CH 1. First order Differential Equations

Def : A Diff eq (D.E) is an equation that contains unknown function & its derivative (y & $y^{(n)}$) and an Independent variable (x or t) or a function in x or t.

ex: ① $y'' + 3y' + e^x y = \sin x$

$(y = f(x))$

② $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

$z = f(x,y)$

* Differential equations are classified according to
 -1- the type of the derivative
 -2- Order
 -3- linearity

Def :

① A D.E is said to be O.D.E (ordinary D.E) if the derivative of the unknown function is ordinary.

② A D.E is called P.D.E if the derivative is partial

ex: $\frac{dy}{dx} + xy = \frac{1}{x}$ → O.D.E

ex: $\frac{\partial f}{\partial x} + xy\frac{\partial f}{\partial y} = \sin xy$ → P.D.E

③ The order of D.E is the highest derivative of the unknown function

ex: $3xy' + (2x^2+1)y = e^x \rightarrow$ ODE of order 1, 1st order ODE

ex: $y'' + 2y' + y = 0 \rightarrow$ 2nd order ODE.

④ The D.E is called linear if it is linear in the unknown function and all of its derivatives.

Rmk: a linear D.E should not contain a term consists of the product of y and any of its derivatives yy', yy'' or $y'y''$

ex: Determine whether the D.E is linear or non-linear.

① $y'' + 2x^3y' + e^x y = \sin x$: 2nd order, linear, ordinary.

② $y' + 3xy = \underbrace{e^{x+y}}_{\text{non linear.}}$: non-linear ODE.

③ $(y'')^3 + 3y' + y = x$: non-linear

④ $y'' + 2yy' = x^2 + 1$: non-linear

⑤ $xy' + y^{-1} = 0$: non-linear

⑤ A D.E of order 1 is separable if it can be written as
$y' = f_1(x) f_2(y)$

* 1st order O.D.E

→ General form : $y' = \frac{dy}{dx} = f(x,y)$

$y'' = f(x, y, y')$

$y''' = f(x, y, y', y'')$

$\vdots$

ex: $y' + 3y = x^2 + e^x$

$y' = \underbrace{x^2 + e^x - 3y}_{f(x,y)}$

→ General form of n^{th} order O.D.E : $y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$

ex: $y''' + 3xy = 0$

ex: Determine which D.E is separable

① $y' = x^2 + x^2 y^3$

$= x^2(1 + y^3)$

$= f_1(x) \cdot f_2(y)$

② $y' = \sin(x \cdot y)$

$\neq f_1(x) \cdot f_2(y)$

because $\sin(x \cdot y) \neq \sin(x)\sin(y)$

③ $y' = e^{x+y}$

$= e^x \cdot e^y$

④ $y' = e^{x+y} + 2$

$= e^x \cdot e^y + 2$

$\neq f_1(x) \cdot f_2(y)$

⑤ $y' + y = \frac{x^2 + y^3 + y}{y^2 + 1}$

$y' = -y + \frac{x^2 + y^3 + y}{y^2 + 1}$

$= \frac{\cancel{-y^3} \cancel{-y} + x^2 + \cancel{y^3} + \cancel{y}}{y^2 + 1}$

$= \frac{x^2}{y^2 + 1} = x^2 \cdot \frac{1}{y^2 + 1}$

Def :

A solution of the n^{th} order D.E $y^{(n)} = F(x, y, y', \dots, y^{(n-1)})$ is a function $f(x)$ that is n^{th}-differentiable on some open interval (a,b) and satisfies the D.E

ex: let $y'' + 3y' + 2y = 0$ determine which of the following is a solution

① $f(x) = x^2$

② $g(x) = e^{-2x}$

③ $h(x) = e^{-x}$

④ $l(x) = 2e^{-2x} - 5e^{-x}$

① $f'(x) = 2x \rightarrow f''(x) = 2$

$f''(x) + 3f'(x) + 2f(x) = 2 + 6x + 2x^2 \neq 0$ for all x

$\therefore f(x) = x^2$ is not a solution!

② $g'(x) = -2e^{-2x} \rightarrow g''(x) = 4e^{-2x}$

$g'' + 3g' + 2g = 4e^{-2x} - 6e^{-2x} + 2e^{-2x} = 0$

$\therefore g(x)$ is a solution.

$x^2, \; 5x^2$

dependent!

③ $h'(x) = -e^{-x} \rightarrow h''(x) = e^{-x}$

$h'' + 3h' + 2h = e^{-x} - 3e^{-x} + 2e^{-x} = 0$

$\therefore h(x)$ is a solution.

Def :

①

The initial value Problem (I.V.P) is a D.E with initial condition(s)

I.V.P :-

$$y' = f(x,y) \quad \text{with} \quad y(a) = b$$

②

The solution of the I.V.P is a function that satisfies the D.E along with the initial condition.

ex: in the previous example if $y'' + 3y' + 2y = 0$, $y(0) = 3$ then $g(x) = e^{-2x}$ and $h(x) = e^{-x}$ are not solutions for the I.V.P but $F(x) = 3e^{-2x}$ is a sol. of the I.V.P.

* Solving Separable 1st order O.D.E :-

Let $y' = f(x,y) \dashrightarrow \circledast$ be a sep. D.E $\left(f(x,y) = f_1(x) \cdot f_2(y) \right)$
now, the general Solution (g.s) can be obtained as follows :-

① $\dfrac{dy}{dx} = f_1(x) \cdot f_2(y)$

② $\dfrac{1}{f_2(y)} dy = f_1(x)\, dx$

③ integrate both sides $\displaystyle\int \frac{1}{f_2(y)} dy = \int f_1(x)\, dx$

$$F_2(y) = F_1(x)$$

Then either $y = G(x)$ OR $H(x,y) = C$

ex: Find the G.S of

-1- $y' - xy^2 = x$

$y' = xy^2 + x \qquad \frac{dy}{dx} = x(y^2+1)$

$\int \frac{1}{y^2+1}\,dy = \int x\,dx$

G.S

$\tan^{-1} y = \frac{x^2}{2} + C \qquad y = \tan\left(\frac{x^2}{2} + C\right)$

G.S: is a formula that includes every single solution for the D.E
one for each value of C

-2- $y\,y' = e^{x+y}$

$\frac{dy}{dx} = e^x \cdot \frac{e^y}{y}$

$\int y e^{-y}\,dy = \int e^x\,dx$

Parts.

G.S.

$-ye^{-y} - e^{-y} = e^x + C \qquad -ye^{-y} - e^{-y} - e^x = C$

-3- $y' - xy = xy^2$

$\frac{dy}{dx} = xy + xy^2 = x(y+y^2)$

$\int \frac{1}{y+y^2}\,dy = \int x\,dx \rightsquigarrow \frac{x^2}{2} + C$

Partial fractunes.

$\frac{1}{y+y^2} = \frac{1}{y(1+y)} = \frac{a}{y} + \frac{b}{y+1}$

$a = 1 \quad , \quad b = -1$

$\int \frac{1}{y^2+y}\,dy = \int \frac{1}{y}\,dy + \int \frac{-1}{y+1}\,dy$

$= \ln|y| - \ln|y+1|$

$= \ln\left|\frac{y}{y+1}\right|$

GS : $\ln\left|\frac{y}{y+1}\right| = \frac{x^2}{2} + C$

$$\left|\frac{y}{y+1}\right| = e^{x^2/2} \cdot k$$

$\frac{y}{y+1} = ke^{x^2/2}$

$y = \frac{ke^{x^2/2}}{1-ke^{x^2/2}}$

$\frac{y}{y+1} = -ke^{x^2/2}$

$y = \frac{-ke^{x^2/2}}{1+ke^{x^2/2}}$

ex: Solve the I.V.P $y' = \frac{y}{x}$, $y(2) = -7$, $x > 0$

$\frac{1}{y}\,dy = \frac{1}{x}\,dx$

$\ln|y| = \ln|x| + C$

$|y| = |x| \cdot k$ $\qquad y = \pm k|x|$ $\qquad y = \pm kx$

$y(2) = -7$ $\rightsquigarrow$ plug in the G.S

$-7 = \pm 2k$ $\rightsquigarrow$ $k = \pm\frac{7}{2}$

$y = \frac{7}{2}x$ $\qquad y = \frac{-7}{2}x$

$\downarrow$ $\qquad\qquad \downarrow$

$y(2) = 7$ not -7 $\qquad$ is the sol of I.V.P

* 1st order ODE that can be made separable.

let $\overbrace{y' = f(x,y)}^{(*)}$ be not separable

if $f(x,y) = g(y/x)$, then D.E can be made separable as follows

let $u = y/x \Rightarrow y = xu \Rightarrow y' = u + xu'$

$\therefore$ (*) becomes:

$u + xu' = g(u)$

$x \frac{du}{dx} = g(u) - u$

$\int \frac{1}{g(u)-u} du = \int \frac{1}{x} dx$

$G(u) = F(x)$

$G(y/x) = F(x)$

ex: solve the following D.E

① $y' = \frac{y^2 - x^2}{xy} \neq f_1(x) \cdot f_2(y)$ not separable.

$= \frac{y^2}{xy} - \frac{x^2}{xy} = \frac{y}{x} - \frac{x}{y}$

$y' = \frac{y}{x} - \frac{1}{y/x} = g(\frac{y}{x})$ it can be made separable

let $u = \frac{y}{x} \Rightarrow y' = u + xu'$

$\cancel{u} + xu' = \cancel{u} - \frac{1}{u}$

$$x\frac{du}{dx} = \frac{-1}{u}$$

$$u\,du = \frac{1}{x}\,dx \Rightarrow \frac{u^2}{2} = -\ln|x| + c$$

$$u = \pm\sqrt{-2\ln|x| + c}$$

$$\frac{y}{x} = \pm\sqrt{-2\ln|x| + c}$$

$$y = \pm x\sqrt{-2\ln|x| + c}$$

② $y' = \dfrac{x+y}{x-y} \neq f_1(x)\cdot f_2(y)$ $\qquad x > 0$

$$= \frac{\frac{x+y}{x}}{\frac{x-y}{x}} = \frac{1+\frac{y}{x}}{1-\frac{y}{x}} = g(y/x)$$

let $u = \dfrac{y}{x} \Rightarrow y' = u + xu'$

$$u + xu' = \frac{1+u}{1-u} \Rightarrow xu' = \frac{1+u}{1-u} - u$$

$$= \frac{1+\cancel{u}-\cancel{u}+u^2}{1-u} \Rightarrow x\frac{du}{dx} = \frac{1+u^2}{1-u}$$

$$\int \frac{1}{x}\,dx = \int \frac{1-u}{1+u^2}\,du$$

$$\ln x + c = \int \frac{1}{1+u^2} - 1/2\,\frac{2u}{1+u^2}$$

$$\tan^{-1}u - \frac{1}{2}\ln(1+u^2) = \ln x + c$$

$$\tan^{-1}\left(\frac{y}{x}\right) - 1/2\,\ln\left(1+\frac{y^2}{x^2}\right) = \ln x + c$$

H.W : Solve :

$$y' = \frac{x+y+3}{x-y-4}$$

$$* \quad \frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$$

Hint : get rid of 3 & -4 $\quad \frac{dz}{dw} = \frac{w+z}{w-z}$

Remark :

let $y' = f(x,y)$ if $f(tx, ty) = f(x,y)$ for any t

then $y' = f(x,y)$ can be made separable

$\Rightarrow \quad \frac{x+y}{x-y} \quad \frac{tx+ty}{tx-ty} = \frac{\cancel{t}}{\cancel{t}} \frac{(x+y)}{(x-y)}$

$\Rightarrow \quad \frac{y^2-x^2}{xy} \quad \frac{t^2y^2-t^2x^2}{t^2xy} = \frac{\cancel{t^2}}{\cancel{t^2}} \frac{y^2-x^2}{xy}$

ex: Solve $y' = \sin(x+y)$ not separable & cant be made separable .

let $u = x+y$

$u' = 1 + y'$

$y' = u' - 1$

$u' - 1 = \sin u$

$\frac{du}{dx} = \sin u + 1$ (sep.)

$\int \frac{1}{\sin u + 1} du = \int dx = x + C$ مراجعة

$\sin u = \frac{2z}{z^2+1}$

$du = \frac{2}{1+z^2} dz$

$\boxed{\text{H.W}}$

$$\frac{1}{\sin u+1}\cdot\frac{\sin u-1}{\sin u-1}=\frac{1-\sin u}{\cos^2 u}$$

$$= \sec^2 u - \sec u.\tan u \quad \dots$$

H.W

$$y' = \sin(x+y) + \cos(x+y) + 2.$$

use. special substitution

* Exact ODE.

A first order ODE of the form $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be exact if :-

Def①: there exists a function $u(x,y)$ with cts partial derivatives s.t $du = M(x,y)dx + N(x,y)dy = 0$

Recall from Calc. 3 :-

if $u = u(x,y)$ s.t u_x & u_y exists & cts. then differential of u is $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

ex : the D.E $y\,dx + x\,dy = 0$ is exact

$u(x,y) = xy$

$\int d(xy) = y\,dx + x\,dy = 0$

$xy = C$

$y = C/x$

Def ②: $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

ex: Determine whether the DE is exact or not if exact, find the solution:

Recall:

If $M(x,y)dx + N(x,y)dy = 0$ is a 1st order ODE that satisfies: $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

then $\exists\, u(x,y) = c$, where c is a constant

s.t ① $U_x = M(x,y)$

② $U_y = N(x,y)$

① $\underbrace{(2x+3)}_{M}dx + \underbrace{(2y-2)}_{N}dy = 0$ $\quad M_y = 0$, $N_x = 0$ $\therefore$ exact

there exist $U(x,y) = C$

s.t ① & ② hold.

integrate both sides of ① w.r.t x

$$\int U_x(x,y)\,dx = \int (2x+3)\,dx + g(y)$$

$$U(x,y) = x^2 + 3x + g(y)$$

Diff both sides w.r.t y, then use ②

$$U_y = 0 + g'(y)$$

$$2y - 2 = g'(y)$$

$$g(y) = \int 2y - 2\,dy + c = y^2 - 2y + C_1$$

$$\therefore U(x,y) = x^2 + 3x + y^2 - 2y + C_1 = C$$

$$\Rightarrow x^2 + 3x + y^2 - 2y = C^*$$

مريم حيدر!

to find $y=f(x)$:- $\quad y^2-2y+(x^2-3x-C)=0$

$$y=\frac{2\mp\sqrt{4-4(x^2+3x-C^*)}}{2}$$

اهمال مرجع!

