*** Differential Equ. :-** Equation containing derivatives (differentials)

1. $y'' + y' = x$ (2nd order D.E)

2. $y''' + y - 1 = 0$ (3rd order D.E)

3. $(x^2+1)\,dx + (y+1)\,dy = 0$ (first order D.E)

4. $y''' + (y'')^2 = x^2$ (3rd order D.E)

→ $x^2 y'' + (\tan x)\,y = e^x$ (2nd order linear D.E)

→ $y'' + y y' = x$ (2nd order non linear D.E)

D.E

- $y(x)$ → ordinary: $y'' + y = 0$
- $u(x,y)$ → partial: $u_{xx} + 1 = 0$

-Ex: Verify $y = \frac{c}{x}$, $x \neq 0$ is solution of $xy' = -y$

$y' = -\frac{c}{x^2}$

$x \cdot -\frac{c}{x^2} \stackrel{?}{=} -\frac{c}{x}$

$-\frac{c}{x} = -\frac{c}{x}$ $\therefore y = \frac{c}{x}$ is a solution

→ $y = \frac{c}{x}$ is called a general solution

→ $y = \frac{1}{x}$ is a particular solution

* Soparable D.E :- هو أحل اشي بحربه

$f(x)\,dx = g(y)\,dy$

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يعني احوله □ = y

Ex :- solve the D.E

$\boxed{1}$ $xy' = -y$

$$x\frac{dy}{dx} = -y$$

$$\frac{dy}{y} = -\frac{dx}{x}$$

$$\int \frac{dy}{y} = \int -\frac{dx}{x}$$

$$\ln|y| = -\ln|x| + C$$

$$\ln|y| + \ln|x| = C$$

$$\ln|yx| = C \iff |yx| = e^{C}$$

$\boxed{2}$ $e^{2x+y}\,dx = e^{2y}\,dy$

$$e^{2x}\cdot e^{y}\,dx = e^{2y}\,dy$$

$$\int e^{2x}\,dx = \int e^{y}\,dy \rightarrow \frac{1}{2}e^{2x} = e^{y} + C$$

$\boxed{3}$ $y' = 1 + 4y^2$, $y(1) = 0$ ← هاي يعني ما بده (1,0) فيعوض C

This is problem is called Initial value problem (IVP)

$$\frac{dy}{dx} = 1 + 4y^2$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

$$\int \frac{dy}{1+4y^2} = \int \frac{1}{dx}$$

$$\int \frac{dy}{4\left(\frac{1}{4}+y^2\right)} = \int dx$$

$$\int \frac{dy}{\left(\frac{1}{2}\right)^2+y^2} = \int 4\,dx$$

$$2\tan^{-1} 2y = 4x + c$$

$$2\tan^{-1} 0 = 4 + c$$

$$c = -4$$

$$\tan^{-1} 2y = 8x - 8$$

$$2y = \tan(8x-8)$$

$$y = \frac{\tan(8x-8)}{2}$$

$\boxed{4}$ $(\underbrace{xy+x}+y+1)\,dx = (y^2-1)\,dy$

$(x(y+1)+(y+1))\,dx = (y^2-1)\,dy$

$(y+1)(x+1)\,dx = (y^2-1)\,dy$

$$\int (x+1)\,dx = \int \frac{(y-1)\cancel{(y+1)}}{\cancel{(y+1)}}\,dy$$

* Reduction to separable D.E

$$y' = f\left(\frac{y}{x}\right)$$

كيف بحلها :-

$\boxed{1}$ let $\frac{y}{x} = u$

$y = xu$ — بشتقها

$\boxed{2}$ $y' = xu' + u$

D.E اذا كان كلهم نفس — كيف اعرف اني احل كليها :-

Ex 8- Solve

$\boxed{1}$ $(x^2 e^{\frac{2y}{x}} + xy)\, dx = x^2\, dy$

$$\frac{dy}{dx} = e^{\frac{2y}{x}} + \frac{y}{x}$$

$$y' = e^{2(\frac{y}{x})} + \frac{y}{x} \qquad \text{let} \to \frac{y}{x} = u$$

$$x u' + \cancel{u} = e^{2u} + \cancel{u} \qquad y' = x u' + u$$

$$x \frac{du}{dx} = e^{2u}$$

$$\frac{du}{e^{2u}} = \frac{dx}{x}$$

$$\int e^{-2u}\, du = \int \frac{dx}{x}$$

$$-\frac{1}{2} e^{-2u} = \ln|x| + c \Rightarrow -\frac{1}{2} e^{-2\frac{y}{x}} = \ln|x| + c$$

$$\boxed{2} \quad \frac{dy}{dx} = \frac{\overset{\text{2-D.E}}{3y^2} - \overset{\text{2-D.E}}{x^2}}{\underset{\text{2-D.E}}{2xy}}$$

$$\frac{dy}{dx} = \frac{3y}{2x} - \frac{1}{\frac{y}{x}}$$

let → $\frac{y}{x} = u$

$y' = xu' + u$

$$x\frac{du}{dx} + u = \frac{3}{2}u - \frac{1}{u}$$

$$x\frac{du}{dx} = \frac{1}{2}u - \frac{1}{u}$$

$$\frac{du}{\frac{1}{2}u - \frac{1}{u}} = \frac{dx}{x}$$

$$\frac{du}{\frac{1}{u}\left(\frac{1}{2}u^2 - 1\right)} = \frac{dx}{x}$$

$$\int \frac{u}{\frac{1}{2}u^2 - 1}\,du = \int \frac{dx}{x} \rightarrow \ln\left|\frac{1}{2}u^2 - 1\right| = \ln|x| + c$$

$$\ln\left|\frac{1}{2}\frac{y}{x} - 1\right| = \ln|x| + c$$

* Exact D.E :-

$$M(x,y)\,.dx + N(x,y)\,dy = 0$$

$$f(x,y) = x^2 + xy + 5 + y^2 \rightarrow \frac{df}{dx} = 2x + y$$

$$\int 2xy\,.dx = yx^2 + f(y)$$

$$\frac{dM}{dy} = \frac{dN}{dx} \Rightarrow \text{Exact}$$

* The solution is $\boxed{u(x,y) = c}$

$$\rightarrow \frac{du}{dx} = M$$

$$\int \frac{du}{dx}\,.dx = \int M\,dx$$

$$u(x,y) = \int M(x,y)\,dx + k(y)$$

$$\rightarrow \frac{du}{dy} = N$$

$$\int \frac{du}{dy}\, dy = \int N\, dy$$

$$u = \int N dy + K(x)$$

Ex:- solve

$$\boxed{1}\quad \underbrace{\sin(x+y)}_{M}\, dx + \underbrace{\left[2y + 3y^2 + \sin(x+y)\right]}_{N} dy = 0$$

$$\frac{dM}{dy} = \cos(x+y)$$

$$\boxed{\frac{dM}{dy} = \frac{dN}{dx}} \Rightarrow \text{Exact}$$

$$\frac{dN}{dx} = 0 + 0 + \cos(x+y)$$

The solution is $u(x,y) = C$

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$u(x,y) = \int M(x,y)\,dx + k(y)$

$u(x,y) = \int \sin(x+y)\,dx + k(y)$

$\boxed{u(x,y) = -\cos(x+y) + k(y)}$

هسا مشكلة k(y) لازم
اجد قيمتها شو رح
اعمل بشتق بالنسبة ل y

Diff. w. to y

$\frac{du}{dy} = \sin(x+y) + k'(y)$

$2y+3y^2+\cancel{\sin(x+y)} = \cancel{\sin(x+y)} + k'(y)$

$\int k'(y) = \int 2y+3y^2$

$k(y) = y^2 + y^3$

The solution is $u(x,y) = c$

$-\cos(x+y) + y^2 + y^3 = c$

$\boxed{2}$ $\underbrace{[1+\sin x \tan y]}_{M}\,dx\ \underbrace{-\cos x \sec^2 y}_{N}\,dy = 0$

$$\left.\begin{aligned} \frac{dM}{dy} &= \sec^2 y \sin x \\ \frac{dN}{dx} &= +\sin x \sec^2 y \end{aligned}\right\} \Longrightarrow \text{Exact}$$

The solution is $u(x,y) = C$

$$u(x,y) = \int M\,dx + k(y)$$

$$u(x,y) = \int 1+\sin x \tan y \cdot dx + k(y)$$

$$\boxed{u(x,y) = x + -\cos x \tan y + k(y)}$$

$$\frac{du}{dy} = -\sec^2 y \cos x + k'(y)$$

$$\cancel{-\cos x \sec^2 y} = \cancel{-\sec^2 y \cos x} + k'(y)$$

$$\int k'(y) = \int 0 \longrightarrow k(y) = 0$$

لو كان $y(0)=\pi$

هاي يعني $(0,\pi)$ لازم اجد قيمة C

The solution is $u(x,y) = C$

** $\boxed{x + -\cos x \tan y = C}$

* Integrating Factor :-

$M(x,y)\,.dx + N(x,y)\,.dy = 0$

→ suppore $\frac{dM}{dy} \neq \frac{dN}{dx}$

- $\mu(x)$:-

$$\frac{\frac{dM}{dy} - \frac{dN}{dx}}{N} = R(x)$$

مالازم يكون في y

- $\mu(y)$:

ما لازم يكون في x

$$\frac{\frac{dM}{dy} - \frac{dN}{dx}}{-M} = R(y)$$

$\mu(x) = e^{\int R(x)\,.dx}$

$\mu(y) = e^{\int R(y)\,.dy}$

Then :-

$\mu\, M(x,y)\,dx + \mu\, N(x,y)\,dy = 0$ is Exact

* EX : solve

$\boxed{1}$ $\underbrace{(e^{x+y} + y\,e^{y})}_{M}\,dx + \underbrace{(x e^{y} - 1)}_{N}\,dy = 0$ $\quad y(0) = -$

$$\frac{dM}{dy} = e^{x+y} + ye^{y} + e^{y}$$

$\therefore$ not Exact

$$\frac{dN}{dx} = e^{y}$$

$$\frac{dM}{dy} - \frac{dN}{dx} = e^{x+y} + ye^{y}$$

$$\frac{\frac{dM}{dy} - \frac{dN}{dx}}{-M} = \frac{\cancel{e^{x+y} + ye^{y}}}{-(\cancel{e^{x+y} + ye^{y}})} = -1 = R(y)$$

$$\mu(y) = e^{\int R(y)\,dy} = e^{\int -1\,dy} = e^{-y}$$

→ Multiply the D.E. by e^{-y}

$$(e^{-y} e^{x+y} + ye^{y} e^{-y})\,dx + (xe^{y} e^{-y} - e^{-y})\,dy = 0$$

$$\underbrace{(e^{x} + y)}_{M}\,dx + \underbrace{(x - e^{-y})}_{N}\,dy = 0$$

$\frac{dM}{dy} = 1$ $\qquad$ $\frac{dN}{dx} = 1.$ $\quad$ ∴ Exact

The solution is $u(x,y) = C$

$u = \int M(x,y)\,dx + k(y)$

$u = \int e^x + y \cdot dx + k(y)$

$u = e^x + yx + k(y)$

$\frac{du}{dy} = x + k'(y)$

$\cancel{x} - e^{-y} = \cancel{x} + k'(y) \Rightarrow k(y) = e^{-y}$

The solution is :-

$u(x,y) = C$

$e^x + yx + e^{-y} = C \rightarrow e^x + yx + e^{-y} = e + 1$

$e^0 + -1 \times 0 + e^1 = C$

$\boxed{e + 1 = C}$

$\boxed{2}$ $\underbrace{(x^2+y^2)}_{M}dx \underbrace{-2xy}_{N} dy = 0$

$\frac{dM}{dy} = 2y \qquad \frac{dN}{dx} = -2y \qquad \therefore$ not Exact

$\frac{dM}{dy} - \frac{dN}{dx} = 4y$

$\frac{\frac{dM}{dy} - \frac{dN}{dx}}{N} = \frac{4y}{-2xy} = \frac{-2}{x} = R(x)$

$M(x) = e^{\int R(x)\,dx} = e^{-2\ln|x|} = x^{-2}$

$\Rightarrow$ Multiply by x^{-2}

$\underbrace{1 + \frac{y^2}{x^2}}_{M} dx + \underbrace{\frac{-2y}{x}}_{N} dy = 0$

$\frac{dM}{dy} = \frac{2y}{x^2} \qquad \frac{dN}{dx} = +\frac{2y}{x^2} \qquad \therefore$ Exact

$\therefore$

* First-order linear O.D.E :-

$y' + P(x)\, y = r(x)$ ⇐ الشكل العام إلها

Evalute $e^{\int P(x).dx}$

$$\boxed{e^{\int P(x).dx} \cdot y = \int e^{\int P(x).dx} . r(x)\, dx + c}$$ ← solution

$xy' = 2y + x^2 e^x$

$xy' - 2y = x^2 e^x$

$y' - \frac{2}{x} y = x e^x$

Ex :- Solve

$\boxed{1}$ $y' + \tan x \; y = \sin 2x$; $y(0) = 1$

$$e^{\int P(x) \cdot dx} = e^{\int \tan x \cdot dx} = e^{\int \frac{\sin x}{\cos x} \cdot dx} = e^{-\ln|\cos x|} = \sec x$$

$$(\sec x) \cdot y = \int \sec x \cdot \sin 2x \cdot dx + C$$

$$(\sec x) \cdot y = \int \frac{1}{\cancel{\cos x}} \cdot \sin x \; \cancel{\cos x} \cdot dx + C$$

$$y \sec x = -2 \cos x + C$$

$\boxed{\text{Set}}$ $x = 0$, $y = 1$

$$(\sec 0) \cdot 1 = -2 \cos 0 + c$$

$$\left(\frac{1}{\cos 0}\right) \cdot 1 = -2 + c \Rightarrow c = 3$$

جواب نهائي $\Rightarrow$ $(\sec x) y = -2 \cos x + 3$

$\boxed{2}$ $xy' - 2y = x^3 e^x$

Divid by x $\rightarrow$ $y' - \frac{2}{x} y = x^2 e^x$

$e^{\int p(x)\cdot dx} = e^{\int -\frac{2}{x} \cdot dx} = e^{-2\ln|x|} = \cancel{e}^{\ln(x)^{-2}} = \frac{1}{x^2}$

$e^{\int p(x)\, dx} \cdot y = \int e^{\int p(x)\cdot dx} \cdot r(x) \cdot dx + c$

$x^{-2} \cdot y = \int x^{-2}\, x^2 e^x \cdot dx + c$

$x^{-2} y = e^x + c$

$\boxed{3}$ $y\,dx = (2x + y^3 e^y)\, dy$

مشان هيك فكرنا

linear of x

$y \frac{dx}{dy} = 2x + y^3 e^y$

$x' = \frac{2}{y} x + y^2 e^y$

$\underbrace{x' - \frac{2}{y}}_{p(y)} x = \underbrace{y^2 e^y}_{r(y)}$ $\Leftarrow$ First order linear D.E

بدل $y \Leftarrow x$

$xy' - 2y = x^3 e^x$

$x\frac{dx}{dy} - 2y = x^3 e^x$

لو

$x\,dx = (x^3 e^x + 2y)\, dy$

$(2y + x^3 e^x)\, dy - x\,dx = 0$

$$e^{\int p(x)\cdot dx} = e^{\int \frac{2}{y} dy} = e^{\ln y^{-2}} = y^{-2}.$$

$$e^{\int p(x)\, dy} \cdot x = \int e^{\int P(x)\, dy} \cdot r(y)\, dy + c$$

$$y^{-2} x = \int e^{x} \cdot dy + c$$

$$\boxed{\frac{x}{y^2} = e^{x} + c}$$

* Bernoli Equ. 8-

$$y' + p(x) y = q(x)\, y^{n} \quad \text{—} (*)$$

$, n = 0, 1 \rightarrow$ يكون liner

$n \neq 0, 1 \rightarrow$ يكون مش liner

The equation nun linear for $n \neq 0, 1$ whic is called Bernuli Equ.

$\rightarrow$ let $u = y^{1-n}(x)$

$$u' = (1-n)\, y^{-n} \cdot y'$$

Multiply (*) by $(1-n)\, y^{-n}$

$$(1-n)\, y^{-n} y' + (1-n)\, p(x)\, y^{1-n} = (1-n)\, q(x)$$

$$u' + (1-n)\, p(x)\, u = r(x) \rightarrow$$

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$$u' + P(x)\,u = r(x)$$

linear $\leftarrow$ u بالنسبة لـ

$$e^{\int P(x)\,dx} \cdot u = \int e^{\int P(x)\,dx} \cdot r(x)\,dx + c$$

EX: solve

1. $x^2 y' + 2xy = y^3$

$$y' + \frac{2}{x}y = \frac{1}{x^2}y^3$$

$$u = y^{1-n} \Rightarrow \boxed{u = y^{-2}}$$

then $\rightarrow$ $u' + (1-n)\,P(x)\,u = (1-n)\,q(x)$

$$u' + \underbrace{-2 \cdot \frac{2}{x}}_{P(x)}\,u = \underbrace{\frac{-2}{x^2}}_{r(x)}$$

First order linear

$$e^{\int P(x)\,dx} = e^{\int \frac{-4}{x}\,dx} = e^{-4\ln|x|} = e^{\ln(x)^{-4}} = x^{-4}$$

$$\Rightarrow x^{-4} \cdot u = \int x^{-4} \times \frac{-2}{x^2}\,dx + c$$

$x^{-4} \cdot u = \frac{-2}{-5} x^{-5} + C$

$x^{-4} \cdot y^{-2} = \frac{2}{5} x^{-5} + C$

$\boxed{2}$ $y' = \frac{1}{6e^{y} - 2x}$ مقلوبها !!

