<u>H.W</u> Solve: $y'' + 2y' + y = 4e^{-x} \ln x$