② $\underbrace{(e^x\sin y-2y\sin x)}_{M}dx+\underbrace{(e^x\cos y+2\cos x)}_{N}dy=0$

$M_y=e^x\cos y-2\sin x \;=\; N_x=e^x\cos y-2\sin x$

$\therefore$ exact !

$\exists\, u(x,y)=C$ st ① $u_x=M$ ② $u_y=N$

from ② : $\int u_y\,dy=\int N(x,y)\,dy+g(x)$

$u(x,y)=\int(e^x\cos y+2\cos x)\,dy+g(x)$

$u(x,y)=e^x\sin y+2y\cos x+g(x)$

ممكن تحتاج تبدأ من ② قبل ① .. عادي!

Diff both sides w.r.t x use ①

$u_x=e^x\sin y-2y\sin x+g'(x)$

$e^x\cancel{\sin y}-\cancel{2y\sin x}=e^x\cancel{\sin y}-\cancel{2y\sin x}+g'(x)$

$g'(x)=0 \leadsto g(x)=C_1$

$u(x,y)=e^x\sin y+2y\cos x=C^*$

H.W ③ $(3x^2-2xy+2)dx+(6y^2-x^2+3)dy=0$

ex: Is $y\,dx + x\,dy = 0$ exact?

yes: $d(x,y) = y\,dx + x\,dy = 0$

$d(xy) = y\,dx + x\,dy = 0$

$xy = C$

$y = C/x$

ex: Is $-y\,dx + x\,dy = 0$ exact?

No $\nexists\, u(x,y) = C$: $du = -y\,dx + x\,dy = 0$

Q) can we find a function $F(x)$ or $F(y)$ s.t If we multiply the D.E (not exact) be F, it becomes exact?

Ans. Yes, we can consider $F(x) = \frac{1}{x^2}$

$\frac{1}{x^2}(-y\,dx + x\,dy = 0)$

$\frac{-y}{x^2}dx + \frac{1}{x}dy = 0$

the new D.E becomes exact:

$d\left(\frac{y}{x}\right) = \frac{-y}{x^2}dx + \frac{1}{x}dy = 0$

$\int d\left(\frac{y}{x}\right) = \int 0\, d\left(\frac{y}{x}\right)$ $\quad \frac{y}{x} = C \Rightarrow y = Cx$

Such function $F(x) = \frac{1}{x^2}$ is called integrating factor (I.F)

Q) How to find the I.F?

Ans. let $M(x,y)dx + N(x,y)dy = 0$ be non exact D.E ($M_y \neq N_x$)

① if $R = \frac{M_y - N_x}{N}$ depends only on x, then the I.F

$$F(x) = e^{\int R(x)dx}$$

② if $R^* = \frac{N_x - M_y}{M}$ depends only on y, then the I.F

$$F(y) = e^{\int R^*(y)dy}$$

ex Multiply the D.E by F to get exact ODE

* find the G.S of:-

Recall:

If $M(x,y)dx + N(x,y)dy = 0$ is not exact, then we can find an integration factor $F(x)$ or $F(y)$ so that when we multiply the D.E by it, it becomes exact i.e ① ②

① $2x\tan y\,dx + \sec^2 y\,dy = 0$

$M_y = 2x\sec^2 y \neq N_x = 0$

$$R(x) = \frac{2x\sec^2 y - 0}{\sec^2 y} = 2x$$

$$F(x) = e^{\int 2x\,dx} = e^{x^2}$$

$\Rightarrow 2xe^{x^2}\tan y\,dx + e^{x^2}\sec^2 y\,dy = 0 \rightsquigarrow$ is exact D.E (check)

$\exists\, u(x,y) = C$ s.t

① $u_x = 2xe^{x^2}\tan y$

② $u_y = e^{x^2}\sec^2 y$

$u(x,y) = e^{x^2}\tan y = C$

From ② $u(x,y) = \int u_y\,dy = \int e^{x^2}\sec^2 y\,dy + g(y) = e^{x^2}\tan y + g(x)$

Diff both sides w.r.t x & use ①

$$u_x = 2xe^{x^2}\tan y + g'(x)$$

$$2xe^{x^2}\tan y = 2xe^{x^2}\tan y + g'(x)$$

$$\therefore \; g'(x) = 0 \rightsquigarrow g(x) = C$$

$$u(x,y) = e^{x^2}\tan y = C^* \qquad y = \tan^{-1}(C^* e^{-x^2})$$

② $(3x^2y + 2xy + y^3)\,dx + (x^2+y^2)\,dy = 0$

$$\underbrace{(3x^2y + 2xy + y^3)}_{M}\,dx + \underbrace{(x^2+y^2)}_{N}\,dy = 0$$

$$\left.\begin{aligned} M_y &= 3x^2 + 2x + 3y^2 \\ N_x &= 2x \end{aligned}\right\} \text{not exact}$$

$$R(x) = \frac{3x^2 + \cancel{2x} + 3y^2 - \cancel{2x}}{x^2+y^2} = \frac{3x^2+3y^2}{x^2+y^2} = 3$$

$$F(x) = e^{\int 3\,dx} = e^{3x}$$

$$(3x^2ye^{3x} + 2xye^{3x} + y^3e^{3x})\,dx + (x^2e^{3x} + y^2e^{3x})\,dy = 0 \quad \text{exact (check!)}$$

$$\exists\, u(x,y) = C$$

① $u_x = 3x^2ye^{3x} + 2xye^{3x} + y^3e^{3x}$

② $u_y = x^2e^{3x} + y^2e^{3x}$

From ②: $u(x,y) = \int u_y\,dy = \int x^2e^{3x} + y^2e^{3x}\,dy + g(x)$

$$= x^2e^{3x}y + \frac{1}{3}y^3e^{3x} + g(x)$$

diff both sides w.r.t x and use ①

$$u_x = \underbrace{2xye^{3x} + x^2\cdot 3ye^{3x} + y^3e^{3x}}_{u_x} + g'(x) \qquad \cancel{u_x} = \cancel{u_x} + g'(x)$$

$g'(x) = 0 \rightsquigarrow g(x) = C$

$\therefore u(x,y) = x^2 e^{3x} y + 1/3\, y^3 e^{3x} + C_1 = C$

③ $dx + \left(\frac{x}{y} - \sin y\right) dy = 0$

step 1. $M(x,y) = 1$, $N(x,y) = \frac{x}{y} - \sin y$

step 2 $M_y = 0$, $N_x = \frac{1}{y}$

step 3 not exact!

step 4 $R(x) = \dfrac{0 - \frac{1}{y}}{\frac{x}{y} - \sin y} \neq$ fun. of x

$R^*(y) = \dfrac{\frac{1}{y} - 0}{1} = \frac{1}{y}$

step 5 $F(y) = e^{\int \frac{1}{y} dy} = e^{\ln y} = y$

step ~~7~~ 6 check for exactness $\rightsquigarrow$ exact

step 6 mult. the D.E by y $y\,dx + (x - y\sin y) dy$

step 8 let $u(x,y) = C$ s.t ① $u_x = y$ ② $u_y = x - y\sin y$

step 9 use step 8 to find $u(x,y)$

From ① $u(x,y) = \int u_x\, dx = \int y\, dx + g(y)$

$u(x,y) = yx + g(\cancel{x}y)$

diff both sides w.r.t y & use (2)

$u_y = x + g'(y)$

$\cancel{x} - y\sin y = \cancel{x} + g'(y)$

$\therefore\ g'(y) = -y\sin y \Rightarrow g(y) = \underbrace{\int -y\sin y\, dy}_{\text{by parts}}$

$g(y) = y\cos y - \sin y$

$\therefore\ u(x,y) = xy + y\cos y - \sin y = C$

step 10 SMILE

(4) $y' = e^{2x} + y - 1$

(5) $y\,dx + (2xy - e^{-2y})\,dy = 0$

(6) $(e^{x+y} + ye^{y})\,dx + (xe^{y} - 1)\,dy = 0 \qquad y(0) = -1$

First order linear ODE

→ Standard form:

$$(*) \quad y' + P(x)y = g(x) = \begin{cases} 0 \rightarrow \text{Homogenious} \\ \neq 0 \rightarrow \text{non-homogenious} \end{cases}$$

where $P(x)$ & $g(x)$ are cts on some open interval (a,b)

→ The GS of $(*)$ can be obtained by

$$\boxed{y(x) = \frac{1}{F(x)} \cdot \left[\int F(x) \cdot g(x)\,dx + c\right]}$$

where $F(x)$ is called the I.F of $(*)$ & given by $\boxed{F(x) = e^{\int P(x)\,dx}}$

$$F(x)y' + F(x)p(x) = F(x)g(x)$$
$$(F(x)y)' = F(x)g(x)$$
$$F(x)y = \int F(x)g(x)\,dx + c$$
$$y = \frac{1}{F}\left[\int Fg\,dx + c\right]$$

ex: Find the GS:

① $y' + \underbrace{2x}_{P(x)}y = \underbrace{2xe^{-x^2}}_{g(x)}$

IF: $F(x) = e^{\int 2x\,dx} = e^{x^2}$

$$\underbrace{e^{x^2}y' + 2x\,e^{x^2}y}_{(e^{x^2}y)'} = 2x$$

$$y(x) = \frac{1}{e^{x^2}}\left[\int 2xe^{-x^2} \cdot e^{x^2}\,dx + c\right]$$

$$= e^{-x^2}\left[x^2 + c\right]$$

(2) $(1+x^2)y' + 4xy = (1+x^2)^{-2}$

$$y' + \underbrace{\frac{4x}{(1+x^2)}}_{P(x)} y = \underbrace{(1+x^2)^{-3}}_{g(x)}$$

$$I.F : F(x) = e^{\int \frac{4x}{1+x^2} dx} = e^{2\ln(1+x^2)} = (1+x^2)^2$$

$$G.S = y(x) = \frac{1}{(1+x^2)^2}\left[\int \cancel{(1+x^2)^2} \cdot (1+x^2)^{-3}\, dx + c\right]$$

$$= \frac{1}{(1+x^2)^2}\left[\tan^{-1}x + c\right]$$

ex: solve the I.V.P

$$y' + \underbrace{\frac{2}{x}}_{P(x)} y = \underbrace{\frac{\cos x}{x^2}}_{g(x)}, \quad x > 0, \quad y\left(\frac{\pi}{2}\right) = 0$$

$$F(x) = e^{\int \frac{2}{x} dx} = x^2$$

$$y(x) = \frac{1}{x^2}\left[\int x^2 \cdot \frac{\cos x}{x^2}\, dx + c\right]$$

$$y(x) = \frac{1}{x^2}\left[\sin x + c\right]$$

$x = \pi/2 \rightarrow y = 0$

$$0 = \frac{1}{\pi^2/4}\left[1 + c\right] \Rightarrow c = -1$$

Particular sol. : $y(x) = \frac{1}{x^2}\left[\sin x - 1\right]$

- Bernoulli's D.E

→ general form:

(**) $y' + P(x)\,y = g(x)\,y^n \quad , \; n \neq 0,1$

is first order non linear of special form.

→ To solve (**)

let $u = y^{1-n} \Rightarrow u' = (1-n)\,y^{-n} \cdot y'$

multiply (**) by $(1-n)\,y^{-n}$

$$(1-n)\,y^{-n} \cdot y' + (1-n)\,P(x)\,y^{-n} \cdot y = (1-n)\,g(x)\,\cancel{y^{-n}}\,\cancel{y^{n}}$$

$u' + (1-n)\,P(x)\,u = (1-n)\,g(x)$ → is 1st order linear in u

ex: find the G.S of:

① $x^2 y' + 2xy = y^3 \quad , \; x > 0$

$y' + \frac{2}{x}\,y = \frac{1}{x^2}\,y^3$ ⇒ Bernoully

let $u = y^{1-3} = y^{-2} \Rightarrow u' = -2y^{-3} \cdot y'$

$\Rightarrow -2y^{-3}y' - \frac{4}{x}\,y^{-3}y = \frac{-2}{x^2}\,y^{-3}y^3$

$$\boxed{u' - \frac{4}{x}\,u = \frac{-2}{x^2}}$$

$$F(x) = e^{\int -\frac{4}{x}\,dx} = x^{-4}$$

$$u(x) = \frac{1}{x^{-4}} \cdot \left[\int x^{-4} \cdot \frac{-2}{x^2}\,dx + c\right] = x^4\left[\frac{2}{5}x^{-5} + c\right]$$

$$y^{-2} = u = \frac{2}{5x} + Cx^4$$

$$y = \pm\sqrt{\frac{1}{\frac{2}{5x} + Cx^4}}$$

(2) $\dfrac{dy}{dx} = \dfrac{x}{x^2y + y^3}$ $\neq f(x/y)$ not seperable

$$-x\,dx + (x^2y + y^3)\,dy = 0$$

$M_y = 0 \neq N_x = 2xy$ not exact!

$R(x) = \dfrac{0 - 2xy}{x^2y + y^2}$ doesn't depend on x

$R^*(y) = \dfrac{2xy - 0}{-x} = -2y$ can be made exact!

$$\Rightarrow \frac{dx}{dy} = \frac{x^2y - y^3}{x} = yx + y^3x^{-1}$$

$$x'(y) - yX(y) = y^3x^{-1}$$

Bernoulli in x

$u = x^{1-(-1)} = x^2 \Rightarrow u' = 2xx'$

$$2xx' - 2yx^2 = 2y^3xx^{-1}$$

$$u' - 2yu = 2y^3$$

I.F : $F(u) = e^{\int -2y\,dy} = e^{-y^2}$

$$u(y) = \frac{1}{e^{-y^2}} \cdot \left[\underline{\int e^{-y^2} \cdot 2y^3 \; dy} + c \right]$$

by substitution then by parts ......