$x' = 6e^{y} - 2x$

$x' + \underbrace{2}_{P(y)}x = \underbrace{6e^{y}}_{R(y)}$ first order linear

Bernoulli وليس

$e^{\int P(y) \cdot dy} = e^{\int 2 \cdot dy} = e^{2y}$

$\rightarrow e^{2y} \cdot x = \int e^{2y} \cdot 6e^{y} \cdot dy + C$

$e^{2y} \cdot x = \int e^{3y} \cdot 6 \; dy + C$

$e^{2y} \cdot x = \frac{6}{3} e^{3y} + C$

$\boxed{3}$ $x\,dx + (x^{2}y - y)\,dy = 0$

$\boxed{A}$ $x\,dx = -(x^{2}y - y)\,dy$

$x\,dx = -y(x^{2} - 1)\,dy$

$\int \frac{x}{(x^{2}-1)} \cdot dx = \int -y\,dy$ soprable

$-\frac{1}{2} \ln|1 - x^{2}| = \frac{y^{2}}{2} + C$

حل آخر $\boxed{B}$

$$\frac{dx}{dy} = \frac{y - x^2 y}{x}$$

$$x' = \frac{y}{x} - xy$$

$$x' + \underbrace{y}_{p(x)} x = \underbrace{y}_{q(x)} x^{\boxed{-1}\, n}$$

let → $u = x^{1-n} \to u = x^2$

then $u' + (1-n)\,p(y)\,u = (1-n)\,q(y)$

$$u' + \underbrace{(2\cdot y)}_{p(y)} u = \underbrace{2\cdot y}_{r(y)}$$

$$e^{\int p(y)\cdot dy} = e^{\int 2\cdot y\cdot dy} = e^{y^2}$$

$\rightarrow e^{y^2}\cdot u = \int e^{y^2}\cdot 2y \cdot dy$

$e^{y^2}\cdot u = e^{y^2} + c$

$$\boxed{e^{y^2}\cdot x^2 = e^{y^2} + c}$$

$\boxed{4}$ $2xyy' + (x-1)y^2 = x^2 e^x$ devid by $2xy$

$$y' + \frac{x-1}{2x} y = \frac{x e^x}{2y}$$

$$y' + \underbrace{\frac{1}{2}\left(1 - \frac{1}{x}\right)}_{p(x)} y = \underbrace{\frac{1}{2} x e^x}_{q(x)} y^{\boxed{-1}\, n}$$

⋮

كمل الحل

Bernoulli Revition

Ex : solve

$\boxed{1}$ $2\sin y \cos x \, dx + \cos y \sin x \, dy = 0$

$2\sin y \cos x \, dx = -\cos y \sin x \, dy$

$\int 2 \frac{\cos x}{\sin x} dx = \int - \frac{\cos y}{\sin y} . dy$

$2 \ln|\sin x| = - \ln|\sin y| + c$

$\ln(\sin x)^2 + \ln(\sin y) = c$

$\ln(\sin^2 x . \sin y) = c$

$\sin^2 x \sin y = e^c$ #

$\boxed{2}$ $\underbrace{(\cos 2y - \sin x)}_{M} dx - \underbrace{2 \tan x \sin 2y}_{N} dy = 0$

$\frac{dM}{dy} = 2\sin 2y$ $\qquad \frac{dN}{dx} = -2\sec^2 x \sin 2y$ $\qquad$ ∴ Not exact

$\frac{dM}{dy} - \frac{dN}{dx} = -2\sin 2y + 2\sec^2 x \sin 2y$

$= 2\sin 2y (\sec^2 x - 1)$

$= 2\sin 2y \tan^2 x$

$1 + \tan^2 x = \sec^2 x$

$\sec^2 x - 1 = \tan^2 x$

$$\frac{\frac{dM}{dy} - \frac{dN}{dx}}{N} = \frac{2\sin 2y \tan^2 x}{-2 \tan x \sin 2y}$$

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$$\boxed{P(x) = -\tan x}$$

$$M(x) = e^{\int P(x)\,dx} = e^{\int -\frac{\sin x}{\cos x}\,dx} = e^{\ln|\cos x|} = \cos x$$

$\boxed{3}$ $$\frac{dy}{dx} = \frac{2y + \sqrt{x^2 - y^2}}{2x}$$ الـ degree متساوية

$$\frac{dy}{dx} = \frac{2y}{2x} + \frac{\sqrt{x^2-y^2}}{2\sqrt{x^2}}$$

$$\frac{dy}{dx} = \frac{y}{x} + \frac{1}{2}\sqrt{\frac{x^2-y^2}{x^2}}$$

$$\frac{dy}{dx} = \frac{y}{x} + \frac{1}{2}\sqrt{1-\left(\frac{y}{x}\right)^2}$$

Let → $u = \frac{y}{x}$

أكمل

$\boxed{4}$ $$(x^2y + xy - y)\,dx + (x^2y - 2x^2)\,dy = 0$$

$$y(x^2 + x - 1)\,dx + x^2(y-2)\,dy = 0$$

$$y(x^2 + x - 1)\,dx = -x^2(y-2)\,dy$$

$$\int \frac{(x^2+x-1)}{x^2}\,dx = \int \frac{2-y}{y}\,dy$$

$$\int 1 + \frac{1}{x} - \frac{1}{x^2}\,dx = \int \frac{2}{y} - 1\,dy$$

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* بعد التعديل كده السؤال كمثال *

$$(xy+y^2)dx+(xy-2x^2)dy=0$$

$$(xy+y^2)\,dx=(2x^2-xy)dy$$

$$\frac{dy}{d}=\frac{xy+y^2}{2x^2-xy}$$

$$\frac{dy}{dx}=$$

* Second-order ODE :-

consider :-

$$y'' + p(x)\, y' + q(x)\, y = 0 \quad \text{—} (*)$$

This equ. is a 2^{nd}-order linear homgencous D.E

→ If y_1, y_2 are linearly independent solutions of (*) then $c_1 y_1, c_2 y_2$ &

$c_1 y_1 + c_2 y_2$ are solution of (*)

هاد كلو صحيح اذا كانت معادلة

homgencous بس

dependent :-

$$\frac{y_1}{y_2} = C$$

يعني لو عرفتي وحدة منهم

بتعرفي ثانية

* Reduction of order :-

$$F(x, y', y'') = 0 \qquad y\text{-missed}$$

let → $u = y'$

$u' = y''$

Ex :- Solve : $x^2 y'' + 2x y' - 1 = 0$ y-missed

let → $u = y'$

$u' = y''$

By substitution :-

$$x^2 u' + 2xu = 1$$

$$u' + \underbrace{\frac{2}{x}}_{P(x)} u = \underbrace{\frac{1}{x^2}}_{r(x)}$$

$$e^{\int P(x)\cdot dx} \cdot u = \int e^{\int P(x)\cdot dx} \cdot r(x) \cdot dx + c_1$$

$$e^{\int \frac{2}{x}\cdot dx} \cdot u = \int e^{\int \frac{2}{x}\cdot dx} \cdot \frac{1}{x^2} \cdot dx + c_1$$

$$e^{\ln x^2} \cdot u = \int e^{\ln x^2} \cdot \frac{1}{x^2} \cdot dx + c_1$$

$$x^2 \cdot u = \int 1 \cdot dx + c_1$$

$$x^2 \cdot u = x + c_1$$

$$u = \frac{x + c_1}{x^2}$$

$$\frac{dy}{dx} = \frac{1}{x} + \frac{c_1}{x^2}$$

$$\int dy = \int \left(\frac{1}{x} + \frac{c_1}{x^2}\right) \cdot dx$$

$$y = \ln|x| - \frac{c_1}{x} + c_2$$

H.W:- solve $xy'' + y' = 1$

$\Rightarrow F(y, y'', y') = 0$ x-missed

let $\rightarrow$ $u = y'$

$u' = y''$ $\Rightarrow F(y, u, u')$ $\frac{du}{dx}, y, u$ هسا هون في ٣ متغيرات

$$u' = \frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}$$
$$= u \cdot \frac{du}{dy}$$

Ex: solve :- $y y'' + (y')^2 = 0$ x-missed

let $\rightarrow$ $u = y'$

$u' = y''$

$u' = u \cdot \frac{du}{dy}$ $\rightarrow y u' + u^2 = 0$

$$y \cdot u \cdot \frac{du}{dy} + u^2 = 0$$
$$y \cdot u \cdot \frac{du}{dy} = -u^2$$
$$\int \frac{1}{u}\, du = \int -\frac{1}{y}\, dy$$
$$\ln|u| = -\ln|y| + c_1$$
$$\ln|u| + \ln|y| = c_1$$
$$\ln|uy| = c_1$$
$$uy = \underbrace{e^{c_1}}$$

عدد ثابت

$$y \frac{dy}{dx} = c_1$$
$$\int y\, dy = \int c_1 \cdot dx$$
$$\frac{1}{2} y^2 = c_1 x + c_2$$

* Consider

$$\boxed{1}\quad y'' + p(x)y' + q(x)y = 0 \quad \text{—— } (*)$$

Given y_1 we can find y_2 as:-

$$y_2 = y_1 \int \frac{e^{-\int p(x)\cdot dx}}{y_1^2} \cdot dx$$

Ex:- Find the general solution of

$$(x^2 - x)y'' - xy' + y = 0 \qquad \boxed{y_1 = x}$$

$$y'' - \underbrace{\frac{x}{x^2 - x}}_{P(x)} y' + \frac{1}{x^2 - x} y = 0$$

$$-\int p(x)\cdot dx = -\int \frac{-x}{x^2 - x} dx = -\int \frac{-\cancel{x}}{\cancel{x}(x-1)} dx$$

$$= +\ln|x-1|$$

$$e^{-\int p(x)\cdot dx} = e^{\ln|x-1|} = |x-1|$$

$$y_2 = y_1 \int \frac{e^{-\int p(x)\cdot dx}}{y_1^2} dx$$

$$= x \int \frac{x-1}{x^2} dx$$

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$$y_2 = x \int \frac{1}{x} - \frac{1}{x^2} \, dx$$

$$= x \left(\ln|x| + \frac{1}{x} \right) \qquad = x\ln|x| + 1$$

→ The general solution $y = c_1 y_1 + c_2 y_2$

$$= c_1 x + c_2 (x\ln|x| + 1)$$

* 2^{nd}-order linear Homogeneous O.D.E :-
with constant coefficient

$$a y'' + b y' + c y = 0 \quad —— ⊛$$

The solution of equ ⊛ is given by $\boxed{y(x) = e^{rx}}$

characteristic equ :: $a r^2 + b r + c = 0$ معادلة تربيعية

هاد عمين $b^2 - 4ac$

اذا كانا جوابه موجب بكون في حلين مختلفين

اذا كان جواب صفر بكون في حلين متشابهين

اذا كان الجواب سالب زمان بكون ما في حل موجب الها حل

Ex: Solve :-

Case I (Distinct routs)

$\boxed{1}$ $y'' - 5y' + 6y = 0$

char. equ. $\rightarrow$ $r^2 - 5r + 6 = 0$

$(r-2)(r-3) = 0$

$r = 2, 3$

$\therefore y = e^{2x}, \; y = e^{3x}$

The general solution $y(x) = c_1 e^{2x} + c_2 e^{3x}$

Basic of solution :- $\{e^{2x}, e^{3x}\}$

$\boxed{2}$ $4y'' - 8y' = 0$

char. equ $\rightarrow$ $4r^2 - 8r = 0$

$r^2 - 2r = 0$

$r(r-2) = 0$

$r = 0, 2$

$\therefore y_1 = e^{0x} = 1 \qquad y_2 = e^{2x}$

The general solution :- $y(x) = c_1 + c_2 e^{2x}$

Case II : (Equal Routs)

$\rightarrow r_1 = r_2 = r$

$y_1 = e^{r_1 x} = e^{rx}$

$y_2 = e^{r_2 x} = e^{rx}$

مابنفع حل لانو صح طالع y_2

من الـ نوت ازا $y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2}$

Ex:- Solve: $2y'' + 8y' + 8y = 0$

char. equ. $\rightarrow 2r^2 + 8r + 8 = 0$

$$r^2 + 4r + 4 = 0$$

$$(r+2)(r+2) = 0$$

$$\therefore r = -2, -2$$

$\therefore y_1 = e^{-2x}$ $\qquad y_2 = x e^{-2x}$

The general solution: $y(x) = c_1 e^{-2x} + c_2 x e^{-2x}$

⇒ Complex

$$\boxed{\sqrt{-1} = i}$$

$\sqrt{4} = 2$ $\qquad \sqrt{-4} = \sqrt{4} \cdot \sqrt{-1} = 2i$

$\sqrt{-9} = 3i$

$a=1 \quad b=-2 \quad c=2$

$r^2-2r+2=0 \qquad b^2-4ac = 4-4(1)(2) = -4$

$$r = \frac{-b \mp \sqrt{b^2-4ac}}{2a}$$

$$= \frac{2 \mp \sqrt{-4}}{2}$$

$$= \frac{2 \mp 2i}{2}$$

$$= \boxed{\underbrace{1}_{\lambda} \mp \underbrace{1}_{\mu} i}$$

بتاعة λ μ بتاعة

Case III : complex Roots

$\lambda \pm \mu i$

$y_1 = e^{(\lambda+\mu i)x} \rightarrow y_1 = e^{\lambda x} \cos \mu x$

$y_2 = e^{(\lambda-\mu i)x} \rightarrow y_2 = e^{\lambda x} \sin \mu x$

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Ex:- solve :-

$$y'' - 2y' + 3y = 0$$

$$r^2 - 2r + 3 = 0$$

$$b^2 - 4ac = 4 - 4(1)(3) = -8 < 0$$

$$r = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{+2 \mp \sqrt{-8}}{2}$$

$$= \frac{2 \mp \sqrt{8}\, i}{2}$$

$$= \frac{2 \mp 2\sqrt{2}\, i}{2}$$

$$= \underbrace{1}_{\lambda} \mp \underbrace{\sqrt{2}}_{\mu} i$$

$$y_1 = e^{\lambda x} \cos \mu x$$
$$= e^{x} \cos(\sqrt{2}x)$$

$$y_2 = e^{\lambda x} \sin \mu x$$
$$= e^{x} \sin(\sqrt{2}x)$$

$\{e^{x}\cos(\sqrt{2}x),\ e^{x}\sin(\sqrt{2}x)\}$ → basis of solution

Ex:- Find the O.D.E with solution $y(x) = c_1 e^{2x} + c_2 e^{-x}$:-

$\rightarrow r_1 = 2 \quad , \quad r_2 = -1$

$(r-2)(r+1) = 0$

$r^2 - r - 2(1) = 0$

$y'' - y' - 2y = 0$

$r^2 - (r_1 + r_2) r + (r_1 r_2) = 0$

Ex:- Find a 2^{nd} order homogeneous ODE with basis of solution $\{e^{2x}, xe^{2x}\}$:-

$y(x) = e^{2x}[c_1 + x c_2]$

$r_1 = 2 \qquad r_2 = 2$

$\rightarrow (r-2)(r-2) = 0$

$r^2 - 4r + 4(1) = 0$

$y'' - 4y' + 4y = 0$

Ex:- Find a 2^{nd} order linear homogeneous ODE with basic of solution $\{e^{(2-i)x}, e^{(2+i)x}\}$:- $\{e^{2x}\cos x, e^{2x}\sin x\}$

$r = \lambda \pm \mu i = 2 \pm i$

char. eqn. :→ $r^2 - (r_1+r_2)r - (r_1 r_2) = 0$

$r_1+r_2 = 2+i+2-i = 4$

$r_1 r_2 = (2+i)(2-i) = 4 - i^2 = 5$

$$r^2 - 4r - 5 = 0$$

$$y'' - 4y' + 5y = 0$$

* Euler - Cauchy D.E :-

$$a \underbrace{x^2 y''}_{r(r-1)} + b \underbrace{x y'}_{r} + c \underbrace{y}_{1} = 0$$

$$a r(r-1) + b r + c = 0$$

$\boxed{x^r}$

$r_1 \neq r_2$ real

$y_1 = x^{r_1}$

$y_2 = x^{r_2}$

$r_1 = r_2 = r$

$y_1 = x^r$

$y_2 = (\ln x) x^r$

$r = \lambda \mp \mu i$

$y_1 = x^{\lambda} \cos(\mu$

$y_2 = x^{\lambda} \sin(\mu$

Ex:- Solve :-

1. $2x^2y'' + 3xy' - y = 0$

$2r(r-1) + 3r - 1 = 0$

$2r^2 - 2r + 3r - 1 = 0$

$2r^2 + 1r - 1 = 0$

$(2r-1)(r+1) = 0 \rightarrow 2r-1=0$ or $r+1=0$

$-r$, $+2r$

$r = \frac{1}{2}$, $r = -1$

$y_1 = x^{\frac{1}{2}}$

$y_2 = x^{-1}$

$$\boxed{y(x) = c_1\sqrt{x} + \frac{c_2}{x}}$$

2. $x^2y'' - 3xy' + 4y = 0$

$r(r-1) - 3r + 4 = 0$

$r^2 - 4r + 4 = 0$

$(r-2)(r-2) = 0$

$r = 2, 2 \rightarrow y_1 = x^2$, $y_2 = (\ln x)x^2$

$\rightarrow y(x) = c_1x^2 + c_2(\ln x)x^2$

$\boxed{3}$ $x^2 y'' + 7xy' + 13y = 0$

$r(r-1) + 7r + 13 = 0$

$r^2 + 6r + 13 = 0$

$b^2 - 4ac = 36 - 4 \times 13 = -16 < 0$

$r = \frac{-b \mp \sqrt{b^2-4ac}}{2a}$

$r = \frac{-6 \mp \sqrt{-16}}{2}$

$= \underbrace{-3}_{\lambda} \mp \underbrace{2}_{\mu} i$

$y_1 = x^{-3} \cos(2 \ln x)$

$y(x) = c_1 x^{-3} \cos(2\ln x) + c_2 x^{-3} \sin(2\ln x)$

$y_2 = x^{-3} \sin(2 \ln x)$

$\boxed{4}$ $xy'' + 2y' = 0$

Multiply by x

$x^2 y'' + 2xy' = 0$

$r(r-1) + 2r = 0$

$r^2 + r = 0$

$r(r+1) = 0$ $\quad r = 0, -1$

$y_1 = x^0 = 1$

$y_2 = x^{-1} = \frac{1}{x}$

$y(x) = c_1 (1) + \frac{c_2}{x}$

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* Undetermined Coefficients :-

لازم تكون $r(x)$ $\boxed{1}$ polyminal $x, x^2, \ldots$
$\boxed{2}$ e^x
$\boxed{3}$ sin or cos لا بس x^{-1}

$$y'' + p(x)y' + q(x)\,y = r(x)$$

$\boxed{1}$ $y_h = c_1 y_1 + c_2 y_2$: homogeneous solution

$\boxed{2}$ $y_p(x)$: particular solution — يكون شكله صورة عامة لمعادلة من درجة $r(x)$

$$y(x) = y_h + y_p$$

Ex :- solve $y'' - 4y = 2x^2 + 4$ — ⊛

1- $y'' - 4y = 0$

$r^2 - 4 = 0 \rightarrow r = 2, -2$

$y_1 = e^{-2x}$ $\quad$ $y_2 = e^{2x}$

$y_h = c_1 e^{-2x} + c_2 e^{2x}$

2- let $y_p(x) = Ax^2 + Bx + C$ be solution of ⊛

$y_p'(x) = 2Ax + B$

$y_p'' = 2A$ $\quad$ substitute in ⊛

$$2A - 4\left[Ax^2 + Bx + c\right] = 2x^2 + 4$$

$$-4Ax^2 - 4Bx + 2A - 4c = 2x^2 + 4$$

$-4A = 2 \rightarrow A = -\frac{1}{2}$

$-4B = 0 \rightarrow B = 0$

$2A - 4c = 4 \rightarrow c = -\frac{5}{4} \quad \leftarrow -2 * \frac{1}{2} - 4c = 4$

$\rightarrow y_p(x) = -\frac{1}{2}x^2 - \frac{5}{4}$

$\rightarrow y(x) = y_h + y_p$

$$= c_1 e^{-2x} + c_2 e^{2x} - \frac{1}{2}x^2 - \frac{5}{4}$$

لو اعطانا $y(0) = 1$ $y'(0) = 1$ بتطلعلي C
بنعوضهم بـ general solution

Ex:- Solve $y'' + 4y = 4e^{2x}$ $(*)$

لو كانت $4x$ بتكون y_{pa} بياري $Ax+B$

C |linear| x^1, x^2 ~~$x^2 K$~~

$\boxed{1}$ $y'' + 4y = 0$

$r^2 + 4 = 0$

$r = \pm\sqrt{-4}$

$r = \underbrace{0}_{\alpha} \pm \underbrace{2i}_{\mu}$

لما تكون $N=0$ تسمى pcour

يعني نحلها في $\begin{matrix}\cos\\ \sin\end{matrix}$ بس سريع

$y_1 = e^{Nx}\cos \mu x$

$= \cos 2x$

$\rightarrow y_h = c_1 \cos 2x + c_2 \sin 2x$

$y_2 = \sin 2x$

$\boxed{2}$ $y_{p(x)} = Ae^{2x}$

$y'_p = 2Ae^{2x}$

$y''_p = 4Ae^{2x}$

substitute in $(*)$

$4Ae^{2x} + 4Ae^{2x} = 4e^{2x}$

$8Ae^{2x} = 4e^{2x}$

$8A = 4$ $\rightarrow A = \frac{1}{2}$ $\therefore y_{p(x)} = \frac{1}{2}e^{2x}$

$\rightarrow y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{2}e^{2x}$

$\cos x \sin x = \frac{1}{2}\sin 2x$

$2\sin 2x + 3\cos 2x$

~~$y_p = A\sin 2x + B\cos 2x + \cos 2x$~~

EX:- Solve $y'' - 2y' = 4\sin 2x$ (*)

copy past in $y_{p(x)}$

$\boxed{1}$ $y'' - 2y' = 0$

$r^2 - 2r = 0$

$r(r-2) = 0$, $r = 0, 2$ → $y_1 = 1$

$y_2 = e^{2x}$

$y_h = c_1 + c_2 e^{2x}$

$\boxed{2}$ $y_{p(x)} = A \sin 2x + B \cos 2x$

$y_p' = 2A\cos 2x - 2B \sin 2x$

$y_p'' = -4A\sin(2x) - 4B\cos 2x$ substitution in (*)

$-4A\sin(2x) - 4B\cos 2x - 4A\cos 2x + 4B\sin 2x = 4\sin 2x$

$(-4A + 4B)\sin(2x) + (-4A - 4B)\cos 2x = 4\sin 2x$

$-4A + 4B = 4$ → $B - A = 1$ $\boxed{1}$

$-4A - 4B = 0$ → $B + A = 0$ $\boxed{2}$ +

$2B = 1$

$B = \frac{1}{2}$ → $A = -\frac{1}{2}$

$y_p = -\frac{1}{2}\sin 2x + \frac{1}{2}\cos 2x$

→ $y = c_1 + c_2 e^{2x} + -\frac{1}{2}\sin 2x + \frac{1}{2}\cos 2x$

Ex:- Solve

$y'' - y = 2e^x$

$y'' - y = 0$

$r^2 - 1 = 0$, $r = 1, -1$ $\rightarrow$ $y_1 = e^x$

$y_2 = e^{-x}$

$y_h = c_1 e^x + c_2 e^{-x}$

$y_p(x) = A x e^x$

$y'_p = Ae^x + Axe^x$

$y''_p = Ae^x + Ae^x + Axe^x = 2Ae^x + Axe^x$

By substitution :-

$\cancel{Axe^x} + 2Ae^x - \cancel{Axe^x} = 2e^x$

$2Ae^x = 2e^x$

$\boxed{A = 1}$ $\Rightarrow$ $y_p = xe^x$

$y(x) = y_h + y_p$

$= c_1 e^x + c_2 e^{-x} + x e^x$

Ex:- Dedermine asuitable from for y_p of the method of undermiond coefficent is to be used:

1. $y'' - 3y' + 2y = e^x \sin x$
2. $y'' + 3y' = 2x^4 + x^2 e^{-3x} + \sin 3x$

[1] $y'' - 3y' + 2y = 0$

$r^2 - 3r + 2 = 0$

$(r-1)(r-2) = 0$, $r = 1, 2 \rightarrow y_1 = e^x$

$y_2 = e^{2x}$

$y_h = c_1 e^x + c_2 e^{2x}$

$y_p =$ [illegible] $+ B \cdot e^x \cos x$

[2] $y'' + 3y' = 0$

$r^2 + 3r = 0$

$r(r+3) = 0$, $r = 0, -3 \rightarrow y_1 = 1$

$y_2 = e^{-3x}$

$y_h = c_1 + c_2 e^{-3x}$

$y_{p(x)} = x\left[A_4 x^4 + A_3 x^3 + A_2 x^2 + A_1 x + A_0\right]$

$+ x(Ax^2 + Bx + c)\, e^{-3}$

$+ D \sin x + E \cos x$

بعدما نضرب برجع نقارن ب y_h
لو لا زيه برجع نضرب x

H.W:- solve: $y'' - 2y' + y = 4e^{x}$

y''

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* Variation of parameters :-

$\boxed{1}$ $y'' + P(x)\, y' + q(x)\, y = r(x)$ — ⊛

$\boxed{1}$ $y_h = c_1 y_1 + c_2 y_2$

$\boxed{2}$ $y_{p(x)} = -y_1 \int \frac{y_2 \cdot r}{W} + y_2 \int \frac{y_1\, r}{W}$

$$\boxed{W[y_1, y_2] = y_1 y_2' - y_2 y_1'}$$

Ex:- solve $y'' + y = \sec x$

$y'' + y = 0$

$r^2 + 1 = 0$

$r^2 = -1 \qquad r = \mp\sqrt{-1} \rightarrow y_1 = \cos x$

$\qquad\qquad r = \mp i \qquad\qquad y_2 = \sin x$

$y_h = c_1 \cos x + c_2 \sin x$

$W[y_1, y_2] = y_1 y_2' - y_2 y_1'$

$= \cos x \cos x + \sin x \sin x = 1$

$$y_p = -y_1\int \frac{y_2 r}{W} + y_2\int \frac{y_1 r}{W}$$

$$= -\cos x \int \sin x \cdot \sec x \cdot dx + \sin x \int \cos x \sec x \, dx$$

$$= -\cos x \int \frac{\sin x}{\cos x} \, dx + \sin x \int \frac{\cos x}{\cos x} \, dx$$

$$= \cos \ln|\cos x| + x\sin x$$

$$y_{(x)} = y_h + y_p$$

$$= c_1\cos x + c_2\sin x + \cos \ln|\cos x| + x\sin x$$

Ex :- solve :- $x^2y'' - 4xy' + 6y = 21\,x^{-4}$

$$x^2y'' - 4xy' + 6y = 0$$

$$r(r-1) - 4r + 6 = 0$$

$$r^2 - 5r + 6 = 0$$

$$(r-2)(r-3) = 0 \quad , \quad r = 2, 3 \rightarrow \quad y_1 = x^2$$

$$y_2 = x^3$$

$$\rightarrow y_h = c_1x^2 + c_2x^3$$

$$\rightarrow W[y_1, y_2] = y_1y_2' - y_2y_1'$$

$$= x^2\,3x^2 - x^3\,2x$$

$$= 3x^4 - 2x^4$$

$$= x^4$$

$$\rightarrow r = \frac{21x^{-4}}{x^2}$$

$$= 21x^{-6}$$

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$$y_p(x) = -y_1 \int \frac{y_2 r}{W} + y_2 \int \frac{y_1 r}{W}$$

$$= -x^2 \int \frac{x^3 \cdot 21x^{-6}}{x^4} \cdot dx + x^3 \int \frac{x^2 \cdot 21x^{-6}}{x^4} \cdot dx$$

$$= -x^2 \int 21x^{-7} \cdot dx + 21x^3 \int x^{-8} \cdot dx$$

$$= -21x^2 \frac{x^{-6}}{-6} + 21x^3 \frac{x^{-7}}{-7}$$

$$= \frac{21}{6} x^{-4} + \frac{-21}{7} x^{-4}$$

$$= 3\frac{1}{2} x^{-4} + -3x^{-4}$$

$$= \frac{1}{2} x^{-4}$$

$$y(x) = y_p(x) + y_h$$

$$= c_1x^2 + c_2x^3 + \frac{1}{2}x^{-4}$$

* Higher-order Homogeneous Linear D.E :-

Ex : Solve :

$\boxed{1}$ $y^{(4)} - y = 0$

$\boxed{2}$ $y^{(5)} - 3y^{(4)} + 3y''' - y'' = 0$

$\boxed{3}$ $y^{(4)} + 2y'' + y = 0$

$\boxed{4}$ $\underbrace{x^3 y'''}_{r(r-1)(r-2)} - 3\underbrace{x^2 y''}_{r(r-1)} + 6\underbrace{xy'}_{r} - \underbrace{6y}_{1} = 0$

→ $\boxed{1}$ $r^4 - 1 = 0$

$(r^2-1)(r^2+1) = 0$

$r^2 - 1 = 0 \qquad r = 1, -1 \rightarrow y_1 = e^x , \quad y_2 = e^{-x}$

$r^2 + 1 = 0 \qquad r = \mp i \rightarrow y_4 = \cos x , \quad y_5 = \sin x$

$y_h = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$

$\boxed{2}$ $r^5 - 3r^4 + 3r^3 - r^2 = 0$

$r^2(r^3 - 3r^2 + 3r - \boxed{1}) = 0$

ه

عوامله
نعوض بالمعادلة
اذا صفرة يحقق ✓

$r^2(r-1)(r^2-2r+1) = 0$

$r^2(r-1)(r-1)(r-1) = 0$

$r = 0, 0, 1, 1, 1$

$y_1 = 1$, $y_2 = x$

$y_3 = e^x$, $y_4 = xe^x$, $y_5 = x^2 e^x$

$\Rightarrow y_h = c_1 + c_2 x + c_3 e^x + c_4 x e^x + c_5 x^2 e^x$

$\boxed{1}$

$$\begin{array}{r|l} & r^2 - 2r + 1 \\ \hline (r-1) & r^3 - 3r^2 + 3r - 1 \\ & \underline{-r^3 \pm r^2} \\ & -2r^2 + 3r - 1 \\ & \underline{\pm 2r^2 \mp 2r} \\ & r - 1 \\ & \underline{r - 1} \\ & 0 \end{array}$$

$\boxed{2}$

$r^3 - 1 - 3r^2 + 3r$

$(r-1)(r^2+r+1) - 3r(r-1)$

$(r-1)(r^2+r+1-3r)$

$(r-1)(r-1)(r-1)$

$\boxed{3}$ $r^4 + 2r^2 + 1 = 0$

$(r^2)^2 + 2(r^2) + 1 = 0$

$(r^2+1)(r^2+1) = 0$

$r^2 + 1 = 0$ or $r^2 + 1 = 0$

$r = \mp i$ $\quad$ $r = \mp i$

$y_1 = \cos x$ $\quad$ $y_2 = \sin x$ $\quad$ $y_3 = x\cos x$ $\quad$ $y_4 = x\sin x$

$y(x) = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x$

$\boxed{4}$ $r(r-1)(r-2) - 3r(r-1) + 6r - 6 = 0$

$r(r-1)(r-2) - 3r(r-1) + 6(r-1) = 0$

$(r-1)\left[r(r-2) - 3r + 6\right] = 0$

$(r-1)\left[r^2 - 5r + 6\right] = 0$

$(r-1)(r-2)(r-3) = 0 \qquad r = 1, 2, 3$

$y_1 = x \quad , y_2 = x^2 \quad , y_3 = x^3$

$y = c_1 x + c_2 x^2 + c_3 x^3$

* Higher-order nonhomogenous linear D.E :-

Ex: solve: $y''' + y = (x+1)^2$

$$y''' + y = x^2 + 2x + 1$$

$\boxed{1}$ $y''' + y = 0$

$r^3 + 1 = 0$ $\qquad r^3 = -1 \rightarrow$ ميلي راح

$r = -1$ $\qquad$ احد الاجوبة

$(r+1)(r^2 - r + 1) = 0$

$r + 1 = 0$ or $r^2 - r + 1 = 0$

$r = -1$ $\qquad \hookrightarrow b^2 - 4ac = 1 - 4 = -3$

$$r = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{+1 \mp \sqrt{-3}}{2}$$

$$= \underbrace{\frac{1}{2}}_{N} \mp \underbrace{\frac{\sqrt{3}}{2}}_{M} i$$

$y_1 = e^{-x}$ $\qquad y_2 = e^{Nx} \cos Mx$ $\qquad y_3 = e^{Nx} \sin Mx$

$\qquad = e^{\frac{1}{2}x} \cos \frac{\sqrt{3}}{2} x$ $\qquad = e^{\frac{1}{2}x} \sin \frac{\sqrt{3}}{2} x$

$$y_h = c_1 e^{-x} + c_2 e^{\frac{1}{2}x} \cos \frac{\sqrt{3}}{2} x + c_3 e^{\frac{1}{2}x} \sin \frac{\sqrt{3}}{2} x$$

$\rightarrow y_p = Ax^2 + Bx + C$

$y' = 2Ax + B$

$y'' = 2A$

$y''' = 0$

By sobstition in $(*)$

$Ax^2 + Bx + C = x^2 + 2x + 1$

$A = 1 , B = 2 , C = 1 \rightarrow y_p = x^2 + 2x + 1$

H.W : solve $y'' - y = (x - 1)^2$

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* Variation of parameters :-

$$y''' + a_2 y'' + a_1 y' + a_0 y = r(x)$$

$\boxed{1}$ $y_h = c_1 y_1 + c_2 y_2 + c_3 y_3$

$\boxed{2}$ $y_p = y_1 \int \frac{W_1}{W} r + y_2 \int \frac{W_2}{W} r + y_3 \int \frac{W_3}{W} r$

$$W[y_1, y_2, y_3] = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & 1 & y_3'' \end{vmatrix}$$

$$W[y_1, y_2, y_3, y_4] = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

$$W_3 = \begin{vmatrix} y_1 & y_2 & 0 & y_4 \\ y_1' & y_2' & 0 & y_4' \\ y_1'' & y_2'' & 0 & y_4'' \\ y_1''' & y_2''' & 1 & y_4''' \end{vmatrix}$$

* Remark :- $W[y_1, y_2, y_3] \neq 0$

y_1, y_2, y_3 are linearly independent

لو طلع = 0 بكون في خطأ في الـ y
و بكون dependent ..

→ Ex :- $x^3 y''' - 3x^2 y'' + 6xy' - 6y = x^2$ ..

$y_1 = x_1$, $y_2 = x^2$, $y_3 = x^3$ طلعتهم قبل أكم صفحة !!

$$W[y_1, y_2, y_3] = \begin{array}{c} \begin{matrix} x & x^2 & x^3 \end{matrix} \\ \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} \end{array}$$

$$= x \begin{vmatrix} 2x & 3x^2 \\ 2 & 6x \end{vmatrix} - x^2 \begin{vmatrix} 1 & 3x^2 \\ 0 & 6x \end{vmatrix} + x^3 \begin{vmatrix} 2x & 3x^2 \\ 2 & 6x \end{vmatrix}$$

$$= x(12x^2 - 6x^2) - x^2(6x - 0) + x^3(2 - 0)$$

$$= \cancel{6x^3} - \cancel{6x^3} + 2x^3$$

$$= 2x^3$$

+ even = رقم صف + عامود

− odd

$$W_1[x, x^2, x^3] = \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix} \overset{1+3=4}{=} +(1)\begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

$$= 3x^4 - 2x^4$$

$$= x^4$$

$$W_2[x, x^2, x^3] = \begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & 1 & 6x \end{vmatrix} \overset{5}{=} \boxed{-}(1)\begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix}$$

$$= -(3x^3 - x^3)$$

$$= -2x^3$$

$$W_3[x, x^2, x^3] = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} \overset{6}{=} \boxed{+}(1)\begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$$

$$= 2x^2 - x^2$$

$$= x^2$$

$$\Rightarrow W[e^x, e^{2x}, e^{3x}] = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}$$

$$= e^x e^{2x} e^{3x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$$

اعادة تمرين الماضي

Ex :- solve : $x^3y''' - 3x^2y'' + 6xy' - 6y = 24x^5$

$\boxed{1}$ $x^3y''' - 3x^2y'' + 6xy' - 6y = 0$

$y_1 = x,\ y_2 = x^2,\ y_3 = x^3 \rightarrow y_h = c_1x + c_2x^2 + c_3x^3$

$\boxed{2}$ $y_p = y_1\int \frac{W_1}{W}\cdot r + y_2\int \frac{W_2}{W}\cdot r + y_3\int \frac{W_3}{W}\cdot r$

$\rightarrow r(x) = \frac{24x^5}{x^3} = 24x^2$

$y_p = x\int \frac{x^4}{2x^3}\cdot 24x^2\,dx + x^2\int \frac{-2x^3}{2x^3}\,24x^2\,dx + x^3\int \frac{x^2}{2x^3}\,24x^2\,dx$

$= 12x\int x^3\,dx - 24x^2\int x^2\,dx + 12x^3\int x\,dx$

$= 12x\,\frac{x^4}{4} - 24x^2\,\frac{x^3}{3} + 12x^3\,\frac{x^2}{2}$

$= 3x^5 - 8x^5 + 6x^5$

$= x^5$

$\rightarrow y = y_h + y_p$

$= c_1x + c_2x^2 + c_3x^3 + x^5$

Ex :- Use undetermined coefficients to solve:

$x^2y'' - xy' + y = \ln x$ (*)

1. $x^2y'' - xy' + y = 0$ ← ما في داعي نحل h

$r(r-1) - r + 1 = 0$

$r^2 - 2r + 1 = 0$

$(r-1)(r-1) = 0$

$\boxed{r = 1} \rightarrow y_1 = x \quad , \quad y_2 = x\ln x$

$y_h = c_1x + c_2x\ln x$

→ let $x = e^t \rightarrow \ln x = t$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{1}{x} \cdot \frac{dy}{dt}\right]$$

$$= \frac{-1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dx}\left[\frac{dy}{dt}\right]$$

$$= \frac{-1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2} \cdot \frac{dt}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}$$

Substitute in (*)

$$x^2 \cdot \frac{1}{x^2}\left[\frac{d^2y}{dt^2} - \frac{dy}{dt}\right] - 1\,\frac{1}{x}\,\frac{dy}{dt} + y = t$$

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = t$$

$$y'' - 2y' + y = t \qquad *$$

[1] $r^2 - 2r + 1 = 0$

$(r-1)(r-1) = 0 \qquad r = 1, 1 \qquad y_1 = e^t, \quad y_2 = te^t$

$y_h = c_1 e^t + c_2 te^t$

نفس الحل مرة

[2] let $y_p(t) = At + B$

$y'_p(t) = A$

$y'' = 0$

Substitute in *

$-2A + At + B = t$

$At + (B - 2A) = t$

$\boxed{A = 1} \qquad B - 2A = 0$

$\boxed{B = 2}$

$y_p = t + 2$

$\therefore y = c_1 x + c_2 x \ln x + \ln x + 2$

$D = \frac{d}{dx}$: Differential operater

$(D^2 - 2D - I)\, y = 0$

$y'' - 2y - y = 0$

$(x^2 D^2 - xD - 2I)\, y = 0$

$x^2 y'' - xy' - 2y = 0$

* Basics in Matrices and vectors

$ax + by = c$
$dx + ey = f$

matrices form → $\begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}$

$$A \tilde{X} = \tilde{g}$$

Given amatrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$AX = \lambda X$

$\boxed{X \neq 0}$

→ λ is called an eigen value
→ X is called a corresponding eigenvector

$AX - \lambda X = 0$
$(A - \lambda I) X = 0$
$|A - \lambda I| = 0$

$A\overset{\neq 0}{X} = 0$

A Is not invertible ⇒ $|A| = 0$

A^{-1} not exist

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→ let $X \neq 0$

$AX = 0$ $\qquad$ A^{-1} does not exist

$|A| = 0$

* $AX = \lambda X$

$AX - \lambda X = 0$

$(A - \lambda I)X = 0$

→ $|A - \lambda I| = 0$

*** Ex:** Find the eigenvalues and the corresponding eigenvectors for the matrix

$$A = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}$$

→ $A - \lambda I$

$$\begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\begin{bmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{bmatrix}$$

$$\begin{vmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{vmatrix} = 0$$

$(3-\lambda)(-2-\lambda) + 4 = 0$

$\times -1$ $(\lambda - 3)(\lambda + 2) + 4 = 0$

$\lambda^2 - \lambda - 6 + 4 = 0$

$\lambda^2 - \lambda - 2 = 0$

$(\lambda - 2)(\lambda + 1) = 0$

$\lambda = 2, -1$ eigen value

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$$\begin{bmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\lambda = 2$

$$\begin{bmatrix} 1 & -1 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 - x_2 \\ 4x_1 - 4x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 - x_2 = 0$