* Summary of Chapter 1

1st order ODE

- linear: $y' + p(x)\,y = g(x)$
  - $g(x) \neq 0$ → I.F method: $F(x) = e^{\int p(x)\,dx}$ → $y = \frac{1}{F}\left(\int F.g\,dx + c\right)$
  - $g(x) = 0$ → separable : $y' + p(x)\,y = 0$
- non linear
  - Bernolly $y' + P(x)y = g(x)\,y^n$ (can be made linear $u = y^{1-n}$)
  - separable: $y' = f(x).g(y)$
  - can be made sep. $y' = F\left(\frac{y}{x}\right)$
  - exact ($M_y = N_x$)
  - can be made exact

CH2. 2nd order ODE

→ General form: $y'' + P(x)\,y' + q(x)\,y = g(x)$

where $p(x)$, $q(x)$ and $g(x)$ are cts on some open interval.

$\boxed{1}$ If $g(x) = 0$ → homogenous 2nd order ODE

case 1 with constant coefficints.

case 2 with non-constant coefficints.

$\boxed{2}$ $g(x) \neq 0$ → non homogenous.

case 1
case 2

Ⓘ undetermained coeff.

Ⓘⅰ Variation of parameters

$\boxed{0}$

→ 2nd order ODE that can be reduced to first order.

→ important definitions & facts.

Def ①: Two functions $y_1(x)$ & $y_2(x)$ are called proportional if $\exists$ a constant $k \neq 0$ s.t $\frac{y_1(x)}{y_2(x)} = k$ $\quad y_1(x) = k\,y_2(x)$

ex: $y_1(x) = e^{2x}$, $y_2(x) = 4e^{2x}$ } proportional.

Def 2: $y_1(x)$ & $y_2(x)$ are linearly independent if y_1 & y_2 are not Proportional.

ex: $y_1(x) = e^x$, $y_2(x) = x$ $\quad \frac{e^x}{x} \neq$ constant $\qquad y_1 = e^{4x}$, $y_2 = e^{7x}$ $\quad \frac{y_1}{y_2} = e^{-3x} \neq$ constant

• **Important facts:** let $y'' + P(x)y' + q(x)y = g(x)$ ...(*)

① if $y_1(x)$ & $y_2(x)$ are solutions of (*) then $c_1 y_1(x)$, $c_2 y_2(x)$, & $c_1 y_1(x) + c_2 y_2(x)$ are also solutions of (*)

ex: let $y'' - 5y' + 4y = 0$ H.W

determine wether the functions are solutions.

1. $y_1(x) = e^{4x}$ 2. $y_2(x) = e^{x}$ 3. $y_3(x) = 5e^{4x}$ 4. $y_4(x) = 7e^{x}$ 5. $y_5(x) = 3e^{4x} - 2e^{x}$

② if $y_1(x)$ & $y_2(x)$ are linearly Independante solutions of (*) then $y_1(x)$ & $y_2(x)$ form a basis for the solution space of (*)

(every solution of (*) $f(x)$ can be written as $f(x) = c_1 y_1(x) + c_2 y_2(x)$

explination: $\mathbb{R}^3 : \{(x_1, x_2, x_3) : x_1, x_2, x_3 \in \mathbb{R}\}$

$(2,-1,5) : 2(1,0,0) + -1(0,1,0) + 5(0,0,1)$

$\therefore (1,0,0), (0,1,0), (0,0,1)$ form a basis for $\mathbb{R}^3$.

③ if $y_1(x)$, $y_2(x)$ are linearly Independent Solutions of (*), then the G.S is $y(x) = c_1 y_1(x) + c_2 y_2(x)$

• **Solving 2nd order ODE that can be reduced to 1st order ODE**

1 → y is missing :-

ex: Solve $y'' - \frac{1}{x} y' = 0$ let $u = y'$ $u' = y''$ $x > 0$

$u' - \frac{1}{x} u = 0 \rightarrow \frac{1}{u} du = \frac{1}{x} dx$

$\ln|u| = \ln x + k_1 \rightarrow |u| = e^{\ln x + k_1} = e^{\ln x} \cdot \underbrace{e^{k_1}}_{c}$

$y' = u = \pm c_1 x \rightarrow y = \int \pm c_1 x \, dx + c_2 \rightarrow y = \pm \frac{c_1}{2} x^2 + c_2$

ex: $y'' = 1 + (y')^2$ let $u = y' \Rightarrow u' = y''$

X missing + Y missing
حلها بطريقة Y missing

$u' = 1 + u^2$

$\frac{1}{1+u^2} du = dx \rightarrow \tan^{-1} u = x + c_1$

$u = \tan(x + c_1) \rightarrow y' = \tan(x + c_1) \rightarrow y = \ln|\sec(x + c_1)| + c_2$

2- X is missing : $u = y' \Rightarrow y'' = \frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} \Rightarrow y'' = u \frac{du}{dy}$

ex: find the G.S of

① $y y'' = 3(y')^2$

let $y' = u$ $\quad y'' = u \frac{du}{dy}$

$y \cdot u \frac{du}{dy} = 3u^{\cancel{2}}$

$\int \frac{1}{u} du = \int \frac{3}{y} dy$

$\ln u = 3 \ln y + c = \ln y^3 + c$

$u = y^3 \cdot c_1 \rightarrow \frac{dy}{dx} = c_1 y^3$

$\int \frac{1}{y^3} dy = \int c_1 dx$

$-\frac{2}{y^2} = c_1 x + c_2$

$y = \pm \sqrt{\frac{-2}{c_1 x + c_2}}$

② $y'' - yy' = 0 \qquad y' = u \rightarrow y'' = u\frac{du}{dy}$

$\cancel{u}\frac{du}{dy} - y\cancel{u} = 0 \rightsquigarrow \frac{du}{dy} = y \rightsquigarrow \int du = \int y\,dy \rightsquigarrow u = \frac{y^2}{2} + c_1$

$\Rightarrow \frac{dy}{dx} = 1/2\,(y^2 + c^*)$

$\int \frac{1}{y^2+c^*}\,dy = \int 1/2\,dx \rightsquigarrow \underbrace{\int \frac{1}{y^2+c^*}\,dy}_{I} = 1/2x + c^{**}$

(I) we have 3 cases:

- case-1 $c^* = 0$

$I = \int \frac{1}{y^2}\,dy = \frac{-1}{y} \rightsquigarrow \frac{-1}{y} = 1/2x + c^{**} \rightarrow y = \frac{-1}{1/2x + c^{**}}$

- case-2 $c^* > 0$, then $c^* = k^2$ for some $k \in \mathbb{R} - \{0\}$

$I = \int \frac{1}{y^2+k^2}\,dy = \frac{1}{k^2}\tan^{-1}\left(\frac{y}{k}\right)$

$\frac{1}{k}\tan^{-1}\left(\frac{y}{k}\right) = 1/2x + c^{**} \rightarrow y = k\tan\left(\frac{k}{2}x + kc^{*}\right)$

- case-3 $c^* < 0$, then $c^* = -k^2 \quad k \neq 0$

$I = \int \frac{1}{y^2-k^2}\,dy$

$\frac{1}{y^2 + -k^2} = \frac{a}{y-k} + \frac{b}{y+k} \rightarrow 1 = a(y+k) + k(y-k)$

$y = k \rightarrow a = \frac{1}{2k}$

$y = -k \rightarrow b = \frac{-1}{2k}$

* Let $y'' + P(x)y' + q(x)y = 0$ ... $(*)$
Let $y_1(x)$ be a given solution of $(*)$
to find a second solution $y_2(x)$ so that $y_1(x)$ & $y_2(x)$ form basis of $(*)$ (are L. independent) :-

Assume $\dfrac{y_2(x)}{y_1(x)} = u(x) \neq$ constant $\Rightarrow y_2(x) = y_1(x) \cdot u(x)$

find y_2' & y_2'', then Plug y_2, y_2', y_2'' in $(*)$ and use the fact that $y_1(x)$ is a solution of $(*)$ to get

$$u(x) = \int \frac{1}{y_1^2(x)} \cdot e^{-\int P(x)dx}$$

ex : Let $y_1(x) = \dfrac{\cos x}{x}$ be a solution of $xy'' + 2y' + xy = 0$ find $y_2(x)$ such that y_2 & y_1 are linearly independant -OR- find the G.S.

$y'' + \underbrace{\frac{2}{x}}_{P(x)} y' + y = 0$ $\qquad u(x) = \displaystyle\int \frac{x^2}{\cos^2 x} \cdot e^{-\int \frac{2}{x}dx} dx$

$u = \displaystyle\int \frac{\cancel{x^2}}{\cos^2 x} \cdot \cancel{x^{-2}} \, dx \rightarrow \int \sec^2 x \, dx = \tan x$ $+C \to$ zero

$y_2(x) = \dfrac{\cos x}{x} \cdot \tan x = \dfrac{\sin x}{x}$ $\qquad$ G.S: $c_1 \dfrac{\cos x}{x} + c_2 \dfrac{\sin x}{x}$

ex : find G.S of $x^2y'' + xy' - 4y = 0$, $x > 0$ given that $y_1(x) = x^2$ is a solution

$y'' + \frac{1}{x} y' - \frac{4}{x^2} y = 0$

$u(x) = \displaystyle\int \frac{1}{x^{2 \cdot 2}} \cdot e^{-\int \frac{1}{x}dx} dx = \int \frac{1}{x^5} dx = \frac{-4}{x^4} + C$ $\to$ zero we can assume $C=0$ because we are looking for one solution

$y_2(x) = y_1(x) \cdot u(x)$ $\quad y_2 = x^2 \cdot \frac{-4}{x^4} = \frac{-4}{x^2}$

G.S: $y = c_1 x^2 + c_2 \frac{1}{x^2}$

{1} 2nd order ODE with constant coefficient

'General form: $ay'' + by' + cy = g(x)$, where a, b, c are constants

case 1: homogenous $(g(x)=0)$

$ay'' + by' + cy = 0$ --- $(*)$

let $y = e^{rx}$ want: To find the value of r so that $y = e^{rx}$ satisfies $(*)$

$y' = re^{rx} \rightarrow y'' = r^2 e^{rx}$

plug y, y', y'' in $(*)$

$\Rightarrow ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0$

$e^{rx}(ar^2 + br + c) = 0$

but $e^{rx} \neq 0$ for all x $\quad \boxed{\therefore ar^2 + br + c = 0}$ caractraristic equasion.

$ay' + by = 0$

$a\frac{dy}{dx} = by$

$\frac{1}{y} dy = \frac{b}{a} dx$

$\ln y = \frac{-b}{a} x + c$

$y = e^{-\frac{b}{a}x} \cdot k$

• now we have 3 cases:

case 1: if $D = b^2 - 4ac > 0$ then we have 2-distinct real roots, namely:

$$\boxed{\left.\begin{aligned} r_1 &= \frac{-b+\sqrt{D}}{2a} \\ r_2 &= \frac{-b-\sqrt{D}}{2a} \end{aligned}\right\} \quad r_1 \neq r_2}$$

$y_1 = e^{r_1 x}$ $\quad y_2 = e^{r_2 x}$ are solutions of $(*)$

since $\frac{y_1}{y_2} = \frac{e^{r_1 x}}{e^{r_2 x}} = e^{(r_1 - r_2)x} \neq$ constant

$\therefore$ y_1 & y_2 are two indep. sol. for $(*)$ means.

$$\boxed{GS: \; y = C_1 e^{r_1 x} + C_2 e^{r_2 x}}$$

ex: find the G.S of:

① $y'' - 4y' + 3y = 0$ $\quad$ let $y = e^{rx}$

Char. equ. $\quad r^2 - 4r + 3 = 0 \;\rightarrow\;$ Case 1

$(r-1)(r-3) = 0$

$r_1 = 1 \;,\; r_2 = 3$

$G.S = \quad y = C_1 e^{x} + C_2 e^{3x}$

② $2y'' + 3y' - 2y = 0$ $\quad$ let $y = e^{rx}$

Char. equ. $\quad 2r^2 + 3r - 2 = 0 \;\rightarrow\; D = 9 - 4(2)(-2) = 25 > 0$

$r_1 = \frac{-3+\sqrt{25}}{4} = \frac{1}{2}$ $\qquad r_2 = \frac{-3-\sqrt{25}}{4} = -2$

$G.S \; : y = C_1 e^{\frac{1}{2}x} + C_2 e^{-2x}$

Case 2: If $D = b^2 - 4ac = 0$, then we have two repeated real roots:

$$r_1 = r_2 = \frac{-b}{2a} = r$$

$\therefore \boxed{y_1 = e^{rx}}$ is a solution → can we find y_2 so that $y = y_1 c_1 + y_2 c_2$ is a general solution? Yes!

let $y_2 = y_1 \cdot u(x)$

$$u(x) = \int \frac{1}{y_1^2} e^{\int -p(x)dx} dx$$

$$\frac{ay''}{a} + \frac{by'}{a} + \frac{c}{a} = 0$$

$$y'' + \frac{b}{a}y' + \frac{c}{a} = 0$$

$$p(x) = \frac{b}{a}$$

$$u(x) = \int \frac{1}{e^{2rx}} \cdot e^{-\int \frac{b}{a}dx} dx$$

$$= \int e^{-2rx} \cdot e^{-\frac{b}{a}x} dx = \int e^{\frac{b}{a}x} \cdot e^{-\frac{b}{a}x} dx = x$$

$$\boxed{y_2(x) = x \cdot e^{rx}}$$

$$\therefore \boxed{G.S: \ y = c_1 e^{rx} + c_2 x e^{rx}}$$

ex: find the G.S of

① $y'' - 6y' + 6y = 0$

Char. equ: $r^2 - 6r + 9 = 0$ → $D = 0$ → $r_1 = r_2 = \frac{6}{2} = 3$

$$G.S: \ y = c_1 e^{3x} + c_2 x e^{3x}$$

→ Case $\underline{3}$: If $D = b^2 - 4ac < 0$, then we have two distinct complex roots

Remark: Complex roots comes as pairs of complex conjugates If $z = a + bi$ is a solution of $\alpha z^2 + \beta z + \gamma = 0$, then $\bar{z} = a - bi$ is also a solution.