$4x_1 - 4x_2 = 0$

$x_1 - x_2 = 0 \longrightarrow$

$x_1 = x_2$

$$X^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

or

$$X = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$\vdots$

∞ ناخذ اي اشي

$\lambda = -1$

$$\begin{bmatrix} 4 & -1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4x_1 - x_2 \\ 4x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$4x_1 - x_2 = 0 \longrightarrow X^{(2)} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$x_2 = 4x_1$

→ $|A - \lambda X| = 0$ → eigen values

$[A - \lambda X] = 0$ → eigen vectors

* system of Diff. Equations 2X2

$y_1' = a_{11} y_1 + a_{12} y_2$

$y_2' = a_{21} y_1 + a_{22} y_2$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

1. Find eigen values : $\lambda_1 \quad \lambda_2$
2. Find corresponding eigen vectors :- $X^{(1)}, X^{(2)}$ ← Indepent لازم يكونو

OR x

→ $y^{(1)} = e^{\lambda_1 \boxed{t}} X^{(1)}$

$y^{(2)} = e^{\lambda_2 t} X^{(2)}$

Ex :- solve

$y_1' = -3y_1 + y_2$

$y_2' = y_1 - 3y_2$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$y' = A\underline{y}$$

$\boxed{1}$ $|A - \lambda I| = 0$

$$\begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = 0$$

$$(-3-\lambda)(-3-\lambda) - 1 = 0$$

$$(\lambda+3)(\lambda+3) - 1 = 0$$

$$\lambda^2 + 6\lambda + 8 = 0$$

$$(\lambda+2)(\lambda+4) = 0 \quad \Rightarrow \lambda = -2, -4.$$

$\boxed{2}$ $[A - \lambda I]\, X = 0$

$$\begin{bmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

\# $\lambda = -2$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x_1 + x_2 = 0$
$x_1 - x_2 = 0$ $\Rightarrow x_1 = x_2$ $\therefore X^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore y_1 = e^{\lambda_1 t} X^{(1)} = e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

\# $\lambda = -4$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + x_2 = 0$
$x_1 + x_2 = 0$ $\Rightarrow x_1 = -x_2$ $X^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$y^{(2)} = e^{\lambda_2 t} X^{(2)}$$

$$= e^{-4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\rightarrow \quad y = c_1 y^{(1)} + c_2 y^{(2)} = c_1 e^{-2t} \begin{vmatrix} 1 \\ 1 \end{vmatrix} + c_2 e^{-4t} \begin{vmatrix} 1 \\ -1 \end{vmatrix}$$

Ex:- solve

$$y_1' = 3y_1 - y_2$$
$$y_2' = 4y_1 - 2y_2$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$\boxed{1}$ $|A - \lambda I| = 0$

$$\begin{bmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{bmatrix} = 0$$

$$(3-\lambda)(-2-\lambda) + 4 = 0$$
$$(\lambda-3)(\lambda+2) + 4 = 0$$
$$\lambda^2 - \lambda - 2 = 0 \quad \Rightarrow \quad \lambda = 2, -1$$

2) $[A - I\lambda]\, X = 0$

$$\begin{bmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

\# $\lambda = 2$

$$\begin{bmatrix} 1 & -1 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 - x_2 = 0 \Rightarrow x_1 = x_2 \therefore X^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$y^{(1)} = e^{\lambda_1 t} X^{(1)}$

$= e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

\# $\lambda = -1$

$$\begin{bmatrix} 4 & -1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$4x_1 - x_2 = 0 \Rightarrow x_2 = 4x_1$

$$X^{(2)} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$y^{(2)} = e^{\lambda_2 t} X^{(2)}$$
$$= e^{-t} \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$\rightarrow y = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

* Repeated eigen value :-

Ex :- Solve

$$y_1' = y_1 - y_2$$

$$y_2' = y_1 + 3y_2$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\underline{y}' = A \underline{y}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(3-\lambda) + 1 = 0$$

$$(\lambda-1)(\lambda-3) + 1 = 0$$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$(\lambda-2)(\lambda-2) = 0 \rightarrow \lambda = 2, 2 \quad : \text{eigenvalues}$$

$[A - \lambda I]\, x = 0$

$$\begin{bmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow N=2$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + x_2 = 0$

$-x_1 - x_2 = 0$ $\Rightarrow x_2 = -x_1$

$$X^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad y^{(1)} = e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\rightarrow$ we lock for a generalized eigenvector

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$x_1 + x_2 = -1$ $\rightarrow$ $x_2 = -1 - x_1$

هون كان لو
ح خد $X=0$ لأنه
تانية ما بتساوي كمان صفر

$$X^{2} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \qquad y^{(2)} = t e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t}$$

Ex :- solve

$$y' = \begin{array}{c} A \\ \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \end{array} y$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 \\ -4 & -2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(-2-\lambda) + 4 = 0$$

$$(\lambda-2)(\lambda+2) + 4 = 0$$

$$\lambda^2 - 4 + 4 = 0$$

$$\boxed{\lambda = 0, 0}$$

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} 2-\lambda & 1 \\ -4 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \lambda_1 = 0$

$$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x_1 + x_2 = 0 \Rightarrow x_2 = -2x_1$$

$$-4x_1 - 2x_2 = 0$$

$$\therefore X^{(1)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad y^{(1)} = e^{0t} X^{(1)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$\Rightarrow \lambda_2 = 0$

$$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

بدل ما نخليها تساوي صفر
هنا بتصير تساوي
$X^{(1)}$ مشان طريقة الحل
بتختلف.

$$2x_1 + x_2 = 1 \Rightarrow x_2 = 1 - 2x_1$$

$$\therefore X^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad y^{2} = t\begin{bmatrix} 1 \\ -2 \end{bmatrix} + e^{0}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

لو كانت $\lambda_2 = \lambda_1$ ⇐

مش نطلع $X^{(2)}$ بنعمل

$-[A - \lambda I]X = X^{(1)} -$

⇐ $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$2x_1 + 0 = 0$

$\boxed{x_1 = 0}$

ما في معلومة عن x_2 من: Free أي قيمة

$X = \begin{bmatrix} 0 \\ \boxed{1} \end{bmatrix}$ ← أي شي

* Complex Roots :-

Ex:- solve :

$y' = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} y$

$-|A - \lambda I| = 0$

$\begin{vmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = 0$

→ $(-1-\lambda)(-1-\lambda) + 1 = 0$

$(\lambda+1)(\lambda+1) + 1 = 0$

$\lambda^2 - 2\lambda + 2 = 0$

$b^2 - 4ac = 4 - 4 \cdot 2$

$= -4 < 0$

$$\lambda = \frac{-b \mp \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2}$$

$$= -1 \pm i$$

$\lambda_1 = -1+i \qquad \lambda_2 = -1-i$

$\lambda_1 = -1+i$

$$\begin{bmatrix} -1+1-i & 1 \\ -1 & -1+1-i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-ix_1 + x_2 = 0 \Rightarrow x_2 = +ix_1$

نفس الاشي لو ضربنا بـ i

$-x_1 - ix_2 = 0$

$-ix_1 - \underline{i^2}\, x_2$

$-ix_1 - -1x_2$

$-ix_1 + x_2$

نفس معادلة الاولى

$$\therefore X^{(1)} = \begin{bmatrix} 1 \\ +i \end{bmatrix}$$

$$y^{(1)} = e^{\lambda t} \cdot X^{(1)}$$

$$= e^{(-1+i)t} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

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$\lambda_2 = -1-i$

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$i x_1 + x_2 = 0 \qquad x_2 = -i x_1$

$$\therefore X^{(2)} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\therefore y^2 = e^{\lambda t} X^2$$
$$= e^{(-1-i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$\Rightarrow e^{it} = \cos t + i \sin t$

$e^{-it} = \cos t - i \sin t$

$$y^{(1)} = e^{(-1+i)t} \begin{bmatrix} -1 \\ i \end{bmatrix}$$

$$= e^{-t} e^{it} \begin{bmatrix} -1 \\ i \end{bmatrix}$$

$$= e^{-t} [\cos t + i \sin t] \begin{bmatrix} -1 \\ i \end{bmatrix}$$

$$= e^{-t}\begin{bmatrix} -\cos t - i\sin t \\ i\cos t - \sin t \end{bmatrix}$$

$$= e^{-t}\left(\begin{bmatrix} -\cos t \\ -\sin t \end{bmatrix} + i\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}\right)$$

$$\underbrace{e^{-t}\begin{bmatrix} -\cos t \\ -\sin t \end{bmatrix}}_{a} + i\,\underbrace{e^{-t}\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}}_{b}$$

$$y^{(1)} = e^{-t}\begin{bmatrix} -\cos t \\ -\sin t \end{bmatrix}, \quad y^{(2)} = e^{-t}\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$$

** undeterminal **
** Coefficients **

Ex: solve

$$y_1' = 6y_1 + y_2 + 6t$$

$$y_2' = 4y_1 + 3y_2 - 10t + 4$$

$$\underline{y}' = \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix} \underline{y} + \begin{bmatrix} 6t \\ -10t + 4 \end{bmatrix}$$

$$y' = Ay + \begin{bmatrix} 6 \\ -10 \end{bmatrix} t + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

Find y_h

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 6-\lambda & 1 \\ 4 & 3-\lambda \end{vmatrix} = 0$$

$$(\lambda-6)(\lambda-3)-4=0$$

$$\lambda^2-9\lambda+18-4=0$$

$$\lambda^2-9\lambda+14=0$$

$$(\lambda-2)(\lambda-7)=0 \rightarrow \lambda=2,\ \lambda=7$$

(III) $[A-\lambda I]X=0$

$$\begin{bmatrix} 6-\lambda & 1 \\ 4 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\rightarrow \lambda=2$

$$\begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4x_1+x_2=0 \rightarrow x_2=-4x_1$$

- $X^{(1)}=\begin{bmatrix} 1 \\ -4 \end{bmatrix}$ - $y^{(1)}=e^{2t}\begin{bmatrix} 1 \\ -4 \end{bmatrix}$

$\rightarrow \lambda = 7$

$$\begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x_1 + x_2 = 0 \Rightarrow x_2 = x_1$

$- x^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad - y^2 = e^{7t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\Rightarrow y_h = c_1 e^{2t} \begin{bmatrix} 1 \\ -4 \end{bmatrix} + c_2 e^{7t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Find y_p

$y_p = at + b \Rightarrow y' = a$

$$y' = Ay + \begin{bmatrix} 6 \\ -10 \end{bmatrix} t + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$a = A(at+b) + \begin{bmatrix} 6 \\ -10 \end{bmatrix} t + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} a_1 t \\ a_2 t \end{bmatrix} + \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} 6t \\ -10t \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 6a_1t + 1a_2t \\ 4a_1t + 3a_2t \end{bmatrix} + \begin{bmatrix} 6b_1 + b_2 \\ 4b_1 + 3b_2 \end{bmatrix} + \begin{bmatrix} 6t \\ -10t + 4 \end{bmatrix}$$

$$\underline{(6a_1 + a_2 + 6)}\,t + \underline{6b_1 + b_2}$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 6a_1t + a_2t + 6b_1 + b_2 + 6t \\ 4a_1t + 3a_2t + 4b_1 + 3b_2 - 10t + 4 \end{bmatrix}$$

$$\underline{(4a_1 + 3a_2 - 10)}\,t + \underline{4b_1 + 3b_2 + 4}$$

$a_1 = 6b_1 + b_2 \quad - \boxed{3}$ $\qquad\qquad 0 = 6a_1 + a_2 + 6 \quad - (1)$

$a_2 = 4b_1 + 3b_2 + 4 \quad - \boxed{4}$ $\qquad\qquad 0 = 4a_1 + 3a_2 - 10 \quad - (2)$

$\Rightarrow -3\boxed{1} + \boxed{2}$

$-14a_1 - 28 = 0$

$\boxed{a_1 = -2}$ $\quad \boxed{1} \Rightarrow \quad -12 + a_2 + 6 = 0$

$\boxed{a_2 = 6}$

$-2 = 6b_1 + b_2 \quad - \boxed{3}$

$6 = 4b_1 + 3b_2 + 4 \quad - \boxed{4}$

$-3\boxed{3} + \boxed{4}$

$-14b_1 = 8$

$\boxed{b_1 = -\frac{4}{7}}$ $\quad \boxed{3} \Rightarrow \quad -\frac{4}{7} \cdot 6 + b_2 = -2$

$\boxed{b_2 = \frac{10}{7}}$

$$\Rightarrow y_p = at + b$$

$$= \begin{bmatrix} -2 \\ 0 \end{bmatrix} t + \begin{bmatrix} -\frac{4}{7} \\ \frac{10}{7} \end{bmatrix}$$

$$* \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow |A| = ad - bc$$

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-4} \begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix}$$

** Variation of ** ** parameters **

$\rightarrow y(t) = \underbrace{Y(t)\ c}_{\downarrow} + Y(t) \int Y^{-1}(s)\ g(s)\ ds$

$y_h \rightarrow \begin{bmatrix} \vdots & \vdots \\ y_1 & y_2 \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

Ex :- solve by varition of parameters

$$\underline{y}' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \underline{y} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$$

Find we solve : $\underline{y}' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \underline{y}$

$|A - \lambda I| = 0$

$$\begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = 0$$

$(\lambda+3)(3+\lambda) - 1 = 0 \quad \rightarrow \lambda = -2, -4$

$[A - \lambda I]X = 0$

$$\begin{bmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\lambda = -2$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x_1 + x_2 = 0$

$\boxed{x_1 = x_2} \Rightarrow X^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow y^{(1)} = e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -4$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + x_2 = 0 \Rightarrow \boxed{x_2 = -x_1} \Rightarrow X^{2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$y^{(2)} = e^{-4t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad * \; Y = \begin{bmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{bmatrix}$$

$$-\; |Y| = -e^{-6t} - e^{-6t} = -2e^{-6t}$$

$$-\; \frac{1}{|Y|} = \frac{1}{-2e^{-6t}} = -\frac{1}{2} e^{6t}$$

$$\Rightarrow Y^{-1}(t) = -\frac{1}{2} e^{6t}\begin{bmatrix} -e^{-4t} & -e^{-4t} \\ -e^{-2t} & e^{-2t} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} e^{2t} & e^{2t} \\ e^{4t} & -e^{4t} \end{bmatrix}$$

$$-\; Y^{-1}(t) \cdot g(t)$$

$$= \frac{1}{2}\begin{bmatrix} e^{2t} & e^{2t} \\ e^{4t} & -e^{4t} \end{bmatrix} \cdot \begin{bmatrix} -6e^{-2t} \\ 2e^{-2t} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -6+2 \\ -6e^{2t} + 2e^{2t} \end{bmatrix} = \begin{bmatrix} -2 \\ -4e^{2t} \end{bmatrix}$$

$\Rightarrow \int Y^{-1}(t)\, g(t)\, dt$

$$\int \begin{bmatrix} -2 \\ -4e^{2t} \end{bmatrix} dt = \begin{bmatrix} -2t \\ -2e^{2t} \end{bmatrix}$$

$y_p \Rightarrow Y(t) \int Y^{-1}(s)\, g(s)\, ds$

$$\begin{bmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{bmatrix} \begin{bmatrix} -2t \\ -2e^{2t} \end{bmatrix} = \begin{bmatrix} -2te^{-2t} - 2e^{-2t} \\ -2te^{-2t} + 2e^{-2t} \end{bmatrix}$$

$$y(t) = y_h + y_p$$

$$= Y(t)\, c + Y(t) \int Y^{-1}(s)\, g(s)\, ds$$

$$= \begin{bmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} -2te^{-2t} - 2e^{-2t} \\ -2te^{-2t} + 2e^{-2t} \end{bmatrix}$$

$\Rightarrow y_p = (\underline{a}t + \underline{b})\, e^{-2t}$ where $a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

EX :- consider :

$y_1' = 5y_1 + 3y_2 - 2e^{2t} + 1$

$y_2' = -y_1 + y_2 + e^{2t} - 5t + 7$

[a] Find the homogenous solution of the system $y^{(h)}$

[b] Determine a solution from for $y^{(p)}$ if the undetermined coefficients is to be used

بس بده صوره العامة بس

$$\underline{y}' = \begin{bmatrix} 5 & 3 \\ -1 & 1 \end{bmatrix} \underline{y} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} 0 \\ -5 \end{bmatrix} t + \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$y^h \rightarrow |A - \lambda I| = 0$

$$\begin{vmatrix} 5-\lambda & 3 \\ -1 & 1-\lambda \end{vmatrix} = 0$$

$(\lambda - 5)(\lambda - 1) + 3 = 0$

$\lambda^2 - 6\lambda + 8 = 0$

$(\lambda - 2)(\lambda - 4) = 0 \longrightarrow \lambda = 2, 4$

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} 5-\lambda & 3 \\ -1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\boxed{\lambda = 2}$$

$$\begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x_1 - x_2 = 0$$

$$\boxed{x_2 = -x_1} \rightarrow X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow y^{(1)} = e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\boxed{\lambda = 4}$$

$$\begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 + 3x_2 = 0$$

$$\boxed{x_1 = -3x_2} \rightarrow X_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \Rightarrow y^2 = e^{4t} \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$y^h = c_1 e^{2t}\begin{bmatrix}1\\-1\end{bmatrix} + c_2 e^{4t}\begin{bmatrix}-3\\1\end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} & -3e^{4t} \\ -e^{2t} & e^{4t}\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$$

y^p

$\Rightarrow$ $$y^p = \underline{\vec{a}}\, e^{2t}$$

في حل زيه
زمانكنا نفرض
← x or t ب

$$y^p = (\underline{a}t+\underline{b})\, e^{2t} + \underline{C}t + \underline{D}$$

$$= \left(\begin{bmatrix}a_1\\a_2\end{bmatrix} t + \begin{bmatrix}b_1\\b_2\end{bmatrix}\right) e^{2t} + \begin{bmatrix}C_1\\C_2\end{bmatrix} t + \begin{bmatrix}D_1\\D_2\end{bmatrix}$$

Ex:- consider:

$$y_1' = y_1 + y_2 + 10\cos t$$
$$y_2' = 3y_1 - y_2 - 10\sin t$$

$\boxed{a}$ Find the homogenous solution y^h

$\boxed{b}$ Determin asolution from for y^p if the undeterimeed coefficent is to be used

$$y' = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \underline{y} + \begin{bmatrix} 10 \\ 0 \end{bmatrix} \cos t + \begin{bmatrix} 0 \\ -10 \end{bmatrix} \sin t$$

y^h $\rightarrow$ $\boxed{1}$ $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{vmatrix} = 0$$

$(\lambda-1)(\lambda+1) - 3 = 0$

$\lambda^2 - 4 = 0 \rightarrow \lambda = 2, -2$

$\boxed{2}$ $[A - \lambda I]\underline{X} = \underline{0}$

$$\begin{bmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\boxed{\lambda = 2} \rightarrow$

$$\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x_1 + x_2 = 0$

$\boxed{x_1 = x_2} \rightarrow X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad y_1 = e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\boxed{\lambda = -2}$:

$$\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3x_1 + x_2 = 0$

$\boxed{x_2 = -3x_1} \Rightarrow X^2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \Rightarrow y^2 = e^{-2t}\begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$$\Rightarrow y^{(h)} = c_1 e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$y_p \Rightarrow y_p = \underline{a}\cos t + \underline{b}\sin t$

chapter 7

** linear Algebra **

[1] Diagonal Matrix

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}_{2\times 2} \qquad B = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}_{3\times 3}$$

[2] Triangular Matrix

$$A = \begin{bmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & f \end{bmatrix} \qquad B = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$$

- upper triangular
- lower triangular

$$\Rightarrow A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{A\ \text{transpose}} A^t = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

* Elementary Row operations :-

1. interchange of two rows
2. Multiplication of any row with a non-zero constant
3. Addition of a multiple of one row to another

** Gauss - Elimination Method **

لازم يحدد طريقة

Ex:- Use the Gauss Elimination Method to solve:

$x_1 - x_2 + x_3 = 0$

$-x_1 + x_2 - x_3 = 0$

$10x_2 + 25x_3 = 90$

$20x_1 + 10x_2 = 80$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 0 & 10 & 25 \\ 20 & 10 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 90 \\ 80 \end{bmatrix}$$

← بنكون Augmented Matrix: A مع non Homo.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right] \xrightarrow[-20R_1+R_4]{R_1+R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-3R_2+R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

عدد Rows غ صفرية $r = 3$

عدد مجاهيل $n = 3$

$n = r \rightarrow$ unique solution

$X_1 - X_2 + X_3 = 0$

$10X_2 + 25X_3 = 90$

$-95X_3 = -190 \Rightarrow X_3 = \frac{190}{95} = 2$

$\Rightarrow 10X_2 + 25 \times 2 = 90 \rightarrow X_2 = 4$

$\Rightarrow X_1 - 4 + 2 = 0 \quad X_1 = 2$

EX: use Gauss-Elimination method to solve:-

$3X_1 - 2X_2 + X_3 = 3$

$2X_1 + X_2 + X_3 = 0$

$6X_1 + 2X_2 + 4X_3 = 0$

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 0 \end{array}\right] \xrightarrow{\frac{1}{3}R_1} \left[\begin{array}{ccc|c} \boxed{1} & -\frac{2}{3} & \frac{1}{3} & 1 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 0 \end{array}\right]$$

$$\xrightarrow[-6R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & \frac{1}{3} & 1 \\ 0 & \boxed{\frac{7}{3}} & \frac{1}{3} & -2 \\ 0 & 6 & 2 & -6 \end{array}\right] \xrightarrow{\frac{3}{7}R_2} \left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & \frac{1}{3} & 1 \\ 0 & 1 & \frac{1}{7} & -\frac{6}{7} \\ 0 & 6 & 2 & -6 \end{array}\right]$$

$$\xrightarrow{-6R_2+R_3} \left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & \frac{1}{3} & 1 \\ 0 & 1 & \frac{1}{7} & -\frac{6}{7} \\ 0 & 0 & \frac{8}{7} & \frac{36}{7}-6 \end{array}\right]$$

Ex :- solve :

$x_1 - 2x_2 - 6x_3 = 12$

$2x_1 + 4x_2 + 12x_3 = -17$

$x_1 - 4x_2 - 12x_3 = 22$

ثبت

$$\left[\begin{array}{ccc|c} \underline{1} & -2 & -6 & 12 \\ 2 & 4 & 12 & -17 \\ 1 & -4 & -12 & 22 \end{array}\right] \xrightarrow[-R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & -2 & -6 & 12 \\ 0 & \boxed{8} & 24 & -41 \\ 0 & -2 & -6 & 10 \end{array}\right]$$

$$\xrightarrow{\frac{1}{8}R_2} \left[\begin{array}{ccc|c} 1 & -2 & -6 & 12 \\ 0 & 1 & 3 & -\frac{41}{8} \\ 0 & -2 & -6 & 10 \end{array}\right] \xrightarrow{2R_2+R_3} \begin{array}{c} \begin{array}{cc} A & \quad\quad\quad B \end{array} \\ \left[\begin{array}{ccc|c} 1 & -2 & -6 & 12 \\ 0 & 1 & 3 & -\frac{41}{8} \\ 0 & 0 & 0 & -\frac{1}{4} \end{array}\right] \end{array}$$

هاد لتبع يعني دارة يوما شاذ هاد

$0 = \frac{-1}{4}$

No solution

$R(A) = 2$ عدد صفوف يلي مش كلها صفر

$R(A|B) = 3$

$R(A) \neq R(A|B)$ هاد يعني ما في solution

Ex:- solve:

$x - 2y + z = -6$

$2x - 3y = -7$

$-x + 3y - 3z = 11$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & -6 \\ 2 & -3 & 0 & -7 \\ -1 & 3 & -3 & 11 \end{array}\right] \xrightarrow[R_1 + R_3]{-2R_1 + R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & -6 \\ 0 & 1 & -2 & 5 \\ 0 & 1 & -2 & 5 \end{array}\right]$$

$$\xrightarrow{-R_2 + R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & -6 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R(A) = 2 \quad \therefore R(A|B) = 2$

$n = 3$

$n - R = 1$ ← عدد المتغيرات ال Free يعني اي قيمة

$0 = 0$ ✓

$y - 2z = 5 \rightarrow y = 2z + 5$

$x - 2y + z = -6 \rightarrow x = 2y - z - 6$

$\boxed{z = t}$ ← any values

$y = 2t + 5$

$x = 4t - t + 10 - 6 = 3t + 4$

$t \in \mathbb{R}$

$\boxed{\text{Infinite solution}}$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} n = 3 \\ r = 1 \end{array} \quad n - r = 2$$

$$x - 2y + z = -6$$

Take $y = t$ $\quad t, s \in \mathbb{R}$

$z = s$

$\Downarrow$

$x = 2t - s - 6$

$\infty \ldots$	$t = 1$, $s = 0$, $x = -4$	$t = 0$, $s = 0$, $x = -6$	$t = 0$, $s = 1$, $x = -7$

$\Leftarrow$ solutions

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 1 & 3 \end{bmatrix}$$

3×3

upper trinangular

main diagonal

lower trinangular

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Diagonal Matrix

Idantity

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}_{2\times 3} \qquad B = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}_{3\times 3}$$

$$A \times B = \begin{bmatrix} 2\times2+0\times0+1\times1 & -1\times2+0\times0+1\times1 & 1\times2+2\times0+4\times1 \\ 2\times1+0\times1+1\times0 & -1\times1+-1\times0+1\times0 & 1\times1+2\times-1+0\times4 \end{bmatrix}_{2\times3}$$

$$= \begin{bmatrix} 5 & -1 & 6 \\ 2 & -1 & -1 \end{bmatrix}_{2\times3}$$

BA not well defined

Ex:- consider the system :

$x - y + 2z = 4$

$3x - 2y + 9z = 14$

$2x - 4y + az = b$

Find all posible value of a and b so that the system

1. has a unique solution
2. has no solution
3. has infinitly many solution

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 3 & -2 & 9 & 14 \\ 2 & -4 & a & b \end{array}\right] \xrightarrow[-2R_1+R_3]{-3R_1+R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & -2 & a-4 & b-8 \end{array}\right]$$

$$\xrightarrow{2R_2+R_3} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & a+2 & b-4 \end{array}\right]$$

1. $2+a \neq 0$ $\quad b \in \mathbb{R}$

   $a \neq -2$

2. $2+a = 0$ $\quad b-4 \neq 0$

   $a = -2$ $\quad b \neq 4$

$\boxed{8}$ $2+a=0$ & $b-4=0$

$a=-2$, $b=4$

** Cramer's Rule **

Ex:- use Cramer's Rule to solve the system:

$$4x_1 + 3x_2 = \boxed{12}$$
$$2x_1 + 5x_2 = \boxed{-8}$$

تستخدم هاي طريقة
لما يكون ٣ متغيرات
ب ٣ معادلات

$$\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 12 \\ -8 \end{bmatrix}$$

$$D = \begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix} = 14$$

$$D_1 = \begin{vmatrix} 12 & 3 \\ -8 & 5 \end{vmatrix} = 60 + 24 = 84$$

$$D_2 = \begin{vmatrix} 4 & 12 \\ 2 & -8 \end{vmatrix} = -32 - 24 = -56$$

$$x_1 = \frac{D_1}{D} = \frac{84}{14} = 6$$

$$x_2 = \frac{D_2}{D} = \frac{-56}{14} = -4$$

EX:- use cramer's rule to solve the system:

$$x + y = 2$$
$$3y - z = -4$$
$$x + z = 3$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$$

$$D = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 3 & -1 \\ 1 & 0 & 1 \end{vmatrix} = 1(3+0) - 1(0+1) + 0(0-3)$$

$$= 3 - 1$$

$$= 2$$

$$D_1 = \begin{vmatrix} 2 & 1 & 0 \\ -4 & 3 & -1 \\ 3 & 0 & 1 \end{vmatrix} = 2(3+0) - 1(-4+3) + 0(0-9)$$

$$= 6 + 1$$

$$= 7$$

$$D_2 = \begin{vmatrix} 1 & 2 & 0 \\ 0 & -4 & -1 \\ 1 & 3 & 1 \end{vmatrix} = 1(-4+3) - 2(+1) + 0(0+4)$$

$$= -1 - 2$$

$$= -3$$

$$D_3 = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 3 & -4 \\ 1 & 0 & 3 \end{vmatrix} = 1(9-0) - 0(3-0) + 1(-4-6)$$

$$= 9 + -10$$

$$= -1$$

$$x_1 = \frac{D_1}{D} = \frac{7}{2} \qquad y = \frac{D_2}{D} = \frac{-3}{2}$$

$$z = \frac{D_3}{D} = \frac{-1}{2}$$

* Properties of Determinents :-

1. The determinent of any triangular Matrix is the product of the diagonel entre $\boxed{|A| = \det(A)}$

EX :-

let $A = \begin{bmatrix} 10 & 20 & 30 \\ 0 & 4 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ Find $|A|$

$|A| = 10 \times 4 \times 2 = 80$

2. - Interchage any two rows, change the sign of the determinent

غيرة row → odd → يضرب بسالب

→ even → ما في داعي اضرب

- Multiplication of any row with a constant a, change the value of the determinent by multiplying the determinent by a

اذا ضرب اي صف بعدد

لازم اضرب الجواب بنفس عدد

EX: let A be 3x3 matrix $|A| = 3$ Find $|2A|$:-

$|2A| = 2 \times 2 \times 2\,|A| = 2 \times 2 \times 2 \times 3 = 24$

Remark : let A, B, betwen Matrics Then.

$$|AB| = |A|\,|B|$$

* Invers of Matrices :-

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\det A = |A|$$

EX:- let $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$ Find A^{-1}

$$|A| = \begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix} = 4+6 = 10$$

$$A^{-1} = \frac{1}{10} \begin{bmatrix} 4 & +2 \\ -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0.4 & 0.2 \\ -0.3 & 0.1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

معلومة :- لو ضربنا $A \cdot A^{-1}$ يكون حيل جواب

$$A \cdot A^{-1} \qquad \therefore$$

EX:- Use Guass-Jordan elimination to find A^{-1} for

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & -8 \\ -3 & -5 & 8 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 5 & -8 & 0 & 1 & 0 \\ -3 & -5 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow[3R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 1 & -1 & 3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-R_2+R_3]{-2R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 5 & -2 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -1 & 1 \end{array}\right] \xrightarrow[-R_3+R_1]{2R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 8 & -1 & 2 \\ 0 & 0 & 1 & 5 & -1 & 1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} 0 & -1 & -1 \\ 8 & -1 & 2 \\ 5 & -1 & 1 \end{bmatrix}$$

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Finding A^{-1} by Adjoint method :-

$$A^{-1} = \frac{1}{|A|} \cdot adj(A)$$

بخلي الصفوف كاعمود

$$adj(A) = [C_{ij}]^{\boxed{T}} \quad , \quad C_{ij} : \text{Cofactors}$$

Ex :- use the adjoint Method to find A^{-1} for

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & -8 \\ -3 & -5 & 8 \end{bmatrix}$$

cofactors $= C_{11}$

$$|A| = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 5 & -8 \\ -3 & -5 & 8 \end{vmatrix} = (1)\begin{vmatrix} 5 & -8 \\ -5 & 8 \end{vmatrix} - 2\begin{vmatrix} 2 & -8 \\ -3 & 8 \end{vmatrix} - 3\begin{vmatrix} 2 & 5 \\ -3 & -5 \end{vmatrix}$$

$$= 0 - 2(16 - 24) - 3(-10 + 15)$$

$$= 16 - 15$$

$$= 1$$

$$C_{11} = \begin{vmatrix} 5 & -8 \\ -5 & 8 \end{vmatrix} = 0$$

[illegible]

$$C_{12} = \boxed{-}\begin{vmatrix} 2 & -8 \\ -3 & 8 \end{vmatrix} = 8$$

$$C_{13} = \begin{vmatrix} 2 & 5 \\ -3 & -5 \end{vmatrix} = 5$$

rows + cols is
- odd
+ even

$$C_{21} = \boxed{-}\begin{vmatrix} 2 & -3 \\ -5 & 8 \end{vmatrix} = -1$$

$$C_{22} = \begin{vmatrix} 1 & -3 \\ -3 & 8 \end{vmatrix} = -1$$

$$C_{23} = -\begin{vmatrix} 1 & 2 \\ -3 & -5 \end{vmatrix} = -1$$

$$C_{31} = \begin{vmatrix} 2 & -3 \\ 5 & -8 \end{vmatrix} = -1$$

$$c_{32} = -\begin{vmatrix} 1 & 2 \\ -3 & -8 \end{vmatrix} = 2$$

$$c_{33} = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 1$$

$$[c_{ij}] = \begin{bmatrix} 0 & 8 & 5 \\ -1 & -1 & -1 \\ -1 & 2 & 1 \end{bmatrix}$$

$$\operatorname{adj}(A) = [c_{ij}]^{T} = \begin{bmatrix} 0 & -1 & -1 \\ 8 & -1 & 2 \\ 5 & -1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \operatorname{adj}(A) = \begin{bmatrix} 0 & -1 & -1 \\ 8 & -1 & 2 \\ 5 & -1 & 1 \end{bmatrix}$$

Ex :- Find all aigenvalues and aigenvactors for

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$(1-\lambda)\left((1-\lambda)(1-\lambda) - 1\right) - 1\left(-1 + (1-\lambda)\right) + 1\left(1 - (1-\lambda)\right) = 0$

$(1-\lambda)(\lambda^2 - 2\lambda + \cancel{1} - \cancel{1}) + 2\lambda = 0$

$(\lambda - 1)(\lambda^2 - 2\lambda) - 2\lambda = 0$

$\lambda^3 - 2\lambda^2 - \lambda^2 + \cancel{2\lambda} - \cancel{2\lambda} = 0$

$\lambda^3 - 3\lambda^2 = 0$

$\lambda^2(\lambda - 3) = 0$

$\lambda = 0, 0, 3$

$\boxed{\lambda = 3}$

$[A - \lambda I]\, X = \underline{0}$

$$\begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \xrightarrow[R_1 \,\&\, R_3]{\text{Replace}} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[2R_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] \xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$n = 3$

$r = 2$

$n - r = 3 - 2 = \boxed{1}$

$\rightarrow -3x_2 + 3x_3 = 0 \quad - \quad \boxed{x_2 = x_3}$

$x_1 + x_2 - 2x_3 = 0 \quad \boxed{x_1 = 2x_3 - x_2}$

$$X^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

لو كان سؤال solve بشكل عام.
هيك بكون الجواب

$x_1 = t$

$x_2 = t \quad t \in \mathbb{R}$

$x_3 = t$

$\boxed{\lambda = 0.0}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow[-R_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$n = 3$

$r = 1$

$3 - 1 = 2$ values free

$x_1 + x_2 + x_3 = 0 \rightarrow \boxed{x_3 = -x_1 - x_2}$

$$X^{(2)} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad X^{(3)} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$x_1 = 1$, $x_2 = 0$ $\qquad$ $x_1 = 0$, $x_2 = 1$

لو كان solve

$x_1 = t$

$x_2 = s \quad t, s \in \mathbb{R}$

$x_3 = -t - s$

لازم يكونوا Independent

يعني القيم يساويهم لازم ابيه

** Revision **

- Determinent

$|A| = det(A)$

1. $|AB| = |A||B|$
2. If A is $n \times n$: $|2A| = 2^n |A|$
3. $|A^{-1}| = \frac{1}{|A|}$
4. $|I| = 1$ $\quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad |I| = 1 \times 1 \times 1 = 1$
5. $|A| = |A^T|$

- Transpuse

1. $(kA)^T = kA^T$
2. $(A+B)^T = A^T + B^T$
8. $(AB)^T = B^T A^T$ $\Rightarrow$ not $A^T . B^T$ X

- Inverse

$(AB)^{-1} = B^{-1} A^{-1}$ $\Rightarrow$ not $A^{-1} . B^{-1}$ X

أي تغيير بين صف أو عامود بنضرب بسالب

Ex : Find det (A) if

$$A = \begin{bmatrix} 2 & 3 & 7 & 9 & 7 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 7 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \end{bmatrix} \rightarrow (-1) \begin{bmatrix} 3 & 2 & 7 & 9 & 7 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 3 & 1 & 7 & 9 & 7 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \end{bmatrix} \rightarrow (-1) \begin{bmatrix} 3 & 1 & 7 & 9 & 7 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 3 & 1 & 7 & 9 & 7 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$\rightarrow |A| = 3 \times 1 \times 7 \times 4 \times 3 \times 2$

then + عدد مرات التغيير = 4

Ex:- If A is 5×3 matrix AB is 5×7 matrix what is the size of B

$$\underbrace{A_{5\times 3} \cdot B_{\boxed{3}\times\boxed{7}}}_{\text{متساويين}} = \underbrace{AB}_{5\times 7}$$

$B_{3\times7}$

Ex:- given that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$

Find:

1. $\begin{vmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{vmatrix} = 7$

اذا جمع صفين او عامودين
عادي ما بأثر على قيمة matrix

2. $\begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix} = 7$

+

بدل مرتين ما بأثر على قيمة
بس الإشارة

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$\boxed{3}$ $\begin{vmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{vmatrix} = (3)^1 \cdot 7 = 21$

لو ضربنا كلهم بنضرب القيمة بـ $(3)^3$

$B(\bar{A}) = (B)^3 |A|$

$\boxed{4}$ $\begin{vmatrix} a & b & c \\ a & b & c \\ g & h & i \end{vmatrix} = zero$

انا احدى صفوف متساوية بتكون

$det = zero$

ونفس الاشي لو كانت الاعمدة متساوية

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Ex: let A, B be 3×3 matrices with $\det(A)=4$
$\det(B)=-3$. Find:

1. $\det(AB)$

2. $\det(A^{-1})$

3. $\det(5A)$

4. $\det(A^2) = \det(A) \cdot \det(A) = 16$

5. $\det(A^T)$

→ Remark :

- If $|A| = 0$, then A^{-1} doesn't exist
- $\det |A| \neq 0$ Then

$AX = 0 \Rightarrow \boxed{X = 0}$

$A^{-1}AX = 0$

$IX = 0$

$X = 0$

EX :- det $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & a & 2 \\ 2 & 1 & a \end{bmatrix}$ Find $|A|$

$$\begin{vmatrix} 1 & 1 & 2 \\ 1 & a & 2 \\ 2 & 1 & a \end{vmatrix} \xrightarrow[-2R_1+R_3]{-R_1+R_2} \begin{vmatrix} 1 & 1 & 2 \\ 0 & a-1 & 0 \\ 0 & -1 & a-4 \end{vmatrix} = (1)(a+1)(a-4)$$

$$= (a-1)(a-4)$$

* linearly Independence :-

$V_1 = (a_1, b_1, d_1)$

$V_2 = (a_2, b_2, d_2)$

$V_3 = (a_3, b_3, d_3)$

$C_1V_1 + C_2V_2 + C_3V_3 = (0,0,0) \rightarrow$ If $c_1 = c_2 = c_3$ then $\{V_1, V_2, V_3\}$ are linearly independent If at least one of $\{c_1, c_2, c_3\}$ not equal zero. then they are dependent.