The general solution is:

$$y(x) = c_1 e^{\alpha x} \cos\beta x + c_2 e^{\alpha x} \sin\beta x$$

where $\alpha = \frac{-b}{2a}$ $\quad \beta = \frac{\sqrt{|D|}}{2a}$

ex: find the general solution of:

① $y'' + 2y' + 2y = 0$

char. eq: $r^2 + 2r + 2 = 0$

$D = 4 - 4(1)2 = -4 < 0$

$\alpha = \frac{-2}{2} = -1$ $\quad \beta = \frac{\sqrt{|-4|}}{2} = 1$

G.S: $y = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x$

Complex numbers :-

$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$

Real Part

Imagenary Part

$x^2 + 1 = 0$

$x = \pm\sqrt{-1}$

$i = \sqrt{-1}$

imagenary number

ex: solve $x^2 + x + 1 = 0$

$D = 1 - 4 = -3 < 0$

$r_1 = \frac{-1 + \sqrt{-3}}{2}$, $\sqrt{-3} = \sqrt{-1} \cdot \sqrt{3}$

$= \frac{-1}{2} + \frac{\sqrt{-3}}{2} = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$

$r_2 = \frac{-1}{2} - i\frac{\sqrt{3}}{2}$

② $y'' + 9y = 0$

Char. equ: $r^2 + 9 = 0$

$D = -36 < 0$

$\alpha = 0$, $\beta = 3$

G.S : $y(x) = c_1 \cos 3x + c_2 \sin 3x$

③ $4y'' - 4y' + y = 0$

Char. equ: $4r^2 - 4r + 1$

$D = 16 - 16 = 0$ case 2

$r = \dfrac{-(-4)}{2(4)} = \dfrac{1}{2}$

$\therefore$ G.S : $y = c_1 e^{1/2 x} + c_2 x e^{1/2 x}$

④ $5y'' + 3y' - y = 0$

Char equ: $5r^2 + 3r - 1 = 0$

$D = 9 - 4(5)(-1) = 29 > 0$

$r_1 = \dfrac{-3 + \sqrt{29}}{10}$ $r_2 = \dfrac{-3 - \sqrt{29}}{10}$

2nd Order I.V.P

Consists of : ① D.E: $y'' + p(x)y' + q(x)y = g(x)$

② I.C : $y(x_0) = y_0$ & $y'(x_0) = y'_0$

ex Solve the I.V.P

$$y'' - 5y' + 6y = 0, \quad y(0) = 1, \quad y'(0) = 2$$

Char eq: $r^2 - 5r + 6 = 0$

$(r-2)(r-3) = 0$

$r = 2 \qquad r = 3$

G.S: $y = c_1 e^{2x} + c_2 e^{3x}$

• $y(0) = 1 \rightsquigarrow 1 = c_1 + c_2$ --- ①

$y'(x) = 2c_1 e^{2x} + 3c_2 e^{3x}$

• $y'(0) = 2 \rightsquigarrow 2 = 2c_1 + 3c_2$ --- ②

Solve 1 & 2 to get $c_1 = 1 \quad c_2 = 0$

∴ the particular solution is $y(x) = e^{2x}$

H.W: Solve the I.V.P

$$y'' + 2y' + 5y = 0 \qquad y(0) = 0, \quad y'(\pi) = 0$$

• 2nd order homogenous with non constant coefficients:

⟶ Cauch - Euler equations:

$$\alpha x^2 y'' + \beta x y' + \gamma y = 0 \qquad x > 0$$

$\boxed{\alpha x y' + \beta y = 0 \quad \vdots \quad y = k x^r}$

Let $y = x^r$
$y' = r x^{r-1}$
$y'' = r(r-1) x^{r-2}$
} Plug y, y', y'' in the D.E

$\alpha x^2 r(r-1) x^{r-2} + \beta x r x^{r-1} + \gamma x^r = 0$

$\Rightarrow x^r (\alpha r(r-1) + \beta \cancel{r} + \gamma) = 0$

but $x > 0$

$\therefore$ char. eq: $\alpha r(r-1) + \beta (r) + \gamma = 0$

Case 1: if $D > 0$, then $\exists r_1, r_2 \in \mathbb{R}$ (roots) s.t $r_1 \neq r_2$

$\boxed{\therefore G.S}$ $y = C_1 x^{r_1} + C_2 x^{r_2}$

Case 2: if $D = 0$, then $\exists r_1, r_2 \in \mathbb{R}$ s.t
$r_1 = r_2 = r$
$\therefore y_1(x) = x^r$
$\Rightarrow y_2(x) = x^r u(x)$

$$u(x) = \int \frac{1}{y_1^2} \cdot e^{-\int P(x)\cdot dx} dx = \ln x$$

$\boxed{\therefore G.S}$ $y = C_1 x^r + C_2 x^r \ln x$

Case 3: if $D < 0$ then

$\boxed{\therefore G.S}$ $y = C_1 x^\alpha \cos(\beta \ln x) + C_2 x^\alpha \sin(\beta \ln)$

where: $\alpha = \frac{-b}{2a}$, $\beta = \frac{\sqrt{|D|}}{2a}$

ex: find the G.S of $x^2\left(y'' - \frac{3}{x}y' + \frac{1}{x^2}y = 0\right)$

① $x^2y'' - 3xy' + y = 0$

Char. equ: $r(r-1) - 3r + 1 = 0$

$r^2 - 4r + 1 = 0$

$D = 16 - 4 = 12 > 0$

$r_1 = \frac{4+\sqrt{12}}{2}$ $\qquad r_2 = \frac{4-\sqrt{12}}{2}$

$2+\sqrt{3}$ $\qquad 2-\sqrt{3}$

G.S: $y = C_1 x^{2+\sqrt{3}} + C_2 x^{2\sqrt{3}}$

② $x^2y'' + xy' + 4y = 0$

Char. equ: $r(r-1) + r + 4 = 0$

$r^2 + 4 = 0$ $\qquad D < 0 \Rightarrow \alpha = 0, \ \beta = 2$

G.S: $y = C_1\cos(2\ln x) + C_2\sin(2\ln x)$

③ $x^3y'' - 3x^2y' + 4xy = 0$ $\qquad x > 0$

$x^2y'' - 3xy' + 4y = 0$

Char. eq: $r(r-1) - 3r + 4 = 0$

$r^2 - 4r + 4 = (r-2)^2 = 0$

$r_1 = r_2 = 2$

G.S: $y = C_1x^2 + C_2x^2\ln x$.

2 non homogenious 2nd order ODE with constant Coeff.

→ standard form :-

$(*)\ y'' + by' + cy = g(x) \neq 0$

→ the G.S of $(*)$ is : $y = y_h + y_p$

where y_h is the G.S of the the corresponding homogenous ODE
$y'' + by' + cy = 0$
$y_h = c_1 y_1 + c_2 y_2$ where y_1 & y_2 are L.independent.

& y_p: is called the particular solution of $(*)$, & its the function that satisfies $y_p'' + by_p' + cy_p = g(x)$

→ why $y = y_h + y_p$ satisfies $(*)$?

$y' = y_h' + y_p'$
$y'' = y_h'' + y_p''$
plug y, y', y'' in $(*)$

$$y'' + by' + cy = \underbrace{y_h'' + by_h' + cy_h}_{=0} + y_p'' + by_p' + cy_p$$
$$= 0 + g(x)$$

→ To find y_p, we have two methods:

① Method of undetermined coefficients.

• This method can be used only if $g(x)$ is
1. exponential
2. $\sin ax$ or $\cos ax$
3. Polynomial
4. product of two or more functions listed in 1→3
5. sum of two or more functions listed in 1→3

The main Idea is to assume y_p to be the general form of $g(x)$ as listed in $1 \to 5$ then plug y_p in $(*)$ & find the values of the coefficints that appear in y_p

ex: Find the general solution of $y'' - 5y' + 6y = 4x^2 + 2$.

G.S: $y = y_h + y_p$

y_h: let $y'' + -5y' + 6 = 0$

$r^2 + -5r + 6 = 0 \to (r-2)(r-3) = 0 \to r_1 = 2, \; r_2 = 3$

$y_h = C_1 e^{2x} + C_2 e^{3x}$

let $y_p = Ax^2 + Bx + C$

$y_p' = 2Ax + B$

$y_p'' = 2A$

plug in $(*)$

$2A - 5(2Ax + B) + 6(Ax^2 + Bx + C) = 4x^2 + 2$.

$6Ax^2 + (6B - 10A)x + (2A - 5B + 6C) = 4x^2 + 2$

$\Rightarrow 6A = 4 \qquad A = 2/3$

$6B - 10A = 0 \qquad B = 20/18 = 10/9$

$2A - 5B + 6C = 2 \qquad C = \frac{28}{27}$

$\therefore y_p = 2/3\, x^2 + 10/9\, x + \frac{28}{27}$

GS: $y = C_1 e^{2x} + C_2 e^{3x} + 2/3\, x^2 + 10/9\, x + 28/27$

$g(x)$	let $y_p =$	example.
① $\beta e^{\alpha x}$	$A e^{\alpha x}$	$g(x) = 4e^{5x}$ $y_p = A e^{5x}$
② Polynomial of degree n	$A_n x^n + A_{n-1} x^{n-1} + \dots + A_1 x + A_0$	
③ $\beta \cos\alpha x$ + ④ $\gamma \sin\alpha x$	$A\cos\alpha x + B\sin\alpha x$	$g(x) = 2\sin 3x$ $y_p = A\cos 3x + B\sin 3x$ $g(x) = \sqrt{2}\cos 3x$ $g(x) = -1/4 \sin 3x + 5\cos 3x$
⑤ poly of degree n. $\begin{cases}\sin\alpha x\\ \cos\alpha x\end{cases}$	$(A_n x^n \dots A_0)\cos\alpha x + (B_n x^n \dots B_0)\sin\alpha x$	
⑥ poly of degree n. $e^{\alpha x}$	$(A_n x^n \dots A_0)\, e^{\alpha x}$	

ex: Find G.S of $y'' + 4y = 3x\cos x$

y_h: $y'' + 4y = 0$

$r^2 + 4 = 0 \qquad \alpha = 0 \qquad \beta = 2$

$y_h = c_1\cos 2x + c_2\sin 2x$

let $y_p = (Ax+B)\cos x + (Cx+D)\sin x$

$y_p' = A\cos x + (Ax+B)(-\sin x) + C\sin x + (Cx+D)\cos x$

$y_p'' = -A\sin x + A(-\sin x) + (Ax+B)(-\cos x) + C(\cos x) + (Cx+D).(-\sin x)$

$= (-A)x\cos x + (-C)x\sin x + (-B+2C)\cos x + (-2A-D)\sin x$

$y_p'' + 4y_p = 3x\cos x$

$-Ax\cos x - Cx\sin x + (2C-B)\cos x + (-2A-D)\sin x + 4((Ax+B)\cos x + (Cx+D)\sin x) = 3x\cos x$

$(3A)x\cos x + (3C)x\sin x + (2C+3B)\cos x + (-2A+3D)\sin x = 3x\cos x$

coefficients in both sides are equal $\Rightarrow$

$3A = 3 \rightarrow A = 3$

$3C = 0 \rightarrow C = 0$

$2C + 3B = 0 \rightarrow B = 0$

$-2A + 3D = 0 \rightarrow D = 2/3$

$\therefore y_p = \cos x + 2/3 \sin x$

$G.S: y_h + y_p = C_1 \cos 2x + C_2 \sin 2x + \cos x + 2/3 \sin x$

Remark :

① (very important)

when you write y_p in general form of $g(x)$ (as in the table) if one of the terms in y_p appears in y_h then you have to moltiply y_p by x

ex: In the following ODE, write a suitable form for y_p, if the methed of undetermined coefficients to be used.

① $y'' - 4y' + 4y = 3e^{2x}$

y_h: $r^2 - 4r + 4 = 0 \rightarrow r_1 = r_2 = 2$

$y_h = C_1 e^{2x} + C_2 x e^{2x}$

$y_p = A e^{2x}$ ✗

$y_p = A x e^{2x}$ ✗

$y_p = A x^2 e^{2x}$ ✓

② $y'' + 9y = x\sin^2(3/2\, x)$

y_h: $r^2 + 9 = 0$ complex roots

$\alpha = 0$, $\beta = 3$

$y_h = C_1\cos 3x + C_2 \sin 3x$

$g(x) = x\sin^2(3/2\, x) = x \cdot 1/2\,(1-\cos 3x)$

$= 1/2\, x - 1/2\, x\cos 3x$

$y_p = (Ax+B) + \big((Cx+D)\cos 3x + (Ex+F)\sin 3x\big) \cdot x$

Remark:

② IF $y'' + by' + cy = g_1(x) + g_2(x) + \cdots + g_n(x)$

then the G.S $y = y_h + Y_p$ where

$Y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_n}$

y_{p_1}: is the particular solution of $y'' + by' + cy = g_1(x)$

y_{p_2}: is the particular solution of $y'' + by' + cy = g_2(x)$

⋮

y_{p_n}: " " " " $y'' + by' + cy = g_n(x)$

③ $y'' - 4y' + 3y = e^{2x} + 3xe^x + \cos x$

y_h: $r^2 - 4r + 3 = 0$ $(r-3)(r-1) = 0$ $r_1 = 3$, $r_2 = 1$

$y_h = c_1e^{3x} + c_2e^x$

$Y_p = y_{p_1} + y_{p_2} + y_{p_3}$

y_{p_1} = $y'' - 4y' + 3y = e^{2x}$

$y_{p_1} = A\,e^{2x}$

y_{P_2}: $y'' - 4y' + 3y = 3x e^x$

$y_{P_2} = ((\cancel{A}_C x + \cancel{B}_D) e^x) x$

y_{P_3}: $y'' - 4y' + 3y = \cos x$

$y_{P_3} = \cancel{A}_E \cos x + \cancel{B}_F \sin x$

METHOD 2: can be used with any $g(x)$ (Variation of parameters)