← اذا كان عدد vector (V) أكبر من dimention $R^{\boxed{3}}$ يكون dependent

In $\mathbb{R}^3$

$\Rightarrow$ $V_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\quad V_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\quad V_3 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$

~~اذا [illegible]~~
~~$R^{\boxed{3}}$ [illegible]~~ أكبر

Is $\{V_1, V_2, V_3\}$ linearly independent?

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\det(A) = 2\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -2 \neq 0$$ ← independent

*
$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{-2R_1+R_2} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -5 & 1 \end{bmatrix}$$

اذا ما تصفر
احد الصفوف
يعني independent

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 2 \end{bmatrix} \xrightarrow{-2R_1+R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ & & \end{bmatrix}$$

لانه صفر
احد صفوف ⇐
صار يعني
dependent

** Series Solution **

$\sum_{n=0}^{\infty} a_n x^n \Rightarrow$ Maclurin series

$\sum_{n=0}^{\infty} a_n (x-x_0)^n \Rightarrow$ Tayler Series

we search for asolution of the form $\boxed{y(x) = \sum_{n=0}^{\infty} a_n x^n}$

← هاي مجهولات

Ex :- use the series method to solve $y' - y = 0$ --- (*)

[series in power of x
solve about $x_0 = 0$]

let $y = \sum_{n=0}^{\infty} a_n x^n$ be solution

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

→ لأنه لما تعوض $n=0$ رح
تصفر ما إلها داعي

Substitute in (*) :

1 بنوحد ال power

2 بنوحد ال $\sum_{n=\square}$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0} a_n x^n = 0 \qquad n = n+1$$

$$\sum_{n=0} (n+1) a_{n+1} x^n - \sum_{n=0} a_n x^n = 0$$

أي زيادة داخل sum. بصير
نقصان Σ بالخارج

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$$\sum_{n=0} \left[(n+1)\, a_{n+1} - a_n\right] x^n = \boxed{0}$$

$$\sum_{n=0} \left[(n+1)\, a_{n+1} - a_n\right] x^n = \sum_{n=0} 0\, x^n$$

$\therefore$ $\boxed{(n+1)\, a_{n+1} - a_n = 0}$ $\quad n=0,1,2,\ldots$ → recurrence relation

$$a_{n+1} = \frac{a_n}{n+1} \qquad n=0,1,2,3,\ldots$$

$n=0 \rightarrow a_1 = \frac{a_0}{1} = \frac{a_0}{1!}$

$n=1 \rightarrow a_2 = \frac{a_1}{2} = \frac{a_0}{2} = \frac{a_0}{2!}$

$n=2 \rightarrow a_3 = \frac{a_2}{3} = \frac{a_0}{2\times 3} = \frac{a_0}{3!}$

$n=3 \rightarrow a_4 = \frac{a_3}{4} = \frac{a_0}{2\times 3\times 4} = \frac{a_0}{4!}$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$= a_0 + \frac{a_0}{1!} x + \frac{a_0}{2!} x^2 + \frac{a_0}{3!} x^3 + \cdots$$

$$= a_0 \left[1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right]$$

$$= a_0 \boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

$$= a_0 e^x$$

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or in Powers of x

EX:- Find a series solution near $x_0=0$ for

$$(1-x^2)y'' - 2xy' + 2y = 0$$

remark :- $a(x) = 1-x^2$

$b(x) = -2x$

$c(x) = 2$

$a(x_0) = a(0) = 1 \neq 0$ $\quad$ $x_0=0$ is an ordinary point

let $y(x) = \sum_{n=0} a_n x^n$ be a solution

$$y' = \sum_{n=1} n a_n x^{n-1}$$

$$y'' = \sum_{n=2} n(n-1) a_n x^{n-2}$$

By substitution :-

$$(1-x^2) \sum_{n=2} n(n-1) a_n x^{n-2} - 2x \sum_{n=1} n a_n x^{n-1} + 2 \sum_{n=0} a_n x^n = 0$$

$$\sum_{n=2} n(n-1) a_n x^{n-2} - \sum_{n=2} n(n-1) a_n x^n - 2 \sum_{n=1} n a_n x^n + \sum_{n=0} 2 a_n x^n = 0$$

الـPower عادي بتغيرزه مابدي 😂 بس $\sum_{n=\square}$ لازم

الصغير ارفعه للكبير

$$\sum_{\underline{n=0}} (n+2)(n+1)\, a_{n+2}\, x^n - \sum_{\boxed{n=2}} n(n-1)\, a_n x^n - \sum_{\underline{n=1}} 2na_n x^n + \sum_{\underline{n=0}} 2a_n x^n = 0$$

2 2 2

$$\underbrace{2a_2 x^0 + 6a_3 x}\; \underbrace{-\cancel{2a_1} x} + \underbrace{2a_0 + \cancel{2a_1 x}} + \sum_{n=2} \Big[(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + 2a_n \Big] x^n = 0$$

$$(2a_2+2a_0)+6a_3x + \sum_{n=2} \Big[(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + 2a_n \Big] x^n = 0$$

لانها بتساوي صفر كل معاملات X

وثوابت تساوي صفر

$2a_2 + 2a_0 = 0 \Rightarrow a_2 = -a_0$

$6a_3 = 0 \Rightarrow a_3 = 0$

$(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + 2a_n = 0 \qquad n \geq 2$

~~[illegible]~~ الأكبر

كل طرف لحاله 😂

$(n+2)(n+1)a_{n+2} = n(n-1)a_n + 2na_n - 2a_n$

$(n+2)(n+1)\, a_{n+2} = [n(n-1) + 2n - 2]\, a_n$

$(n+2)(n+1)\, a_{n+2} = [n^2 + n - 2]\, a_n$

$$a_{n+2} = \frac{(\cancel{n+2})(n-1)\, a_n}{(\cancel{n+2})(n+1)}$$

$$a_{n+2} = \frac{(n-1)}{(n+1)}\, a_n \qquad n \geq 2$$

$y(x) = a_0 + a_1x + a_2x^2 + a_3\cancel{x^3} + a_4x^4 + a_5x^5 + a_6x^6 + \dots$

$\boxed{n=2}$ $a_4 = \frac{1}{3} a_2 = -\frac{1}{3} a_0$

$\boxed{n=3}$ $a_5 = \frac{2}{4} a_3 = 0$

$\boxed{n=4}$ $a_6 = \frac{3}{5} a_4 = \frac{\cancel{3}}{5} \cdot -\frac{1}{\cancel{3}} a_0 = -\frac{1}{5} a_0$

$\Rightarrow y(x) = a_0 + a_1 x - a_0 x^2 - \frac{1}{3} a_0 x^4 - \frac{1}{5} a_0 x^6 \dots$

$= \underbrace{a_0 \left[1 - x^2 - \frac{1}{3} x^4 - \frac{1}{5} x^6 \dots\right]} + \underbrace{a_1 x}$

* $e^x = \sum_{n=0} \frac{x^n}{n!}$

* $\frac{1}{1-x} = \sum_{n=0} x^n$

* $\cos x = \sum_{n=0} (-1)^n \frac{x^{2n}}{(2n)!}$

* $\sin x = \sum_{n=0} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

Ex :- Find a series solution near $x=0$ for

$$y'' - xy = 0$$

let $y = \sum_{n=0} a_n x^n$

$$y' = \sum_{n=1} n a_n x^{n-1}$$

$$y'' = \sum_{n=2} n(n-1) a_n x^{n-2}$$

$$\sum_{n=2} n(n-1) a_n x^{n-2} - \sum_{n=0} a_n x^{n+1} = 0$$

$$\sum_{n=0} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1} a_{n-1} x^n = 0$$

$$2a_2 + \sum_{n=1} [(n+2)(n+1) a_{n+2} - a_{n-1}] x^n = 0$$

$2a_2 = 0 \Rightarrow a_2 = 0$

$(n+2)(n+1) a_{n+2} - a_{n-1} = 0 \qquad n \geq 1$

$$a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}$$

$\boxed{n=1}$ $a_3 = \frac{a_0}{6}$

$\boxed{n=2}$ $a_4 = \frac{a_1}{12}$

$\boxed{n=3}$ $a_5 = \frac{a_2}{20} = 0$

$\boxed{n=4}$ $a_6 = \frac{a_3}{30} = \frac{a_0}{180}$

⋮

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* Froberius method :-

$a(x)\,y'' + b(x)\,y' + c(x)\,y = 0$

→ If $a(x_0) \neq 0$ then x_0 is called an oredary point

→ If $a(x_0) = 0$ the x_0 is called an singular point

a singular point
- → regular singular point
- → irregular singular point ← هاي مش مطلوبة

→ a singular point of x is called a regular point if :-

$\boxed{1}$ $\lim\limits_{x \to x_0} (x - x_0)\dfrac{b(x)}{a(x)} < \infty \quad b_0$

$\lim\limits_{x \to x_0} (x - x_0)^2 \dfrac{c(x)}{a(x)} < \infty \quad c_0$

they both must be exist

* The indicial equation is :-

$r(r-1) + b_0 r + c_0 = 0$ ← (we use it when it's regular singular equation)

Ex:- determine all regular singular point to the following equation :-

$$\underbrace{2x(x-2)^2}_{a(x)}\, y'' + \underbrace{3x}_{b(x)}\, y' + \underbrace{(x-2)}_{c(x)}\, y = 0$$

1. we must find the singular point where $a(x) = 0$

$2x(x-2)^2 = 0 \rightarrow \boxed{x = 0}\ .\ \boxed{x = 2}$ singular point

at $x = 0$:-

1. $\lim\limits_{x\to 0} (x-0) \dfrac{3x}{2x(x-2)^2} = \lim\limits_{x\to 0} \dfrac{3x^{\cancel{2}}}{2\cancel{x}(x-2)^2} = \lim\limits_{x\to 0} \dfrac{3x}{2(x-2)^2} = \dfrac{0}{8} = 0$

2. $\lim\limits_{x\to 0} \dfrac{x^2 \cancel{(x-2)}}{2x(x-2)^2} = \lim\limits_{x\to 0} \dfrac{x}{2(x-2)} = 0$

→ both (lim) are = to zero then $x = 0$ is a regular singular point

at $x = 2$:-

1. $\lim\limits_{x\to 2} \dfrac{\cancel{(x-2)}\ 3\cancel{x}}{2\cancel{(x)}(x-2)^{\cancel{2}}} = \lim\limits_{x\to 2} \dfrac{3}{2(x-2)} = \dfrac{3}{0} = \infty$

ما ضايل
باقي أحل

→ then $x_0 = 2$ is an irregular singular point

The indicial equation for $x_0 = 0$ is

$r(r-1) + b_0 r + c_0 = 0 \longrightarrow r(r-1) = 0$

* * Frobenius method * *

when x_0 is regular singular point :-

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \qquad a_0 \neq 0$$

$$y'(x) = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

يعني انو نقطة regular singular

EX:- use Frobenius method : $2xy'' + y' + xy = 0$

$a(x) = 0$

$2x = 0 \rightarrow \boxed{x = 0}$

it's a singular point

وإنه طالب بالسؤال هاي method يعني

أكيد regular.

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y'(x) = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$$

$a_0 \neq 0$

$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} + x \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

بدي اوحد القوى بس
لو بدي اوحدهن $n+r$
رح نبلش $n=1$

$$\sum_{\boxed{n=0}}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r-1} + \sum_{\boxed{n=0}}^{\infty} a_n (n+r) x^{n+r-1} + \sum_{\boxed{n=2}} a_{n-2} x^{n+r-1} = 0$$

$$\left(2r(r-1) a_0 x^{r-1}\right) + \left(2(r+1) r a_1 x^r\right) + \left(a_0 r x^{r-1}\right) + \left(a_1 (r+1) x^r\right) +$$

$$\sum_{n=2}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=2}^{\infty} a_n (n+r) x^{n+r-1} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} = 0$$

$$\left(2r(r-1)+r\right) a_0 x^{r-1} + \left(2(r+1)r + (r+1)\right) a_1 x^r + \sum_{n=2}^{\infty} \left(2(n+r)(n+r-1) a_n + a_n (n+r) + a_{n-2}\right) x^{n+r-1} = 0$$

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$\boxed{a_0 \neq 0}$

$(2r(r-1)\ \ r)\,a_0 = 0$

$(2r(r+1) + r + 1)\,a_1 = 0$

$(2(n+r)(n+r-1)a_n + (n+r)a_n + a_{n-2}) = 0$ reurence relation

$n \geq 2$

$(2r(r-1) + r)\,a_0 = 0$

$\boxed{a_0 \neq 0}$ then $2r(r-1) + r = 0$ Indicial equation

$2r^2 - r = 0$

$r(2r-1) = 0$ $\to r_1 = 0$, $\to r_2 = \frac{1}{2}$

$r_2 - r_1 \to \frac{1}{2} - 0 = \frac{1}{2}$

← اذا كان فرق بين r عدد عدد مش صحيح فيكون في حلين

← اذا كان فرق بين r عدد صحيح يكون في حل واحد

when $\boxed{r=0}$ then :-

$\to [2r(r+1) + r + 1]\,a_1 = 0 \to \boxed{a_1 = 0}$

$\to 2(n+r)(n+r-1)a_n + (n+r)a_n + a_{n-2} = 0$

$[2n(n-1) + n]\,a_n = -a_{n-2}$ $\quad n \geq 2$

$$a_n = -\frac{a_{n-2}}{2n(n-1) + n}$$

$$a_n = -\frac{a_{n-2}}{n(2n-1)}$$

$\boxed{n=2}$ $\quad a_2 = \dfrac{-a_0}{(2)(3)}$

$\boxed{n=3}$ $\quad a_3 = \dfrac{-a_1}{\square} = 0$

$\boxed{n=4}$ $\quad a_4 = \dfrac{-a_2}{(4)(7)} = \dfrac{a_0}{(2)(3)(4)(7)}$

يكفي ٣ قيم
بس يكون كل الفكرة
في نفسه !!

$\boxed{n=5}$ $\quad a_5 = \dfrac{-a_3}{\square} = 0$

$\boxed{n=6}$ $\quad a_6 = \dfrac{-a_4}{(6)(11)} = \dfrac{+a_0}{(2)(4)(6)\ (3)(7)(11)}$

$\Rightarrow a_{2n-1} = 0$ odd

$$a_{2n} = \frac{a_0}{\underbrace{2}_{\text{هاي البداية}} \cdot 4 \cdot \ldots \underbrace{(2n)}_{\text{هاي نهاية}} \cdot \underbrace{(3)}_{\text{بداية}}(7)\ldots\underbrace{(4n-1)}_{\text{نهاية}}}$$ even

هاد سؤال مش مطلوب اطلع صيغة بس ممكن يطلب

$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$

$= a_0 + 0 + \dfrac{-a_0}{(2)(3)} x^2 + 0 + \dfrac{a_0}{(2)(4)\ (3)(7)} x^4 \ldots$

$= a_0 \left[1 - \dfrac{1}{(2)(3)} x^2 + \dfrac{1}{2(4)(3)7} x^4 \ldots \right]$

when $\boxed{r=\frac{1}{2}}$ then :-

$$[2r(r+1)+r+1]\,a_1=0 \longrightarrow a_1=0$$

$$2(n+\tfrac{1}{2})(n-\tfrac{1}{2})a_n+(n+\tfrac{1}{2})a_n+a_{n-2}=0$$

$$2(n^2-\tfrac{1}{4})a_n+(n+\tfrac{1}{2})a_n+a_{n-2}=0$$

$$2n^2a_n-\tfrac{1}{2}a_n+a_nn+\tfrac{1}{2}a_n=-a_{n-2}$$

$$a_n=\frac{-a_{n-2}}{n(2n+1)} \qquad n\geq 2$$

$\boxed{n=2}$ $$a_2=\frac{-a_0}{(2)(5)}$$

$\boxed{n=3}$ $$a_3=\frac{-a_1}{\boxed{}}=0$$

$\boxed{n=4}$ $$a_4=\frac{-a_2}{(4)(9)}=\frac{a_0}{(2)(4)\ (9)(5)}$$

$$a_{2n-1}=0$$

$$a_{2n}=\frac{\pm a_0}{2.4\ldots(2n)\ (5)(9)\cdots(4n+1)}$$

When $r = \frac{1}{2} \rightarrow x^{\frac{1}{2}}$

$$y = \sum_{n=0} a_n x^{n+\frac{1}{2}}$$

$$= x^{\frac{1}{2}} \sum_{n=0} a_n x^n$$

$$y = a_0 + a_1 x_1 + a_2 x^2 + a_4 x^4 \dots$$

$$= x^{\frac{1}{2}} a_0 \left[1 - \frac{x^2}{2 \times 5} \dots \right]$$

لكل r يوجد حل واحد

Ex:- use Froberius Method to solve :-

$$(x^2-x)y'' - xy' + y = 0 \qquad \text{near } x_0 = 0$$

$\rightarrow a(x) = x^2 - x$

$a(x) = 0 \rightarrow x^2 - x = 0$

$x(x-1) = 0 \rightarrow x = 0, 1$

$\boxed{x_0 = 0}$

[1] $\lim\limits_{x\to 0} x \cdot \dfrac{-x}{x^2-x} = \lim\limits_{x\to 0} \dfrac{-x^2}{\cancel{x}(x-1)}$

$= \lim\limits_{x\to 0} \dfrac{-x}{x-1} = \dfrac{0}{-1} = 0 \leftarrow b_0$

[2] $\lim\limits_{x\to 0} x^2 \dfrac{1}{x^2-x} = \lim\limits_{x\to 0} \dfrac{x^2}{\cancel{x}(x-1)} = \dfrac{0}{-1} = 0 \leftarrow C_0$

** The indicial equation is : $r(r-1) + \cancel{b_0} + \cancel{c_0} = 0$

$r(r-1) = 0 \rightarrow r = 1, 0$

$1 - 0 = 1 \in \mathbb{Z}$

فرق بينهم عدد صحيح يعني رح يكون حل واحد وبنوخذ ال r الأكبر

وهنا لو كان $r = 2, 2$ فرقا عدد صحيح فيكون حل واحد

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$\boxed{r=1}$

let $y = \sum_{n=0} a_n x^{n+1}$

$y' = \sum_{n=0} (n+1) a_n x^n$

$y'' = \sum_{\boxed{n=1}} n(n+1) a_n x^{n-1}$

بلشت من الواحد
لانه لو صفر بطلعو ما
إلها داعي

$\Rightarrow (x^2-x)\sum_{n=1} n(n+1)a_n x^{n-1} - x\sum_{n=0}(n+1)a_n x^n + \sum_{n=0} a_n x^{n+1} = 0$

$\sum_{n=1} n(n+1)a_n x^{n+1} - \sum_{n=1} n(n+1)\overset{a_n}{} x^{n} - \sum_{n=0}(n+1)a_n x^{n+1} + \sum_{n=0} a_n x^{n+1} = 0$

$\sum_{n=1} n(n+1)a_n x^{n+1} - \sum_{n=0}(n+1)(n+2)a_{n+1} x^{n+1} - \sum_{n=0}(n+1)a_n x^{n+1} + \sum_{n=0} a_n x^{n+1} = 0$

$\underbrace{-2a_1 x} - \cancel{a_0} x + \cancel{a_0} x + \sum_{n=1}\underline{\left(n(n+1)a_n - (n+1)(n+2)a_{n+1} - (n+1)a_n + a_n\right)} x^{n+1} = 0$

$-2a_1 = 0 \qquad \boxed{a_1 = 0}$

$n(n+1)a_n - (n+1)(n+2)a_{n+1} - (n+1)a_n + a_n = 0 \qquad n \geq 1$

$n^2 a_n + \cancel{n a_n} - \cancel{n a_n} \cancel{-a_n} + \cancel{a_n} = +(n+1)(n+2)a_{n+1}$

$a_{n+1} = \dfrac{n^2 a_n}{(n+1)(n+2)} \qquad n \geq 1$

$\boxed{n=1}$ $\quad a_2 = \dfrac{a_1}{2 \times 3} = 0$

$\boxed{n=2}$ $\quad a_3 = \dfrac{a_2}{\square} = 0$

$\vdots$

$$y = a_0 x + \cancel{a_1 x^2} + \cancel{a_3 x^3} + a_4 x^4 + \cdots$$

$\boxed{y = a_0 x}$ general solution , $a_0 \to$ any constant

$$y_1 = x$$

$$y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx$$

$$\Rightarrow -\int p(x)\,dx = -\int \frac{-x}{(x^2 - x)}\,dx$$

P(x) معامل y' لما y'' معاملها 1

$$= \int \frac{1}{x-1}\,dx = \ln|x-1|$$

$$\Rightarrow e^{-\int p(x)\,dx} = e^{\ln|x-1|} = x-1$$

$$y_2 = x \int \frac{x-1}{x^2} \cdot dx$$

$$= x \int \frac{x}{x^2} - \frac{1}{x^2} \cdot dx$$

$$= x \int \frac{1}{x} - \frac{1}{x^2} \cdot dx$$

$$= x \left[\ln|x| + \frac{1}{x} \right]$$

note $\Rightarrow$ $y'' - xy = 0$ near $\Rightarrow x_0 = 1$

$y = \sum_{n=0} a_n (x-1)^n$ let $t = x - 1$

$y = \sum_{n=0} a_n t^n$

$y'' - (t+1) y = 0$

** laplace Transform **

$$L(f(t)) = \int_0^\infty f(t)\, e^{-st} \cdot dt$$

Ex:- Find $L(1)$

$$L(1) = \int_0^\infty e^{-st}\, dt$$

$$\int_0^T e^{-st}\, dt$$

$$-\frac{e^{-st}}{s} \Big|_0^t$$

$$\frac{e^{-sT}}{-s} + \frac{1}{s}$$

$$\lim_{T\to\infty} \left[\frac{e^{-sT}}{-s} + \frac{1}{s} \right] = 0 + \frac{1}{s} = \frac{1}{s} \qquad s>0$$

$$\lim_{T\to\infty} \frac{e^{-sT}}{-s} = \frac{-e^{-s\infty}}{+s} \qquad \begin{cases} \infty & s<0 \\ 0 & s>0 \end{cases}$$

* $L(1) = \frac{1}{s}$

* $L(e^{at}) = \frac{1}{s-a}$

* $L(t^n) = \frac{n!}{s^{n+1}}$ $\quad n = 0, 1, \ldots$

* $L(\sin at) = \frac{a}{s^2+a^2}$

* $L(\cos at) = \frac{s}{s^2+a^2}$

* $L(\sinh at) = \frac{a}{s^2-a^2}$

* $L(\cosh at) = \frac{s}{s^2-a^2}$

* $L(f(t) + g(t)) = L(f(t)) + L(g(t))$

→ Notation $L(f(t)) = F(s)$

$L(g(t)) = G(s)$

$\Rightarrow L(e^{2t}) = \frac{1}{s-2}$

$L(e^{-3t}) = \frac{1}{s+3}$

$L(t) = \frac{1}{s^2}$ *

$L(t^2) = \frac{2}{s^3}$ * ⇐ الافضل انه نحفظ هدول

$L(t^3) = \frac{6}{s^4}$ *

$\Rightarrow L(t) = \frac{1}{s^2}$

$- L^{-1}\left(\frac{1}{s^2}\right) = t$

$\Rightarrow L(t^3) = \frac{6}{s^4}$

$- L^{-1}\left(\frac{1}{s^4}\right) = \frac{1}{6} L^{-1}\left(\frac{6}{s^4}\right) = \frac{1}{6} t^3$

$\Rightarrow L(f(t)) = F(s)$

$L^{-1}(F(s)) = f(t)$

Ex :- Find $\int_0^\infty \sin 2t\, e^{-st}\, dt$

$$L(\sin 2t) = \frac{2}{s^2+4}$$

$\Rightarrow$ $L(\sinh at) = \frac{a}{s^2-a^2}$ اثبتها :

$L(\sinh at)$

$$= L\left(\frac{e^{at} - e^{-at}}{2}\right)$$

$$= \frac{1}{2}\left[L(e^{at}) - L(e^{-at})\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\ \frac{s+a-(s-a)}{s^2-a^2}$$

$$= \frac{1}{2}\ \frac{2a}{s^2-a^2}$$

$$= \frac{a}{s^2-a^2} \quad \#$$

Theorem :-

$L(f(t)) = F(s)$

$L(f(t)\,e^{at}) = F(s-a)$

$L(f(t))\Big|_{s-a}$

عادي بحسب laplace لل f
بعدين بعدما انتهي بحط بدل
كل $s \leftarrow s-a$

Ex:- Find $L(e^{2t}\sin 4t)$

$$= L(\sin 4t)\Big|_{s-2} = \frac{4}{s^2+16}\Big|_{s-2} = \frac{4}{(s-2)^2+16}$$

Ex:- Find $L(e^{-2t}\cosh 3t)$

$$= L(\cosh 3t)\Big|_{s+2} = \frac{s}{s^2-9}\Big|_{s+2} = \frac{s+2}{(s+2)^2-9}$$

Ex:- Find $L^{-1}\left(\frac{3}{(s-2)^2+9}\right)$ — بعمل shift

$$e^{2t}\,L^{-1}\left(\frac{3}{s^2+9}\right)$$

$$= e^{2t}\sin 3t$$

- $L(f'(t)) = sF(s) - f(0)$

- $L(f''(t)) = s^2 F(s) - sf(0) - f'(0)$

Ex:- solve the IVP :-

$y'' - y = t$ $\qquad y(0) = 1$

$\qquad\qquad y'(0) = 1$

$L(y''(t)) - L(y) = L(t)$

$s^2 Y(s) - sy(0) - y'(0) - Y(s) = \frac{1}{s^2}$

$s^2 Y(s) - s - 1 - Y(s) = \frac{1}{s^2}$

$s^2 Y(s) - Y(s) = \frac{1}{s^2} + s + 1$

$Y(s)(s^2 - 1) = \frac{1}{s^2} + s + 1$

$Y(s) = \frac{1}{s^2(s^2-1)} + \frac{s}{s^2-1} + \frac{1}{s^2-1}$

$\Downarrow$

$y(t) = L^{-1}\left(\frac{1}{s^2(s^2-1)} + \frac{s}{s^2-1} + \frac{1}{s^2-1}\right)$

$= L^{-1}\left(\frac{1}{s^2(s^2-1)}\right) + \cosh t + \sinh t$

* $\cosh t = \frac{e^t - e^{-t}}{2}$

* $L(t f(t)) = -F'(s)$

* $L\left(\int_0^t f(u)\,du\right) = \frac{F(s)}{s}$

$L^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t f(u)\,du$

Ex:- $L(t \sin 4t) = -\left(\frac{4}{s^2+16}\right)'$

$= -4\left(\frac{1}{s^2+16}\right)'$

$= -4\left(-\frac{2s}{(s^2+16)^2}\right)$

$= \frac{8s}{(s^2+16)^2}$

$\Rightarrow L^{-1}\left(\frac{1}{s(s^2-1)}\right) = L^{-1}\left(\frac{\overbrace{\frac{1}{s^2-1}}^{F(s)}}{s}\right)$

$= \int_0^t \sinh u \,.du = \cosh u \Big|_0^t$

$= \cosh t - 1$

Ex:- Find : $L^{-1}\left(\frac{1}{s^2(s^2-1)}\right)$

$$L^{-1}\left(\frac{\boxed{\frac{1}{s(s^2-1)}}^{F(x)}}{s}\right) = \int_0^t \cosh u - 1 \, .du$$

$$= (\sinh u - u)\Big|_0^t$$

$$= \sinh t - t$$

H.W :- Find :- $L^{-1}\left(\frac{1}{s(s^2+1)}\right)$ // $L^{-1}\left(\frac{1}{s^2(s^2+1)}\right)$

Ex:- $L^{-1}\left(\frac{2}{s(s^2+4)}\right)$

$$= L^{-1}\left(\frac{\frac{2}{s^2+4}}{s}\right) = \int_0^t \sin 2u \, .du = \frac{-\cos 2u}{2}\Big|_0^t$$

$$= \frac{\cos 2u}{2}\Big|_t^0$$

$$= \frac{1}{2} - \frac{\cos 2t}{2}$$

Ex: $\boxed{1}$ $L^{-1}\left(\dfrac{s+\boxed{2}}{(s+\boxed{2})^2+4}\right)$

$= e^{2t} L^{-1}\left(\dfrac{s}{s^2+4}\right) = e^{2t}\cos 2t$

$\boxed{2}$ $L^{-1}\left(\dfrac{s+1}{(s-3)^2+4}\right)$ لازم أعمل shift

$$\frac{s+1}{(s-3)^2+4} = \frac{(s-3)+4}{(s-3)^2+4} = \frac{s-3}{(s-3)^2+4} + \frac{4}{(s-3)^2+4}$$

$$= L^{-1}\left(\frac{s-3}{(s-3)^2+4} + \frac{4}{(s-3)^2+4}\right)$$

$$= L^{-1}\left(\frac{s-3}{(s-3)^2+4}\right) + L^{-1}\left(\frac{4}{(s-3)^2+4}\right)$$

$$= e^{3t} L^{-1}\left(\frac{s}{s+4}\right) + 2e^{3t} L^{-1}\left(\frac{2}{s^2+4}\right)$$

$$= e^{3t}\cos 2t + 2e^{3t}\sin 2t$$

EX:- $L^{-1}\left(\dfrac{s}{s^2+2s+2}\right)$ ← مابتتحلل

بس إكمال مربع

$$\frac{s}{(s^2+\boxed{2}s+1)-1+2} = \frac{s}{(s+1)^2+1} = \frac{(s+1)-1}{(s+1)^2+1}$$

$\boxed{2} \to \left(\frac{2}{2}\right)^2$

$$\frac{(s+1)}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}$$

$$\to L^{-1}\left(\frac{(s+1)}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right)$$

$$= L^{-1}\left(\frac{(s+1)}{(s+1)^2+1}\right) - L^{-1}\left(\frac{1}{(s+1)^2+1}\right)$$

$$= e^{-t}\, L^{-1}\left(\frac{s}{s^2+1}\right) - e^{-t}\, L^{-1}\left(\frac{1}{s+1}\right)$$

$$= e^{-t}\cos t - e^{-t}\sin t$$

\#

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اذا في بالمقام معادلة تربيعية في عندي خيارين $\boxed{1}$ إكمال مربع

$\boxed{2}$ تحليل

Ex :- $L^{-1}\left(\dfrac{s-1}{s^2-s-2}\right)$

$$\frac{s-1}{s^2-s-2} = \frac{s-1}{(s-2)\,(s+1)} = \frac{A}{(s-2)} + \frac{B}{(s+1)}$$

$$= \frac{A(s+1) + B(s-2)}{(s-2)(s+1)}$$

$\rightarrow A(s+1) + B(s-2) = s-1$

$\boxed{s=-1}$ $\quad -3B = -2 \quad \rightarrow B = \dfrac{2}{3}$

$\boxed{s=2}$ $\quad 3A = 1 \quad \rightarrow A = \dfrac{1}{3}$

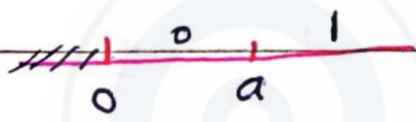
$$\frac{s-1}{s^2-s-2} = \frac{\frac{1}{3}}{(s-2)} + \frac{\frac{2}{3}}{(s+1)}$$

$$L^{-1}\left(\frac{s-1}{s^2-s-2}\right) = L^{-1}\left(\frac{\frac{1}{3}}{(s-2)}\right) + L^{-1}\left(\frac{\frac{2}{3}}{s+1}\right)$$

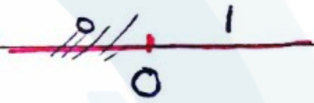
$$= \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

** Unit step function **

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$$



$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$



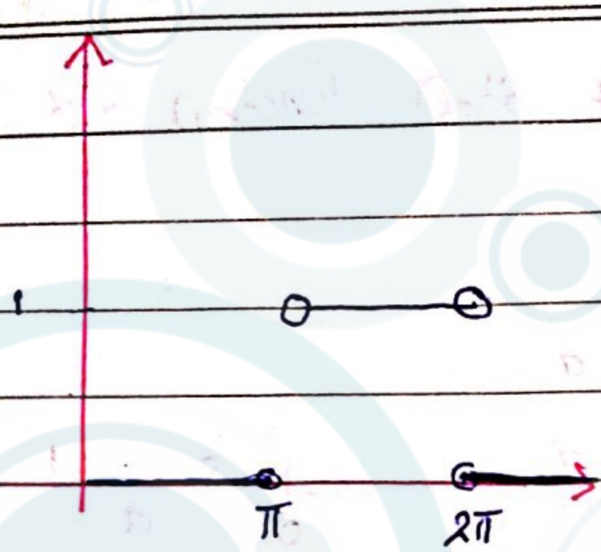
$u(t) = 1 \quad , t \geq 0$

EX:- Sketch $f(t) = u(t-\pi) - u(t-2\pi)$

$u(t-\pi)$

$u(t-2\pi)$

$u(t-\pi) - u(t-2\pi)$



Find:- $f(\frac{\pi}{2}) = 0 - 0 = 0$

$f(3\pi) = 1 - 1 = 0$

$$L(u(t-a)) = \frac{e^{-as}}{s}$$

$$L(u(t)) = \frac{1}{s}$$

$$L(\underline{u(t-a)}\, f(t)) = e^{-as} L(f(t+a))$$

Ex:- $L(u(t-\pi)\sin t) = e^{-\pi s} L(\sin(t+\pi))$

$\sin(\pi + t) = -\sin t$

$\cos(\pi + t) = -\cos t$

$\sin(\pi - t) = \sin t$

$\cos(\pi - t) = -\cos t$

$$= e^{-\pi s} L(-\sin t)$$

$$= -e^{-\pi s} \frac{1}{s^2+1}$$

Ex:- Find the laplace transform of :- **مهم**

$$f(t) = \begin{cases} at^2 & 0 \le t < 1 \\ e^{2t} & 1 \le t < 4 \\ 1 & t \ge 4 \end{cases}$$

$$f(t) = 2t^2 \left(u(t-0) - u(t-1) \right) + e^{2t} \left(u(t-1) - u(t-4) \right) + 1 . u(t-4)$$

$$f(t) = 2t^2 - 2t^2 u(t-1) + e^{2t} u(t-1) - e^{2t} u(t-4) + u(t-4)$$

$$L(f(t)) = L(2t^2) - L(2t^2 u(t-1)) + L(e^{2t} u(t-1)) - L(e^{2t} u(t-4)) + L(u(t-4))$$

① $L(2t^2) = 2L(t^2) = 2 \cdot \frac{2}{s^3}$

$$= \frac{4}{s^3}$$

② $L(2t^2 u(t-1)) = 2e^{-s} \; L((t+a)^2)$

$$= 2e^{-s} \; L((t+1)^2) = 2e^{-s} \; L(t^2 + 2t + 1)$$

$$= 2e^{-s} \left[\frac{2}{s^3} + \frac{2 \cdot 1}{s^2} + \frac{1}{s} \right]$$

$$L(e^{2t}u(t-1)) = e^{-s} L(e^{2(t+1)})$$

$$= e^{-s} L(e^{2t} e^{2})$$

$$= e^{2-s} \frac{1}{s-2}$$

OR

$$L(e^{2t}u(t-1)) = L(u(t-1))_{s-2}$$

$$= \frac{e^{-s}}{s} \Big|_{s-2}$$

$$= \frac{e^{2-s}}{s-2}$$

$$L(u(t-4)) = \frac{e^{-4s}}{s}$$

* $L(u(t-a)\,f(t)) = e^{-as}\, L(f(t+a))$

* $L^{-1}(e^{-as} F(s)) = u(t-a)\, L^{-1}(F(s))_{t-a}$

Ex:- Find

$\boxed{1}$ $L^{-1}\left(\frac{e^{-3s}}{s^2}\right) = u(t-3)\, L^{-1}\left(\frac{1}{s^2}\right)_{t-3}$

$= u(t-3)\ (t-3)$

$\boxed{2}$ $L^{-1}\left(\frac{s e^{-2s}}{s^2+4}\right) = u(t-2)\, L^{-1}\left(\frac{s}{s^2+4}\right)_{t-2}$