$y_h = c_1 y_1(x) + c_2 y_2(x)$ $y_p = -y_1(x)\int \frac{y_2(x)\cdot g(x)}{W(y_1,y_2)(x)}dx + y_2(x)\int \frac{y_1(x)\cdot g(x)}{W(y_1,y_2)(x)}dx$

where:

$g(x)$: non-homogenous Part (when coeff of y'' is 1)

$W(y_1,y_2)(x)$: $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$ is called the wronstian of y_1 & y_2

RMK:

① $W(y_1, y_2) \neq 0$ iff y_1 & y_2 are L. independent

② $W(y_1, y_2) = 0$ iff y_1 & y_2 are L. dependent.

ex: Find the G.S of

$y'' + y = \sec^2 x$

y_h: $r^2 + 1 = 0 \Rightarrow \alpha = 0$, $\beta = 1$

$y_h = c_1 \underbrace{\cos x}_{y_1} + c_2 \underbrace{\sin x}_{y_2}$

$$w(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$y_p = -\cos x \int \frac{\sin x \cdot \sec^2 x}{1}\, dx + \sin x \int \cos x \cdot \sec^2 x\, dx$$

$$= -\cos x \int \tan x \cdot \sec x\, dx + \sin x \int \sec x\, dx$$

$$= -\cos x\,(\sec x) + \sin x \,\ln|\sec x + \tan x|$$

ex: $\underline{1}y'' - 4y' + 4y = x^2 e^x$

الخانة i_{21}
في w_1
الخانة i_{22}
في w_2

$$y'' - 4y' + 4y = 0$$

let $y(x) = e^{mx}$

$$m^2 - 4m + 4 = 0$$

$$(m-2)(m-2) = 0 \quad m_1 = m_2 = 2$$

$$y_h(x) = c_1 e^{2x} + c_2 x e^{2x}$$

let $y_p(x) = e^{2x} u_1 + x e^{2x} u_2$

$$w(e^{2x}, xe^{2x}) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2x e^{2x} \end{vmatrix} = e^{4x} + 2xe^{4x} - 2xe^{4x} - e^{4x}$$

$$w_1 = \begin{vmatrix} 0 & xe^{2x} \\ x^2 e^x & e^{2x} + 2xe^{2x} \end{vmatrix} = -x^3 e^{3x}$$

$$w_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & x^2 e^x \end{vmatrix} = x^2 e^{3x}$$

$$u_1 = -\int \frac{y_2\, r(x)}{w}\, dx$$

$$= -\int \frac{xe^{2x} \cdot x^2 e^x}{e^{4x}}\, dx$$

$$u_2 = \int \frac{y_1\, r(x)}{w}\, dx$$

$$= \int \frac{e^{2x} \cdot x^2 e^x}{e^{4x}}\, dx$$

$$u_1 = \int \frac{w_1}{w}\,dx = \int \frac{-x^3 e^{3x}}{e^{4x}}\,dx = -\int x^3 e^{-x}\,dx$$

$$u_2 = \int \frac{w_2}{w}\,dx = \int \frac{x^2 e^{3x}}{e^{4x}}\,dx = \int x^2 e^{-x}\,dx$$

$$y_p = e^{2x}\int x^3 e^{-x}\,dx + x e^{2x}\int x^2 e^{-x}\,dx$$

G.S : $y(x) = y_h + y_p$

ex $x^2 y'' - 2xy' + 2y = x^3 \cos x$

$$x^2 y'' - 2x y' + 2y = 0$$

let $y = x^m$

$$x^2\, m(m-1)\, x^{m-2} - 2x\, m x^{m-1} + 2x^m = 0$$

$$m^2 - 3m + 2 = 0$$

$$(m-1)(m-2) = 0$$

$$m = 1 \qquad m = 2$$

$$y_h = C_1 x + C_2 x^2 \Rightarrow \begin{array}{l} y_1 = x \\ y_2 = x^2 \end{array}$$

let $y_p = x u_1 + x^2 u_2$

$$w(x, x^2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$$

$$u_1 = -\int \frac{x^2 \cdot x\cos x}{x^2}\,dx = \boxed{-\int \frac{y_2 r(x)}{w}\,dx} = -\int x\cos x\,dx$$

$$u_2 = \int \frac{x \cdot x\cos x}{x^2}\,dx = \int \cos x\,dx = \boxed{\int \frac{y_1 r(x)}{w}\,dx}$$

$$y_p = -x\int x\cos x\,dx + x^2\int \cos x\,dx$$

ex $(x^2D^2 + xD - 4I)y = \frac{1}{x^2}$

$$y' = \frac{dy}{dx} = Dy \quad , \quad y'' = \frac{d^2y}{dx^2} = D^2y \quad , \quad y = Iy$$

$$\Rightarrow x^2y'' + xy' - 4y = \frac{1}{x^2} \; 0$$

let $y = x^m \rightsquigarrow x^2 m(m-1)x^{m-2} + x\,m\,x^{m-1} - 4x^m = 0$

$$m^2 - 4 = 0 \Rightarrow (m-2)(m+2) = 0 \qquad m = \pm 2$$

$y_h(x) = c_1x^2 + c_2x^{-2}$ $\quad y_1 = x^2 \quad y_2 = x^{-2}$

let $y_p(x) = x^2u_1 + x^{-2}u_2$

$$w(x^2, x^{-2}) = \begin{vmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{vmatrix} = -2x^{-1} - 2x^{-1} = -4x^{-1}$$

$$u_1 = -\int \frac{x^{-2} \cdot x^{-4}}{-4x^{-1}} = -1/4 \int x^{-5}\,dx = \frac{-1}{16}\,x^{-4}$$

$$u_2 = \int \frac{x^2 x^{-4}}{-4x^{-1}} = -1/4 \int x^{-1}\,dx = -1/4 \ln|x|$$

$$y_p = x^2\left(\frac{-1}{16}x^{-4}\right) + x^{-2}\left(-1/4 \ln|x|\right)$$

ex 6 $x^2y'' - xy' + y = x\ln|x|$ 　　القسمة على معامل x^2

$x^2y'' - xy' + y = 0$

let $y = x^m \rightsquigarrow x^2 m(m-1)x^{m-2} - x\,m x^{m-1} + m^m = 0$

$m^2 - 2m + 1 = 0$

$(m-1)(m-1) = 0$

$y_h(x) = C_1 x + C_2 x\ln x$

$y_1 = x,\ y_2 = x\ln|x|$

$y_p = x u_1 + x\ln x\, u_2$

$w(x, x\ln|x|) = \begin{vmatrix} x & x\ln|x| \\ 1 & \ln|x|+1 \end{vmatrix} = x\ln|x| + x - x\ln x = x$

substitution

$u = \ln x$

$du = \frac{1}{x}dx$

$u_1 = -\int \frac{x\ln|x| \cdot \frac{\ln|x|}{x}}{x}\,dx = -\int \frac{(\ln x)^2}{x}\,dx$

$u_2 = \int \frac{x \cdot \frac{\ln x}{x}}{x}\,dx = \int \frac{\ln x}{x}\,dx =$

first exam

ex $y'' + y = \tan x$

$y'' + y = 0$

$y_h = C_1\cos x + C_2\sin x$

$y_p = \cos x\, u_1 + \sin x\, u_2$

$$w(\cos x, \sin x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

CHAPTER 3 Higher ORder linear ODE

Let $y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + P_{n-2}(x)\, y^{(n-2)} + \cdots + P_0(x)\, y = g(x)$ --(*)

Standard form of n^{th} order Linear ODE

where $P_{n-1}(x), P_{n-2}(x), \ldots, P_0(x)$ and $g(x)$ all cnts. on (α, β)

Assume $y_1, y_2, \cdots y_n$ are solutions of (*) then :-

① $$W(y_1, y_2, \cdots y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}_{n\times n}$$

recall :-

$$\begin{vmatrix} \overset{+}{a} & \overset{-}{b} & \overset{+}{c} \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

ex : find $W(1, \cos x, \sin x)$

$$= \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = 1$$

② if $w(y_1, y_2, \ldots y_n) \neq 0$ then $y_1 \cdots y_n$ are linearly independent & the g.s of $y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \cdots + P_0(x)\, y = 0$ is

$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$

③ if $w=0$ then they are linearly dependent.

II homogenous linear higher order ODE

(a) with constant coeffecients :- $y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$

let $y = e^{rx}$

Char. equ: $\boxed{r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0 = 0}$ equ. of dig. 1

- solve the Char. eq. for r to get the following cases:-

case 1: distinct real roots

$r_1, r_2, \dots, r_n$ s.t $r_i \neq r_j$ for $i \neq j$

G.S: $y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \dots + c_n e^{r_n x}$

ex: Solve $y^{(4)} - 13y'' + 36y = 0$

Char. equ:- $r^4 + 13r^2 + 36 = 0$

$(r^2)^2 + 13r^2 + 36 = 0$

$(r^2 - 4)(r^2 - 9) = 0 \quad r = \pm 2 \quad r = \pm 3$

G.S : $y = c_1 e^{2x} + c_2 e^{-2x} + c_3 e^{3x} + c_4 e^{-3x}$

case 2: all Roots are real & repeated.

$r = r_1 = r_2 = r_3 = \dots r_n$ $\quad (r-1)^4 = 0$

G.S : $c_1 e^{rx} + c_2 x e^{rx} + c_3 x^2 e^{rx} + \dots C_n x^{n-1} e^{rx}$

ex: $y''' - 3y'' + 3y' - y = 0$

Char. equ: $r^3 - 3r^2 + 3r - 1 = 0$ $\quad$ 1 is a root $\rightarrow$ $r-1$ is a factor.

$r^3 - 3r^2 + 3r - 1 = (r-1)(r-1)(r-1) = 0 \qquad r = 1$

G.S: $y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$

Case $\underline{3}$: Some roots are distinct real and others are repeated & real

$r_1, r_2, \ldots, r_k, r, r, r, \ldots r_n$

G.S: $y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \ldots c_k e^{r_k x} + c_{k+1} e^{rx} + c_{k+2} x e^{rx} + \cdots + c_n x^{n-k-1} e^{rx}$

ex: solve $y''' - 6y'' + 9y' = 0$

char equ: $r^3 - 6r^2 + 9r = 0$

$r(r^2 - 6r + 9) = 0$

$r(r-3)(r-3) = 0 \Rightarrow r_1 = 0 \quad r_2 = r_3 = 3$

G.S : $y = c_1 e^{0x} + c_2 e^{3x} + c_3 x e^{3x}$

ex: $y^{4} - 4y'' + 4y = 0$

$r^4 - 4r^2 + 4 = 0$

$(r^2 - 2)(r^2 - 2) = 0 \qquad r = \pm\sqrt{2}, \quad r = \pm\sqrt{2}$

G.S: $c_1 e^{\sqrt{2}x} + c_2 x e^{\sqrt{2}x} + c_3 e^{-\sqrt{2}x} + c_4 x e^{-\sqrt{2}x}$

case $\underline{4}$: All roots are complex (repeated or not)

ex: solve $y^{(4)} + 9y'' + 20y = 0$

$\alpha = \frac{-b}{2a}$

$\beta = \frac{\sqrt{|D|}}{2a}$

$r^4 + 9r^2 + 20 = 0$

$(r^2 + 5)(r^2 + 4) = 0 \qquad r = \pm i\sqrt{5} \qquad r = \pm 2i$

$\alpha = 0, \beta = \sqrt{5} \qquad \alpha = 0, \beta = 2$

G.S: $y = c_1 e^{0} \cos\sqrt{5}x + c_2 e^{0} \sin\sqrt{5}x + c_3 e^{0} \cos 2x + c_4 e^{0} \sin 2x$

$\Rightarrow r_1 = \alpha_1 + i\beta_1, \quad r_2 = \alpha_1 - i\beta_1, \quad r_3 = \alpha_2 + i\beta_2, \quad r_4 = \alpha_2 - i\beta_2$

$\Rightarrow$ G.S : $c_1 e^{\alpha_1 x} \cos\beta_1 x + c_2 e^{\alpha_1 x} \sin\beta_1 x + c_3 e^{\alpha_2 x} \cos\beta_2 x + c_4 e^{\alpha_2 x} \sin\beta_2 x$

case 5 : some roots are complex & the rest are real (repeated or distinct)

ex: Solve the given D.E.

① $y''' + 9y' = 0$

$r^3 + 9r = 0 \Rightarrow r(r^2+9) = 0 \Rightarrow r_1 = 0 \quad \underbrace{r_2 = 3i \quad r_3 = -3i}_{\alpha = 0 \quad \beta = 3}$

G.S: $y = C_1 e^{0x} + C_2 e^{0x}\cos 3x + C_3 e^{0x}\sin 3x$

$= C_1 + C_2\cos 3x + C_3 \sin 3x$

② If the char. eq of 7th order linear homo. DE has the roots: $1, 2, 3, 3, 3, 2+5i, 2-5i$ write the G.S :-

G.S: $C_1 e^{x} + C_2 e^{2x} + C_3 e^{3x} + C_4 x e^{3x} + C_5 x^2 e^{3x} + C_6 e^{2x}\cos 5x + C_7 e^{2x}\sin 5x$.