$= u(t-2)\ \cos 2(t-2)$

$\boxed{3}$ $L^{-1}\left(\frac{3 e^{-2s}}{s}\right)$

$3\, u(t-2)\ L^{-1}\left(\frac{1}{s}\right)_{t-2}$

$3u(t-2)\ 1$

$3u(t-2)$

** Divar Delta Function **

$$\delta(t-a) = \begin{cases} \infty & t = a \\ 0 & t \neq a \end{cases}$$

1. $\int_0^\infty \delta(t-a)\,dt = 1$

2. $\int_0^\infty \delta(t-a)\,g(t)\,dt = g(a)$

- $L(\delta(t-a)) = e^{-as}$

- $L(\delta(t)) = 1$

Ex:- Find $L(\delta(t-2\pi)\cos t)$

$$L(\delta(t-2\pi)\cos t) = \int_0^\infty \delta(t-2\pi)\,\underbrace{\boxed{\cos t\; e^{-st}}}_{g(t)}\,dt$$

$$= g(2\pi)$$

$$= \cos 2\pi \quad e^{-s2\pi}$$

$$= e^{-2\pi s}$$

3. $L(y'(t)) = s\,Y(s) - y(0)$

4. $L(y''(t)) = s^2\,Y(s) - s\,y(0) - y'(0)$

Ex:- solve: $y' + y = \delta(t-1)$, $y(0) = 1$

$$L(y'(t)) + L(y(t)) = L(\delta(t-1))$$

$$s\,Y(s) - y(0) + Y(s) = e^{-s}$$

$$(s+1)\,Y(s) - 1 = e^{-s}$$

$$Y(s) = \frac{e^{-s} + 1}{s+1} \quad \Leftarrow L \text{ of Solution}$$

$$y(t) = L^{-1}\left(\frac{e^{-s} + 1}{s+1}\right)$$

$$y(t) = L^{-1}\left(\frac{e^{-s}}{s+1}\right) + L^{-1}\left(\frac{1}{s+1}\right)$$

$$= u(t-1)\, L^{-1}\left(\frac{1}{s+1}\right)\Big|_{t-1} + e^{-t}$$

$$= u(t-1)\, e^{-t}\Big|_{t-1} + e^{-t}$$

$\Rightarrow y(t) = u(t-1)\, e^{-(t-1)} + e^{-t}$

Ex:- solve : $y'' + 3y' + 2y = \delta(t-1)$, $y(0) = 0$ $y'(0) = 0$

$L(y''(t)) + 3L(y'(t)) + 2L(y(t)) = L(\delta(t-1))$

$s^2 Y(s) - 0 + 3(sY(s)) + 2Y(s) = e^{-s}$

$(s^2 + 3s + 2) Y(s) = e^{-s}$

$Y(s) = \dfrac{e^{-s}}{s^2+3s+2}$ $\Leftarrow$ L^{-1} of solution

$y(t) = L^{-1}\left(\dfrac{e^{-s}}{s^2+3s+2}\right)$

$= u(t-1) \quad L^{-1}\left(\dfrac{1}{s^2+3s+2}\right)_{t-1}$

$\Rightarrow L^{-1}\left(\dfrac{1}{s^2+3s+2}\right)$ معادلة تربيعية يا إما اكمال مربع أو تحليل

$$\frac{1}{s^2+3s+2} = \frac{1}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1}$$

$$\frac{1}{(s+2)(s+1)} = \frac{A(s+1)}{(s+2)(s+1)} + \frac{B(s+2)}{(s+2)(s+1)}$$

$$1 = A(s+1) + B(s+2)$$

$$\boxed{s=-1} \quad 1 = B \quad \rightarrow \boxed{B=1}$$

$$\boxed{s=-2} \quad 1 = -A \quad \rightarrow \boxed{A=-1}$$

$$L^{-1}\left(\frac{1}{s^2+3s+2}\right)_{t-1} = L^{-1}\left(\frac{-1}{s+2} + \frac{1}{s+1}\right)_{t-1}$$

$$= L^{-1}\left(\frac{-1}{s+2}\right)_{t-1} + L^{-1}\left(\frac{1}{s+1}\right)_{t-1}$$

$$= \left[-e^{-2t} + e^{-t}\right]_{t-1}$$

$$= e^{-1(t-1)} - e^{-2(t-1)}$$

$$\Rightarrow y(t) = u(t-1)\left[e^{-(t-1)} - e^{-2(t-1)}\right]$$

Remark 8-

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$= \frac{\frac{1}{2}}{s} + \frac{-1}{s+1} + \frac{\frac{1}{2}}{s+2}$$

$$L^{-1}\left(\frac{1}{s(s+1)(s+2)}\right) = L^{-1}\left(\frac{\frac{1}{2}}{s}\right) + L^{-1}\left(\frac{-1}{s+1}\right) + L^{-1}\left(\frac{\frac{1}{2}}{s+2}\right)$$

$$= \frac{1}{2} + -e^{-t} + \frac{1}{2}e^{-2t}$$

- $L(1) = \frac{1}{s}$
- $L(5) = \frac{5}{s}$

تعني شرط

زي هيك ما بحكي اني احلها بال L

Ex :- solve the IVP :- L

$$y'' + 3y' + 2y = \begin{cases} 1 & 0 \le t < 1 \\ 0 & t \ge 1 \end{cases}$$

بدوله إن unit F. بي ما بنعرف

نحل إلا بال L

$y(0) = 0$, $y'(0) = 0$

Sol.

$$y'' + 3y' + 2y = 1\,[u(t-0) - u(t-1)] + 0\,[u(t-1)]$$

$$y'' + 3y' + 2y = 1 - u(t-1)$$

Take Laplace to both sides

$$L(y''(t)) + 3L(y'(t)) + 2L(y(t)) = L(1) - L(u(t-1))$$

$$s^2 Y(s) - s\cancel{y(0)} - \cancel{y'(0)} + 3[sY(s) - \cancel{y(0)}] + 2Y(s) = \frac{1}{s} - \frac{e^{-s}}{s}$$

$$(s^2 + 3s + 2)\,Y(s) = \frac{1}{s} - \frac{e^{-s}}{s}$$

$$Y(s) = \frac{1}{s(s^2+3s+2)} - \frac{e^{-s}}{s(s^2+3s+2)}$$

اذا طلب Laplace of the solution

$$y(t) = L^{-1}\left(\frac{1}{s(s^2+3s+2)}\right) - L^{-1}\left(\frac{e^{-s}}{s(s+3s+2)}\right)$$

$$\Rightarrow \frac{1}{s(s^2+3s+2)} = \frac{1}{s(s+1)(s+2)} = \frac{\frac{1}{2}}{s} + \frac{\frac{1}{2}}{s+2} - \frac{1}{s+1}$$

$$L^{-1}\left(\frac{1}{s(s^2+3s+2)}\right) = \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t}$$

حليتها قبل
اكمن صفحة

$$\Rightarrow L^{-1}\left(\frac{e^{-s}}{s(s^2+3s+2)}\right) = u(t-1)\, L^{-1}\left(\frac{1}{s(s^2+3s+2)}\right)_{t-1}$$

$$u(t-1)\left(\frac{1}{2} + \frac{1}{2}e^{-2(t-1)} - e^{-(t-1)}\right)$$

$$\Rightarrow y(t) = \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t} - u(t-1)\left(\frac{1}{2} + \frac{1}{2}e^{-2(t-1)} - e^{-(t-1)}\right)$$

Ex :- solve the IVP :- (use laplace)

$X(0) = 4$

$x'(t) - 2y(t) = 4t$ $\qquad y(0) = -5$

$y'(t) + 2y(t) - 4x(t) = -4t - 2$

لو ماحدد الطريقة ⇒ $y'' + 2y' - \underline{4x'} = -4t - 2$ اشتق معادلة ثانية

من معادلة الاولى

$y'' + 2y' - 4(2y(t) + 4t) = -4t - 2$

$y'' + 2y' - 8y(t) = 16t - 4$

عادي شغل نكمل !!

use laplace ⇒ $L(x'(t)) - 2L(y(t)) = L(4t)$

$L(y'(t)) + 2L(y(t)) - 4L(x(t)) = -4t - 2$

$sX(s) - x(0) - 2Y(s) = 4\frac{1}{s^2}$

$sX(s) - 4 - 2Y(s) = \frac{4}{s^2}$

$sX(s) - 2Y(s) = 4 + \frac{4}{s^2}$ — 1

$sY(s) - y(0) + 2Y(s) - 4X(s) = -\frac{4}{s^2} - \frac{2}{s}$

$(s+2)Y(s) - 4X(s) = -\frac{4}{s^2} - \frac{2}{s} - 5$ ...... ②

* تكملة لسؤال ٥ :-

$(s+2)\boxed{1} \Rightarrow s(s+2)X - 2(s+2)Y = \frac{4}{s^2}(s+2) + 4(s+2)$

$s(s+2)X - 2\cancel{(s+2)}Y = \frac{4}{\cancel{s}} + \frac{8}{s^2} + 4(s+2)$ ... ①

$2\boxed{2} \Rightarrow -8X + 2\cancel{(s+2)}Y = -\frac{8}{s^2} - \frac{4}{\cancel{s}} - 10$ — ②

$1 + 2 \Rightarrow s(s+2)X - 8X = 4(s+2) - 10$

$(s^2+2s-8)X = 4s-2$

$X(s) = \frac{4s-2}{(s+4)(s-2)}$

$x(t) = L^{-1}\left(\frac{4s-2}{(s+4)(s-2)}\right)$

$\frac{4s-2}{(s+4)(s-2)} = \frac{A(s-2) + B(s+4)}{(s+4)(s-2)}$

$4s-2 = A(s-2) + B(s+4)$

$\boxed{s=2} \rightarrow 6 = 6B \qquad \boxed{B=1}$

$\boxed{s=-4} \rightarrow -18 = -6A \qquad \boxed{A=3}$

$$\frac{4s-2}{(s+4)(s-2)} = \frac{3}{(s+4)} + \frac{1}{(s-2)}$$

$$x(t) = L^{-1}\left(\frac{3}{(s+4)}\right) + L^{-1}\left(\frac{1}{s-2}\right)$$

$$= 3e^{-4t} + e^{2t}$$

$$x'(t) = -12e^{-4t} + 2e^{2t}$$

برجع لأول معادلة

$$2y(t) = x'(t) - 4t$$

$$y(t) = \frac{1}{2}\left[-12e^{-4t} + 2e^{2t} - 4t\right]$$

Ex:- Find $L^{-1}\left(\ln\left(1+\frac{a^2}{s^2}\right)\right)$:- $= f(t)$

let $F(s) = \ln\left(\frac{s^2+a^2}{s^2}\right)$

$F(s) = \ln(s^2+a^2) - 2\ln s$

اذا كان داخل L^{-1} صعب بنوخذها وبنشتقها لانه في قاعدة

$L(t f(t)) = -F'(s)$

$L^{-1}(F'(s)) = t f(t)$

$F'(s) = \frac{2s}{s^2+a^2} - \frac{2}{s}$ Tak L^{-1}

$L^{-1}(F'(s)) = L^{-1}\left(\frac{2s}{s^2+a^2}\right) - L^{-1}\left(\frac{2}{s}\right)$

$-t f(t) = 2\cos at - 2$

$f(t) = \frac{2-2\cos at}{t}$

Scanned by CamScanner

$$* \quad L\left(\frac{f(t)}{t}\right) = \int_s^{\infty} F(u)\,.du$$

Ex:- Find $L\left(\frac{\sin t}{\boxed{\cancel{t}}}\right) = \int_s^{\infty} \frac{1}{u^2+1^2}\,.du$

$$= \tan^{-1} u \Big|_s^{\infty}$$

$$= \tan^{-1}\infty - \tan^{-1} s$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

Ex: Solve :-

$t y'' + (1-t) y' + y = 0 \qquad , \; y(0) = 1 \; // \; y'(0) = -1$

$t y'' + -t y' + y' + y = 0$

Take Laplace :-

$L(t y''(t)) + L(y'(t)) - L(t y') + L(y) = 0$

$-\frac{d}{ds}\left[s^2 Y(s) - s y(0) - y'(0) \right] + s Y(s) - y(0) + \frac{d}{ds}\left[s Y(s) - y(0) \right] + Y(s) = 0$

$-\frac{d}{ds}\left[s^2 Y(s) - s + 1 \right] + s Y(s) - 1 + \frac{d}{ds}\left[s Y(s) - 1 \right] + Y(s) = 0$

$-\left[s^2 Y'(s) + 2s Y(s) - 1 \right] + s Y(s) - 1 + s Y'(s) + Y(s) + Y(s) = 0$

$-s^2 Y'(s) + s Y'(s) + -2s Y(s) + s Y(s) + 2 Y(s) + \cancel{1} - \cancel{1} = 0$

$(s - s^2) Y'(s) + (s - 2s + 2) Y(s) = 0$

$(s - s^2) Y'(s) + (2 - s) Y(s) = 0$

$(s - s^2) Y'(s) = (s - 2) Y(s)$

$$\frac{Y'(s)}{Y(s)} = \frac{s-2}{s-s^2}$$

$$\int \frac{1}{Y(s)}\,dy = \int \frac{s-2}{s-s^2}\,ds \quad \rightarrow \frac{s-2}{s-s^2} = \frac{A}{s} + \frac{B}{1-s}$$

$$\ln Y(s) = \ln|s^{-2}| + \ln|s-1| + c \qquad B = -1$$

$$\ln Y(s) = \ln \frac{s-1}{s^2} + c \qquad A = -2$$

$$\rightarrow Y(s) = e^{c} \cdot \frac{s-1}{s^2}$$

$$Y(s) = C * \frac{s-1}{s^2}$$

$$Y(s) = C\left[\frac{1}{s} - \frac{1}{s^2}\right]$$

$$y(t) = C\left(L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s^2}\right)\right)$$

$$y(t) = C[1 - t]$$

لإيجاد C

$$y(0) = C[1-0]$$

$$\boxed{1 = C}$$

$$y(t) = (1 - t)$$

* $L^{-1}\left[\frac{\pi}{2} - \tan^{-1} s\right]$

$F(s) = \frac{\pi}{2} - \tan^{-1} s$

$F'(s) = \frac{-1}{1+s^2}$

$L(-F'(s)) = \left(\frac{1}{1+s^2}\right)$

$t\,f(t) = \sin t$

$\boxed{f(t) = \frac{\sin t}{t}}$

** مراجعة **

$L(1) = \frac{1}{s}$

$L(a) = \frac{a}{s}$

$L(e^{at}) = \frac{1}{s-a}$

$L(t^n) = \frac{n!}{s^{n+1}}$

$L(\sin at) = \frac{a}{s^2+a^2}$

$L(\cos at) = \frac{s}{s^2+a^2}$

$L(\sinh at) = \frac{a}{s^2-a^2}$

$L(\cosh at) = \frac{s}{s^2-a^2}$

$L(\delta(t-a)) = e^{-as}$

$L(u(t-a)) = \frac{e^{-as}}{s}$

$L(u(t-a)\, f(t)) = e^{-as}\, L(f(t+a))$

- $L(t\, f(t)) = -\frac{d}{dt} F(s)$
- $L\left(\int_0^t f(t)\, dt\right) = \frac{F(s)}{s}$
- $L\left(\frac{f(t)}{t}\right) = \int_0^\infty F(s)\, ds$

- $L(f'(t)) = sF(s) - f(0)$
- $L(f''(t)) = s^2 F(s) - sf(0) - f'(0)$

EX :- Find :-

1. $L^{-1}\left(\frac{s}{(s^2+16)^2}\right)$

2. $L^{-1}\left(\frac{s}{(s^2-9)^2}\right)$

3. $L^{-1}\left(\ln \frac{s+a}{s+b}\right)$

4. $L^{-1}\left(\cot^{-1} \frac{s}{\pi}\right)$

5. $L^{-1}\left(\frac{1}{(s^2+4)^2}\right)$

(4) $L^{-1}\left(\underbrace{\cot^{-1}\frac{s}{\pi}}_{F(s)}\right) = f(t)$

let $F(s) = \cot^{-1}\frac{s}{\pi}$

$\tan^{-1} f(t) = \dfrac{f'(t)}{1+(f(t))^2}$

$F'(s) = \dfrac{-\frac{1}{\pi}}{1+\frac{s^2}{\pi^2}}$

$L^{-1}(-F'(s)) = L^{-1}\left(\dfrac{+\frac{1}{\pi}}{1+\frac{s^2}{\pi^2}}\right)$ π^2 ضرب

$L^{-1}(-F'(s)) = L^{-1}\left(\dfrac{\pi}{s^2+\pi^2}\right)$

$t\, f(t) = \sin \pi t$

$f(t) = \dfrac{\sin \pi t}{t}$ #

(1) $L^{-1}\left(\dfrac{s}{(s^2+16)^2}\right)$

let $F(s) = \dfrac{1}{s^2+16}$ هيك من عقله

$$F'(s) = \frac{-2s}{(s^2+16)^2}$$

$$\frac{s}{(s^2+16)^2} = \frac{-1}{2} F'(s)$$

$$L^{-1}\left(\frac{s}{(s^2+16)^2}\right) = \frac{1}{2} L^{-1}\left(-F'(s)\right)$$

$$= \frac{1}{2}\, t\, f(t)$$

$$L^{-1}(F(s)) = L^{-1}\left(\frac{1}{s^2+16}\right)$$

$$f(t) = \frac{1}{4}\sin 4t$$

$$= \frac{1}{2}\, t \cdot \frac{1}{4} \sin 4t$$

5 $L^{-1}\left(\dfrac{1}{(s^2+4)^2}\right)$ هاد سؤال هل نحله by parts

بس ما بدنا ندخل هون هلق ‼

let $F(s) = \dfrac{1}{s^2+4}$

$$F'(s) = \frac{-2s}{(s^2+4)^2}$$

$$\frac{1}{(s^2+4)^2} = -\frac{1}{2}\,\frac{F'(s)}{s}$$

$$L^{-1}\left(\frac{1}{(s^2+4)^2}\right) = \frac{1}{2}\, L^{-1}\left(-\frac{F'(t)}{s}\right)$$

$$= \frac{1}{2}\int_0^t t\, f(t)\, dt \qquad \Leftarrow \quad f(t) = L^{-1}\left(\frac{1}{s^2+4}\right) = \frac{1}{2}\sin 2t$$

$$= \frac{1}{4}\int_0^t t \sin 2t$$

تكامل بالاجزاء :-

t	$\sin 2t$
1	$\oplus \quad -\frac{\cos 2t}{2}$
0	$\ominus \quad -\frac{\sin 2t}{4}$

$$-\frac{1}{2} t\cos 2t + \frac{\sin 2t}{4} \Big|_0^t$$

$$-\frac{1}{2} t\cos 2t + \frac{\sin 2t}{4} - 0$$

$$L^{-1}\left(\frac{1}{(s^2+4)^2}\right) = \frac{1}{4}\left(-\frac{1}{2} t\cos 2t + \frac{\sin 2t}{4}\right)$$

ملاحظات بشكل عام :-

* اذا سؤال حكى (use Laplace) وما كان معطى شروط $y(0)=\_$, $y'(0)=\_$ عادي
بنحل لانه آخر اشي الحد رح ينشطب

* بالنسبة للسؤال series هو بحكيلي انه حلي كـ series
مش انه اكتب لحالي !!

وبالتوفيق