③ $y''' - y'' - y' + y = 0$

$r^3 - r^2 - r + 1 = 0 \Rightarrow r^2(r-1) - (r-1) = 0 \Rightarrow (r^2-1)(r-1) = 0 \Rightarrow (r-1)(r-1)(r+1) = 0$

$r_1 = 1 = r_2 \quad r_3 = -1$

GS: $C_1 e^{x} + C_2 x e^{x} + C_3 e^{-x}$

④ $y^{(4)} - 7y''' + 17y'' - 17y' + \overset{6}{\cancel{12}}y = 0$

$r^4 - 7r^3 + 17r^2 - 17r + 6 = 0$

$= (r-1)(r^3 - 6r^2 + 11r - 6) = 0$

$= (r-1)(r-1)(r^2 - 5r + 6) = 0$

$= (r-1)(r-1)(r-2)(r-3) = 0$

$6 \Rightarrow \pm 1, \pm 2, \pm 3$

Possible rational roots $\Rightarrow \dfrac{\pm 1, \pm 2, \pm 3}{\pm 1}$

$r = 1$ is a root $\Rightarrow r-1$ is a factor

1	1	-7	17	-17	6
		1	-6	11	-6
	1	-6	11	-6	0

$$\begin{array}{c|cccc} 1 & 1 & -6 & 11 & -6 \\ & & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

$r_1 = 1\ ,\ r_2 = 1\ ,\ r_3 = 2\ ,\ r_4 = 3$

$G.S: y = c_1 e^{x} + c_2 x e^{x} + c_3 e^{2x} + c_4 e^{3x}$

⑤ $y''' - y = 0$

$r^3 - 1 = 0 \Rightarrow (r-1)(r^2+r+1) = 0 \Rightarrow r-1=0$ $\quad r^2+r+1=0 \quad D = -3 < 1$

$\boxed{r=1}$ $\quad \alpha = -1/2 \quad \beta = \frac{\sqrt{|-3|}}{2}$

$\boxed{r_2 = -1/2 + i\frac{\sqrt{3}}{2}}$ $\quad \boxed{r_3 = -1/2 - i\frac{\sqrt{3}}{2}}$

$G.S: c_1 e^{x} + c_2 e^{-1/2x} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 e^{-1/2x} \sin\left(\frac{\sqrt{3}}{2}x\right)$

⑥ $y^{(5)} - 2y^{(4)} + 3y''' - 2y'' = 0$

$$\begin{array}{r|l} & r^2 - r + 2 \\ \hline r-1 & r^3 - 2r^2 + 3r - 2 \\ & \mp r^3 \pm r^2 \\ \hline & -r^2 + 3r - 2 \\ & \pm r^2 \mp r \\ \hline & 2r - 2 \\ & \mp 2r \pm 2 \\ \hline & 0 \end{array}$$

$r^5 - 2r^4 + 3r^3 - 2r^2 = 0$

$r^2(r^3 - 2r^2 + 3r - 2) = 0$

$r^2(r-1)(r^2 - r + 2) = 0$

$r^2(r-1)(r^2 - r + 2) = 0 \Rightarrow r^2 = 0\ ,\ r-1=0\ ,\ r^2 - r + 2 = 0$

$r_1 = r_2 = 0\ ,\ r_3 = 1\ ,\ \alpha = \frac{1}{2}\ ,\ \beta = \frac{\sqrt{7}}{2}$

$G.S: c_1 e^{0x} + c_2 x e^{0x} + c_3 e^{x} + c_4 e^{1/2x} \cos\frac{\sqrt{7}}{2} + c_5 e^{1/2x} \sin\frac{\sqrt{7}}{2}.$

2 Non-homogenous linear ODE

- standard form : $y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \dots + P_1(x) y' + P_0(x) y = g(x) \neq 0$

where $P_{n-1}(x), \dots P_0(x), g(x)$ are all cts on (α, B)

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1 ⓐ

ⓑ with non constant coeffecient

cauchy - Euler equations :

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \dots + a_1 x y' + a_0 y = 0$$

let $y = x^r$ then find $y', y'', y''', \dots y^{(n)}$

Now, plug $y', y'' \dots, y^{(n)}$ in the D.E to obtain the char. eq

ex Solve

-1- $x^3 y''' - 3x^2 y'' + 6x y' - 6y = 0 \ , \quad x > 0$

let $y = x^r$ $\quad y' = r x^{r-1}$ $\quad y'' = r(r-1) x^{r-2}$ $\quad y''' = r(r-1)(r-2) x^{r-3}$

Char eq. : $r(r-1)(r-2) - 3r(r-1) + 6r - 6 = 0$

$(r-1)(r(r-2) - 3r + 6) = 0$

$(r-1)(r-2)(r-3) = 0 \quad r_1 = 1 \quad r_2 = 2 \quad r_3 = 3$

G.S: $y = C_1 x + C_2 x^2 + C_3 x^3$

-2- $8x^3y''' + 10x^2y'' + 3xy' - 3y = 0$

char. eq : $8r(r-1)(r-2) + 10r(r-1) + 3r - 3 = 0$

$(r-1)[8r(r-2) + 10r + 3] = 0$

$(r-1)\underbrace{[8r^2 - 6r + 3]}_{\text{complex Roots}} = 0 \quad r_1 = 1 \quad \alpha = \frac{6}{16} \quad \beta = \frac{\sqrt{60}}{16}$

G.S : $y_1 = c_1 x^1 + c_2 x^{3/8} \cos\left(\frac{\sqrt{15}}{8} \ln x\right) + c_3 x^{3/8} \sin\left(\frac{\sqrt{15}}{8} \ln x\right)$

-3- assume 1, 1, 2, 3-i, 3+i are the roots of the char. eq. of Cauchy-Euler homo. D.E

GS : $y = c_1 x + c_2 x \ln x + c_3 x^2 + c_4 x^3 \cos(\ln x) + c_5 x^3 \sin(\ln x)$

$\boxed{2}$ cnts:-

G.S: $y = y_h + y_p$ where y_h is the G.S of $y^{(n)} + P_{n-1} y^{n-1} + \dots y_0 = 0$

- y_p : is the particular solution.

$\boxed{1}$ undetermined coeff. (Same as is ch 2)

ex: solve $y''' - 3y'' + 3y' - y = 2x^2$

$y_h = y''' - 3y'' + 3y' - y = 0 \Rightarrow$ char eq $= y^3 - 3y^2 + 3y - 1 = 0 \Rightarrow r_1 = r_2 = r_3 = 1$

$y_h = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$

$y_p = Ax^2 + Bx + C \quad y'_p = 2Ax + B \quad y''_p = 2A \quad y'''_p = 0$

$0 - 3(2A) + 3(2Ax + B) - (Ax^2 + Bx + C) = 2x^2$

[2] Variation of parameters:

Assume $y_h = c_1y_1 + c_2y_2 \cdots c_ny_n$

$$y_p = \sum_{k=1}^{n} y_k \int \frac{W_k}{W} g(x)\,dx$$

where: $W = W(y_1, y_2 \cdots y_n)$

W_k = is the wronskain with the kth colomn is replaced by the vector $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

ex use any method to solve

$$x^3y''' - 3x^2y'' + 6xy' - 6y = x^4 \ln x, \quad x > 0$$

sol: G.S: $y = y_h + y_p$

$y_h = C_1x + C_2x^2 + C_3x^3$ (previous example)

$$W = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 6x^3 - 6x^3 + 2x^3 = 2x^3$$

$$W_1 = \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix} = x^4 \qquad W_2 = \begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & 1 & 6x \end{vmatrix} = 2x^3$$

$$W_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = x^2 \qquad g(x) = x \ln x$$

$$y_p = y_1\int \frac{W_1}{W} g\,dx + y_2 \int \frac{W_2}{W} g\,dx + y_3 \int \frac{W_3}{W} g\,dx$$

$$= x\int \frac{x^4}{2x^3} x \ln x\,dx + x^2 \int \frac{2x^3}{2x^3} x \ln x\,dx + x^3 \int \frac{x^2}{2x^3} x \ln x\,dx$$

$$= \frac{x}{2}\int x^2 \ln x\,dx + x^2 \int x \ln x\,dx + \frac{x^3}{2}\int \ln x\,dx$$

$$\left.\begin{aligned}\int \ln x\,dx &= x\ln x - x\\ \int x\ln x\,dx &= \frac{x^2}{2}\ln x - \frac{x^2}{4}\\ \int x^2\ln x\,dx &= \frac{x^3}{3}\ln x - \frac{x^3}{9}\end{aligned}\right\}\ \text{by Parts}$$

– Solve the D.E :-

canbe made seprable

أعلى درجة في البسط = أعلى درجة في المقام

$\boxed{1}$ $\dfrac{dy}{dx} = \dfrac{2y + \sqrt{x^2+y^2}}{2x} = \dfrac{2y}{2x} + \dfrac{\sqrt{x^2+y^2}}{2x}$

$= \left(\dfrac{y}{x}\right) + \tfrac{1}{2}\sqrt{\dfrac{x^2}{x^2} + \left(\dfrac{y}{x}\right)^2} = \left(\dfrac{y}{x}\right) + \tfrac{1}{2}\sqrt{1 - \left(\dfrac{y}{x}\right)^2}$

let $u = \dfrac{y}{x} \rightarrow y = xy \rightarrow y' = u + u'x$

$\cancel{u} + xu' = \cancel{u} + \tfrac{1}{2}\sqrt{1+u^2}$

$\displaystyle\int \frac{1}{\sqrt{1-u^2}}\, du = \int \frac{dx}{2x} \Rightarrow \sin^{-1}(u) = \tfrac{1}{2}\ln x + C$

$\dfrac{y}{x} = u = \sin\left(\tfrac{1}{2}\ln x + c\right)$

$\boxed{2}$ $\underbrace{(\cos 2y - \sin x)}_{M} dx \underbrace{- 2\tan x \sin 2y}_{N} dy = 0$, $y(0) = 0$

$M_y = -2\sin 2y \neq N_x = -2\sec^2 x \sin 2y$

Not exact but Must be can be made exact

$R(x) = \dfrac{M_y - N_x}{N} = \dfrac{-2\sin 2y + 2\sec^2 x \sin 2y}{-2\tan x \sin 2y}$

$= \dfrac{\cancel{-2}\,\cancel{\sin 2y}(1 - \sec^2 x)}{\cancel{-2}\tan x\, \cancel{\sin 2y}} = -\tan x$

$F(x) = e^{\int -\tan x\, dx} = e^{-\ln \sec x} = \cos x$

$(\cos 2y \cos x - \cos x \sin x)dx - 2\sin x \sin 2y\, dy = 0 \rightarrow$ exact $\exists\, w(x,y) = C$

st:-

1) $U_x = \cos 2y \cos x - \cos x \sin x$

2) $U_y = -2\sin x \sin 2x$

from 2 $\quad U(x,y) = \int -2\sin x \sin 2y \, dy + g(x) = +2\sin x \cos 2y + g(x)$

diff both sides w.r.t x $\quad U_x = \cos x \cos 2y + g'(x)$

use 1 $\quad \cos 2y \cos x - \cos x \sin x = \cos x \cos 2y + g'(x)$

$\therefore g(x) = -\int \cos x \sin x \, dx = -1/2 \int \sin 2x \, dx = 1/4 \cos 2x$

$\boxed{3}$ Find the value of n, so that $(2xy + 2xe^y)dx + (x^2 + x^n e^y) dy = 0$ is exact

$$M_y = N_x$$

$M_y = 2x + 2xe^y \qquad N_x = 2x + nx^{n-1}e^y$

$M_y = N_x$ iff $\quad n = 2$

$\boxed{4}$ If $F(x) = x$ is an int. factor of $1/2\, x^n dy + (x^2 + y) dx = 0$ then find n

$1/2\, x^{n+1} dy + (x^3 + xy) dx = 0$

$M_y = N_x$

$x = \frac{n+1}{2} x^n \quad \Rightarrow n = 1$

$\boxed{5}$ If $M(x,y)dx + (xe^{xy} + 2xy + \frac{1}{x}) dy = 0$ is exact, find $M(x,y)$

$\frac{\partial M}{\partial y} = M_y = N_x = e^{xy} + xye^{xy} + 2y - \frac{1}{x^2}$

$\therefore M(x,y) = \int e^{xy} + xye^{xy} + 2y - \frac{1}{x^2} \, dy \ldots$

$\boxed{6}$ let $(2xy^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0$
find the integration factor.

$M_y = 8xy^3e^y + 2xy^4e^y + 6xy^2 + 1$
$N_x = 2xy^4e^y - 2xy^2 - 3$

$$R(x) = \frac{M_y - N_x}{N} = \frac{8xy^3e^y + \cancel{2xy^4e^y} + 6xy^2 + 1 - \cancel{2xy^4e^y} + 2xy^2 + 3}{x^2y^4e^y - x^2y^2 - 3x}$$

$$= \frac{8xy^3e^y + 8xy^2 + 4}{x^2y^4e^y - x^2y^2 - 3x} \quad X$$

$$R^*(y) = \frac{2xy^4e^y - 2xy^2 - 3 - 8xy^3e^y - 2xy^4e^y - 6xy^2 - 1}{2xy^4e^y + 2xy^3 + y}$$

$$= \frac{-8xy^2 - 8xy^3e^y - 4}{y(2xy^3e^y - 2xy^2 + 1)} = \frac{-4\cancel{(2xy^2 + 2xy^3e^y + 1)}}{y\cancel{(2xy^3e^y - 2xy^2 + 1)}} = \frac{-4}{y}$$

$$F(y) = e^{\int \frac{-4}{y}dy} = \frac{1}{y^4}$$

$\boxed{7}$ If $y\,dx + [y + \tan(x+y)]\,dy = 0$ find the integrating factor.

a) $F(x) = x$ b) $F(x) = x^2$ c) $F(x,y) = \sin(x+y)$ d) $F(x,y) = \cos(x+y)$

$$\cos(x+y)\,y\,dx + [\cos(x+y)\,y + \sin(x+y)]\,dy = 0$$

$$\Rightarrow M_y = N_x$$

$\boxed{8}$ Solve $y' = \dfrac{x}{x^2y + y^3}$

$\frac{dx}{dy} = \frac{x^2 y + y^3}{x}$ $\Rightarrow$ $x'(y) = xy + \frac{y^3}{x}$

$x' - yx = y^3 x^{-1}$ $\Rightarrow$ $x' + P(y)x = g(y)x^n$ Bernoulli in x.

Let $u = x^{1-(-1)} = x^2$ $\quad u' = 2xx'$

$2xx' - 2yx^2 = 2y^3$ $\Rightarrow$ $u' - 2yu = 2y^3$

$I.F = F(y) = e^{-\int 2y\,dy} = e^{-y^2}$

$\Rightarrow u(y) = \frac{1}{e^{-y^2}}\left[\int e^{-y^2} \cdot 2y^3\, dy + c\right]$ $\rightarrow$ substitution or parts.

let $z = -y^2 \rightarrow \frac{dz}{dy} = -2y$

$\int e^z \cdot 2y^3 \cdot \frac{dz}{-2y} = +\int e^z \cdot +z\, dz$ $\rightarrow$ Parts.

$\boxed{d}$ $y' = \frac{x^2 + 2xy + y^2 + 1}{x^2 + 2xy + y^2 - 1} = \frac{(x+y)^2 + 1}{(x+y)^2 - 1}$ let $z = x + y$

$z' = 1 + y'$

$z' - 1 = \frac{z^2 + 1}{z^2 - 1}$

$z' = \frac{z^2+1}{z^2-1} + \frac{z^2-1}{z^2-1} = \frac{2z^2}{z^2-1} = \frac{dz}{dx}$

$\int \frac{z^2-1}{2z^2}\, dz = \int dx$ $\Rightarrow$ $\int \frac{1}{2} - \frac{1}{2} z^{-2}\, dz = x + C$

$1/2\, z + 1/2\, z^{-1} = x + C$

$(x+y) + (x+y)^{-1} = 2x + C$ *

$\boxed{10}$ solve $2x^2y'' + (y')^3 = 2x\,y'$, $x > 0$

second order + non linear
⇓
missing x OR missing y

missing(y): let $u = y' \Rightarrow u' = y''$

$2x^2u' + u^3 = 2xu \Rightarrow 2x^2u' - 2xu = -u^3$

$u' - \frac{1}{x}u = \frac{-1}{2x^2}u^3$ Bernoulli in U

let $z = U^{-2} \Rightarrow z' = -2U^{-3}u' \Rightarrow -2u^{-3}u' + 2xu^{-2} = \frac{1}{x^2}$

$z' + \frac{2}{x}z = \frac{1}{x^2} \Rightarrow F(x) = e^{\int \frac{2}{x}dx} = x^2$

$z(x) = \frac{1}{x^2}\left[\int x^2 \cdot \frac{1}{x^2}\,dx + C\right] \Rightarrow u^{-2} = \frac{1}{x} + \frac{C}{x^2} \Rightarrow U = \pm\sqrt{\frac{1}{1/x + C/x^2}}$

$y' = \sqrt{\frac{x^2}{x+C}} \Rightarrow y = \int \frac{x}{\sqrt{x+c}}\,dx$

- $C = 0$ $\quad y = \pm\int \frac{x}{\sqrt{x}}\,dx = \pm\int \sqrt{x}\,dx = \pm \frac{x^{3/2}}{\frac{3}{2}} + k$
- $C \neq 0$ $\quad y = \pm\int \frac{x}{\sqrt{x+c}}\,dx \Rightarrow w^2 = x + C$ ....

$\boxed{11}$ $2y^2y'' + 2y(y')^2 = 1$

missing(x): let $y' = u$ $\quad u' = \frac{dy'}{dx} = \frac{du}{dy} \cdot \underbrace{\frac{dy}{dx}}_{u} = y''\frac{du}{dy}$

$2y^2 \cdot u \cdot \frac{du}{dy} + 2yu^2 = 1 \Rightarrow \frac{du}{dy} + \frac{1}{y}u = \frac{1}{2y^2}u^{-1}$ ⇝ Bernoulli in U.

$V = u^{1-(-1)} = u^2 \Rightarrow \frac{dv}{dy} = 2u\frac{du}{dy} \Rightarrow 2u\frac{du}{dy} + \frac{2}{y}u^2 = \frac{1}{y^2}$

$\frac{dv}{dy} + \frac{2}{y}V = \frac{1}{y^2}$ ⇝ linear in V

I.F: $F(y) = e^{\int \frac{2}{y} dy} = y^2$

linear 2nd order missing x, y

non linear → missing y

linear → char eq

$V(y) = \frac{1}{y^2}\left[y^2 \cdot \frac{1}{y^2} dy + C\right] \Rightarrow V = \frac{y+C}{y^2}$

$u = \pm\sqrt{\frac{y+c}{y^2}} \Rightarrow y' = \pm\sqrt{\frac{y+c}{y^2}} \Rightarrow \int \sqrt{\frac{y^2}{y+c}}\, dy = X + C_2$

$\boxed{12}$ let $y'' - y = 4 \sinh x + \frac{x}{\sec^2 x}$ write a sutable form for the perticular solution. y_p

$y'' - y = \cancel{4} \cdot \frac{e^x - e^{-x}}{\cancel{2}} + x \cos^2 x = \underbrace{2e^x - 2e^{-x}}_{g_1(x)} + \underbrace{\frac{x}{2}}_{g_2(x)} + \underbrace{\frac{x}{2}\cos 2x}_{g_3(x)}$

$\cos^2 x \to 1/2(1+\cos 2x)$

let $y'' - y = 0$

$r^2 - 1 = 0 \to r = \pm 1 \to y_h = C_1 e^x + C_2 e^{-x}$

$y_p = y_{p_1} + y_{p_2} + y_{p_3}$

$y_{p_1} = (Ae^x + Be^{-x})x$

$y_{p_2} = Cx + D$

$y_{p_3} = (Ex + F)\cos 2x + (Gx + H)\sin 2x$

$y_p = (Ae^x + Be^{-x})x + (Cx + D) + (Ex + F)\cos 2x + (Gx + H)\sin 2x$

ex $y'' - 4y = e^{2x} + x^2$ find y_p

$y_h = C_1 e^{2x} + C_2 e^{-2x}$

$y_p = Ae^{3x} + Bx^2 + Cx + D$

$*x$

$y_p = y_{p_1} + y_{p_2}$

$y_{p_1}: y'' - 4y = e^{2x} \Rightarrow \underline{Ae^{2x}x}$

$y_{p_2}: y'' - 4y = x^2 \Rightarrow \underline{Bx^2 + Cx + D}$

$\boxed{13}$ Find the D.E whose G.S is $y = c_1 + c_2 e^{3x}$

$r_1 = 0$, $r_2 = 3$ $\rightsquigarrow$ Char eq $(r-0)(r-3) = 0$

$r^2 - 3r = 0$

$\rightarrow y'' - 3y' = 0$

- $y = c_1 e^{2x} + c_2 x e^{2x}$

Char. eq $(r-2)(r-2) = 0 \rightsquigarrow r^2 - 4r + 4 = 0 \rightarrow y'' - 4y' + 4y = 0$

- $y = c_1 x + c_2 x^2$

$(r-1)(r-2) = 0 \rightsquigarrow r^2 - 3r + 2 = 0 \rightsquigarrow r^2 - r - 2r + 2 = 0$

$r(r-1) - 2r - 2 = 0$

$\rightarrow x^2 y'' - 2xy' + 2y = 0$

- $y = \underbrace{c_1 e^x + c_2 e^{-2x}}_{y_h} + \underbrace{2\cos 3x}_{y_p}$

from $y_h = c_1 e^x + c_2 e^{-x} \rightarrow r_1 = 1$, $r_2 = -2$

Char. eq. : $(r-1)(r+2) = 0$

$r^2 + r - 2 = 0$

$y'' + y' - 2y = 0$

$y'' + y' - 2y = g(x)$ $\qquad$ $y''_p + y'_p - 2y_p = g(x)$

$-18\cos 3x + -6\sin 3x + 4\cos 3x = g(x)$

$-22\cos 3x - 6\sin 3x = g(x)$

$\rightarrow \therefore y'' + y' - 2y = -22\cos 3x - 6\sin 3x$

$\boxed{14}$ solve : $x^2y'' + xy' + 4y = x^2 \cos 2\ln x$

y_h : $x^2y'' + xy' + 4y = 0 \Rightarrow r(r-1) + r + 4 = 0$

$r^2 + 4 = 0 \Rightarrow r = \pm 2i \qquad \alpha = 0 , \beta = 2$

$$y_h = \underbrace{C_1\cos(2\ln x)}_{y_1} + C_2 \underbrace{\sin(2\ln x)}_{y_2}$$

$$W = \begin{vmatrix} \cos 2\ln x & \sin 2\ln x \\ \frac{-2\sin(2\ln x)}{x} & \frac{2\cos(2\ln x)}{x} \end{vmatrix} = \frac{2\cos^2(2\ln x)}{x} + \frac{2\sin^2(2\ln x)}{x} = \frac{2}{x}$$

$g(x) = \cos 2\ln x$

$$y_p = y_1\int \frac{y_2 g}{w}\,dx + y_2\int\frac{y_1 g}{w}\,dx = -\cos(2\ln x) \cdot \int \frac{\sin(2\ln x) \cdot \cos(2\ln x)}{2/x}\,dx$$

$$+ \sin(2\ln x) \cdot \int \frac{\cos(\ln x) \cdot \cos(2\ln x)}{2/x}\,dx$$

$\boxed{15}$ solve : $\frac{dy}{dx} = \frac{x+y+3}{x-y-4}$

let $x = X + a \qquad \frac{dy}{dx} = \frac{dY}{dX}$

$y = Y + b$

$u = \tan\frac{x}{2} \Rightarrow \sin x = \frac{2u}{1+u^2}$

$\cos x = \frac{1-u^2}{1+u^2} \Rightarrow dx = \frac{2}{1+u^2}\,du$

we need to find a & b so that

$\Rightarrow \frac{dY}{dX} = \frac{X + Y + a + b + 3}{X - Y + a - b - 4}$

$\left.\begin{array}{l} a + b + 3 = 0 \\ a - b - 4 = 0 \end{array}\right\} \quad \begin{array}{l} a = 1/2 \\ b = -7/2 \end{array}$

$\Rightarrow \frac{dY}{dx} = \frac{X+Y}{X-Y} \Rightarrow \frac{dY}{dx} = \frac{1 + \frac{Y}{X}}{1 - \frac{Y}{X}} \Rightarrow$ let $u = \frac{Y}{X} \Rightarrow$ seprable.

$Y = f(X) + C \qquad y + \frac{7}{2} = f(x - 1/2) + C$

ex solve:

[I] $y^{(4)} + y''' = \sin 2t$

G.S: $y = y_h + y_t$

y_h: $y^{(4)} + y''' = 0$

$r^4 + r^3 = 0$

$r^3(r+1) = 0 \qquad r_1 = r_2 = r_3 = 0 \qquad r_4 = -1$

$y_h = c_1 + c_2 x + c_3 x^2 + c_4 e^{-x}$

let $y_p = A\cos 2x + B\sin 2x$

$y'_p = -2A\sin(2x) + 2B\cos(2x)$

$y''_p = -4A\cos 2x - 4B\sin 2x$

$y'''_p = 8A\sin 2x - 8B\cos 2x$

$y^{(4)}_p = 16A\cos 2x + 16B\sin 2x$

$y_p^{(4)} + y'''_p = \sin 2x$

$(16A - 8B)\cos 2x + (8A + 16B)\sin 2x = \sin 2x$

$2(16A - 8B = 0)$

$8A + 16B = 1$

$40A = 1 \Rightarrow A = 1/40 \Rightarrow B = 1/20$

$y_p = \frac{1}{40}\cos 2x + \frac{1}{2}\sin 2x$

$GS = y_p + y_h$

$\boxed{2}$ $y''' - 3y'' + 3y' - y = \underbrace{e^x}_{g_1(x)} + \underbrace{-x-1}_{g_2(x)}$

$y = y_h + y_p$

y_h: $y''' - 3y'' + 3y' - y = 0$

$r^3 - 3r^2 + 3r - 1 = 0$

$(r-1)(r-1)(r-1) = 0 \quad r_1 = r_2 = r_3 = 1$

$y_h = C_1 e^x + C_2 x e^x + C_3 x^2 e^x$

$y_p = y_{p1} + y_{p2}$

let $y_{p1} = A e^x (x^3)$; $\Rightarrow e^x (Ax + B)$

$y_{p1}' =$

$y_{p1}'' =$

$y_{p1}''' =$

Plug in $y''' - 3y'' + 3y' - y = e^x$ $\quad A = -\frac{1}{6}$

$y_{p1} = -\frac{1}{6} x^3 e^x$

let $y_{p2} = Ax + B$

$y_{p2}' = A$

$y_{p2}'' = 0$

$y_{p2}''' = 0$

Plug in $y''' - 3y'' + 3y' - y = -x - 1$ $\quad A = 1$

$y_{p2} = x + 4$

$Y_p = -\frac{1}{6} x^3 e^x + x + 4$

$GS = Y_p + Y_h$

$\boxed{3}$ $y''' - y'' + y' - y = \sec x \qquad -\pi/2 < x < \pi/2$

$y_h: \quad y''' - y'' + y' - y = 0$

$r^3 - r^2 + r - 1 = 0$

$r^2(r-1) + (r-1) = 0$

$(r-1)(r^2+1) = 0 \rightarrow r_1 = 1 \quad r_2 = i \quad r_3 = -i \rightarrow \alpha = 0 \quad \beta = 1$

$y_h = \underbrace{c_1 e^x}_{1} + \underbrace{c_2 \cos x}_{2} + \underbrace{c_3 \sin x}_{3}$

$$w(y_1, y_2, y_3) = \begin{vmatrix} e^x & \cos x & \sin x \\ e^x & -\sin x & \cos x \\ e^x & -\cos x & -\sin x \end{vmatrix} = e^x - (\cos x)(-e^x \sin x - e^x \cos x) + \sin x(-e^x \cos x + e^x \sin x) = 2e^x$$

$$w_1 = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 1 & -\cos x & -\sin x \end{vmatrix} = 1 \qquad w_2 = \begin{vmatrix} e^x & 0 & \sin x \\ e^x & 0 & \cos x \\ e^x & 1 & -\sin x \end{vmatrix} = -e^x \cos x + e^x \sin x$$

$$w_3 = \begin{vmatrix} e^x & \cos x & 0 \\ e^x & -\sin x & 0 \\ e^x & -\cos x & 1 \end{vmatrix} = -e^x \sin x - e^x \cos x$$

$$y_p = y_1 \underbrace{\int \frac{w_1}{w} g \, dx}_{\text{I}} + y_2 \underbrace{\int \frac{w_2}{w} g \, dx}_{\text{II}} + y_3 \underbrace{\int \frac{w_3}{w} g \, dx}_{\text{III}}$$

$\text{I} = \int \frac{1}{2e^x} \sec x \, dx \Rightarrow$ by parts

$\text{II} = \int \frac{\cancel{e^x}(\sin x - \cos x)}{2\cancel{e^x}} \cdot \sec x = \frac{1}{2} \int \tan x - 1 \, dx$

$\text{III} = \int \frac{-\cancel{e^x}(\sin x + \cos x)}{2\cancel{e^x}} \cdot \sec x$

$\boxed{4}$ $x^3y''' + x^2y'' - 2xy' + 2y = 2x^4 \qquad x>0$

H.W

"Variation Of Parameters" !

Ch7 : linear Algebra.

def:

A matrix: is a rectangular array of numbers (entries) listed in rows & columns between brackets

ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 4 & 0 \end{bmatrix}$

definitions:

$\boxed{1}$ size: number of rows x number of columns

ex: $A = \begin{bmatrix} 1 & 4 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}$ size = 3x2.

in general, An nxm matrix can be written as

$$A = (a_{ij})_{n\times m} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nm} \end{bmatrix}$$

a_{ij}: is the entrey in the i^{th} row & j^{th} column.

$\boxed{2}$ the zero matrix $\bar{O} = (0)_{n\times m}$: all entries are zero.

$\boxed{3}$ square matrix : $n=m$

ex: $A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$

[4] row matrix : $A = (a_{ij})_{1\times m} = [a_{11} \; a_{12} \; \cdots \; a_{1m}]$

[5] Column matrix : $A = (a_{ij})_{n\times 1} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}$

[6] Identity matrix (only for square)

$I_n = (a_{ij})_{n\times n}$, where $a_{ij} = \begin{cases} 1, & i=j \\ 0, & i\neq j \end{cases}$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}_{n\times n}$$

Rmk: if A is a matrix of size $n\times n$ then $A I_n = I_n A = A$

Operations on Matrices :-

[1] Scaler Mult.

IF $A = (a_{ij})_{n\times m}$ & $k \in \mathbb{C}$ then

$kA = (k a_{ij})_{n\times m}$

ex: $A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 4 & 5 \end{bmatrix}$ $\quad 5A = \begin{bmatrix} 10 & 0 & 5 \\ 15 & 20 & 25 \end{bmatrix}$

[2] Addition & Subtraction :

let $A = (a_{ij})_{n\times m}$ & $B = (b_{ij})_{n\times m}$ be two matrices of the Same size

then $A \pm B = (a_{ij} \pm b_{ij})_{n\times m}$

ex: $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix}$ $\quad C = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix}$

[1] $A + B = \begin{bmatrix} 4 & 3 \\ 4 & 4 \end{bmatrix}$ $\qquad$ [2] $A + C =$ undefined

[3] Matrix mult.

let $A = (a_{ij})_{n\times m}$ & $B = (b_{ij})_{s\times r}$

If $m = s$ then $A.B = C = (c_{ij})_{n\times r}$ where $c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$

ex: let $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \end{bmatrix}_{2\times 3}$ $B = \begin{bmatrix} 3 & 1 \\ 4 & 5 \\ -1 & -2 \end{bmatrix}_{3\times 2}$ $C = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}_{2\times 2}$

[1] $A.B = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \end{bmatrix} . \begin{bmatrix} 3 & 1 \\ 4 & 5 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 11 & 11 \\ 5 & 6 \end{bmatrix}$

[3] $A.C$
[4] $C.B$
} undetermined

[2] $B.A = \begin{bmatrix} 3 & 1 \\ 4 & 5 \\ -1 & -2 \end{bmatrix} . \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 9 & 4 \\ 1 & 23 & 20 \\ 1 & -8 & -8 \end{bmatrix}$

→ Matrix multiplication is not comutative $A.B \neq B.A$

• Determinant (only for square)

[1] 2×2: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $|A| = \det A = ad - cb$

[2] 3×3: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$|A| = \det A = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

• Transpose (any size) اطلاعات

if $A = (a_{ij})_{n\times m}$ then $A^T = (a_{ij})_{m\times n}$

ex: $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 4 \end{bmatrix}_{2\times 3}$ $A^T = \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 3 & 4 \end{bmatrix}_{3\times 2}$

Rmk: $\boxed{1}$ $(A \pm B)^T = A^T \pm B^T$

$\boxed{2}$ $(AB)^T = B^T \cdot A^T$

- Trace (only for square matrix)

if $A = (a_{ij})_{n \times n}$ then $tr(A) = \sum_{i=1}^{n} a_{ii}$

ex: $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 4 \\ 5 & 0 & 10 \end{bmatrix} \Rightarrow tr(A) = 2 - 1 + 10 = \underline{\underline{11}}$

Solving linear systems:-

$\boxed{1}$ $\underline{2 \times 2}$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned} \Rightarrow \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{\text{coeff. matrix}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\text{vector of unknowns}} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}}_{\text{constant vector}}$$

$\boxed{2}$ $\underline{3 \times 3}$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \Rightarrow \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$A\vec{x} = \vec{b}$$

- Any system can be written in matrix form.

Recall:

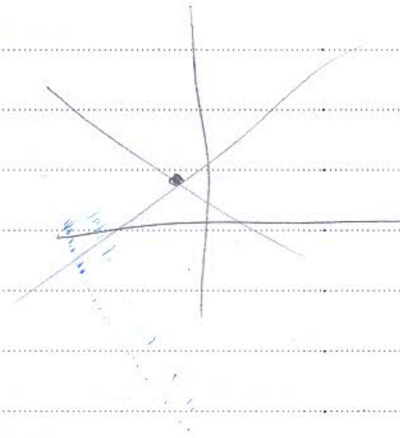
if $A\vec{x} = \vec{b}$ is any system then the system has either

$\boxed{1}$ 1 solution

or $\boxed{2}$ no solution

or $\boxed{3}$ infinitly many solutions

ex: $\left.\begin{array}{l} 2x + 3y = 8 \\ x - y = -1 \end{array}\right\} \Rightarrow$ one solution $(1,2)$



ex: $\left.\begin{array}{l} 2x + 3y = 8 \\ 4x + 6y = 10 \end{array}\right\} \Rightarrow$ has no solutions

ex: $\left.\begin{array}{l} x + 2y = 1 \\ 2x + 4y = 2 \end{array}\right\} \Rightarrow$ infinitly many solutions

$\Rightarrow y = \frac{1-x}{2}$ x is free variable (independent)

let $x = t$, $t \in R$ $\quad y = \frac{1-t}{2}$

- To solve any system we have 3 methods

1. Gaussian elemenation
2. inverse
3. Cramer's rule.

I Guassian elemenation method.

let $A\vec{x} = \vec{b}$ be any system.

step 1: form the augmented matrix $(A \,\vdots\, \vec{b})$ $\left(\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right)$

step 2: Apply a sequence of elementary row operations to transform the augmented matrix into $\left(\begin{array}{ccc|c} 1 & 0 & 0 & k_1 \\ 0 & 1 & 0 & k_2 \\ 0 & 0 & 1 & k_3 \end{array}\right)$

Step 3: $x_1 = k_1 \quad x_2 = k_2 \quad x_3 = k_3$

$$(A \,\vdots\, \vec{b}) \xrightarrow[\text{row operations}]{\text{elemantry}} (I \,\vdots\, \vec{k})$$

* elementary row operations:

1. enterchanging any two rows ~~or~~ columns in the aug. matrix
2. multiPly arow by a constant
3. adding two rows (or subtracting)

ex: solve

$$2x_1 + x_2 - 6x_3 = -14$$
$$x_1 + 2x_2 + x_3 = 8$$
$$-x_1 - 2x_2 + 4x_3 = 7$$

Step 1 :
$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{ccc|c} 2 & 1 & -6 & -14 \\ 1 & 2 & 1 & 8 \\ -1 & -2 & 4 & 7 \end{array}\right)$$

Step 2 : $R_1 \longleftrightarrow R_2$
$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 2 & 1 & -6 & -14 \\ -1 & -2 & 4 & 7 \end{array}\right)$$

Step 3 : $\xrightarrow{-2R_1 + R_2}$ $R_1 + R_3$
$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -3 & -8 & -30 \\ 0 & 0 & 5 & 15 \end{array}\right)$$

Step 4 : $\xrightarrow{-\frac{1}{3}R_2}$
$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 1 & 8/3 & 10 \\ 0 & 0 & 5 & 15 \end{array}\right)$$

Step 5 : $\xrightarrow{-2R_2 + R_1}$
$$\left(\begin{array}{ccc|c} 1 & 0 & -13/3 & -12 \\ 0 & 1 & 8/3 & 10 \\ 0 & 0 & 5 & 15 \end{array}\right)$$

Step 6 : $\xrightarrow{1/5\, R_3}$
$$\left(\begin{array}{ccc|c} 1 & 0 & -13/3 & -12 \\ 0 & 1 & 8/3 & 10 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

Step 7 : $\xrightarrow{-\frac{8}{3}R_3 + R_3}$ $\frac{13}{3}R_3 + R_1$
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$x_1 = 1$

$x_2 = 2$

$x_3 = 3$

ex: Solve the System:

$$3x_1 - 2x_2 + x_3 = -2$$
$$2x_1 + 2x_2 + 3x_3 = 9$$
$$x_1 + x_2 + 4x_3 = 12$$

sol: $\begin{pmatrix} 3 & -2 & 1 \\ 2 & 2 & 3 \\ 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 9 \\ 12 \end{pmatrix} \Rightarrow \left(\begin{array}{ccc|c} 3 & -2 & 1 & -2 \\ 2 & 2 & 3 & 9 \\ 1 & 1 & 4 & 12 \end{array}\right)$

$R_1 \Leftrightarrow R_3$ $\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 4 & 12 \\ 2 & 2 & 3 & 9 \\ 3 & -2 & 1 & -2 \end{array}\right) \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \left(\begin{array}{ccc|c} 1 & 1 & 4 & 12 \\ 0 & 0 & -5 & -15 \\ 0 & -5 & -11 & -38 \end{array}\right)$

$\xrightarrow[-1/5 * R_2]{R_2 \Leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 1 & 4 & 12 \\ 0 & 1 & \frac{11}{5} & 38/5 \\ 0 & 0 & -5 & -15 \end{array}\right) \xrightarrow{-R_2+R_1} \left(\begin{array}{ccc|c} 1 & 0 & +9/5 & 22/5 \\ 0 & 1 & 11/5 & 38/5 \\ 0 & 0 & -5 & -15 \end{array}\right)$

$-1/5 R_3$ $\left(\begin{array}{ccc|c} 1 & 0 & +9/5 & 22/5 \\ 0 & 1 & 11/5 & 38/5 \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow[-9/5 R_3+R_1]{-\frac{11}{5}R_3+R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right)$

$\boxed{x_1 = -1 \quad x_2 = 1 \quad x_3 = 3}$

ex: use G. elemenation method to solve.

1. $2x_1 - x_2 = 1$
   $3x_1 + x_2 = 4$

2. $2x_1 + x_2 = 1$
   $4x_1 + 2x_2 = 2$

3. $x_1 + 3x_2 = 2$
   $3x_1 + 9x_2 = 5$

has no solution $\rightarrow \left(\begin{array}{cc|c} 0 & 0 & 2 \end{array}\right)$

Sol: 2

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 2 & 1 & 1 \\ 4 & 2 & 2 \end{array}\right) \xrightarrow{-2R_1 + R_2} \left(\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right)$$ this system has infenetly many solutions.

$2x_1 + x_2 = 1$

$x_2 = 1 - 2x_1$ $\quad$ x_1 is a free variable

so let $x_1 = t$, $t \in \mathbb{R}$

$x_2 = 1 - 2x_1$ $\quad \therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t \\ 1-2t \end{pmatrix} = \begin{pmatrix} 0+t \\ 1-2t \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} t \\ -2t \end{pmatrix}$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is a solution for the system for any value of $t \in \mathbb{R}$

* Inverese (square)

$\boxed{I}$ inverse of 2×2 matrices.

def: let A be an $n\times n$ matrix, A is invertable iff $\exists$ an $n\times n$ matrix B s.t $AB = BA = I_n$.
we will denote such matrix B by $\boxed{B = A^{-1}}$

Thrm: let A be an $n\times n$ matrix, A^{-1} exist iff $\det(A) \neq 0$
$\equiv$ A^{-1} d.n.e iff $\det(A) = 0$

thrm: let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ s.t $\det(A) = ad - bc \neq 0$ then A^{-1} exists and

$$A^{-1} = \frac{1}{\det(A)} * \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

ex Determine whether A^{-1} exists or not if $A = \begin{pmatrix} 2 & +1 \\ -2 & 4 \end{pmatrix}$

$$\det(A) = 8-(-2) = 10 \neq 0$$

$$A^{-1} = \frac{1}{10}\begin{pmatrix} 4 & -1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 0.4 & -0.1 \\ 0.2 & 0.2 \end{pmatrix}$$

check: $AA^{-1} = \begin{pmatrix} 2 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 0.4 & -0.1 \\ 0.2 & 0.2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$(A \mid I_3) \xrightarrow[\text{row operations}]{\text{elementary}} (I_3 \mid A^{-1})$$

Inverse of 3x3 matrix :

let A be 3×3 matrix. To find the inverse of A:

step 1 : if $\det(A) \neq 0$, then

step 2 : form the augmentary matrix $(A \mid I_3)$

step 3 : use the elementary row operations to obtain $(I_3 \mid B)$

$\therefore B = A^{-1}$

ex let $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ -1 & 2 & 2 \end{pmatrix}$ find A^{-1} if exists

$$\det(A) = -10 - 2.4 + (-3) = -21 \neq 0$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & 4 & 0 & 1 & 0 \\ -1 & 2 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_1+R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & 4 & 0 & 1 & 0 \\ 0 & 4 & 5 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\substack{4R_2+R_3 \\ -1\times R_2}]{2R_2+R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 11 & 1 & 2 & 0 \\ 0 & 1 & -4 & 0 & -1 & 0 \\ 0 & 0 & 21 & 1 & 4 & 1 \end{array}\right)$$

$$\xrightarrow{\frac{1}{21}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 11 & 1 & 2 & 0 \\ 0 & 1 & -4 & 0 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{21} & \frac{4}{21} & \frac{1}{21} \end{array}\right) \xrightarrow[-11R_3+R_1]{4R_3+R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{10}{21} & \frac{-2}{21} & \frac{-11}{21} \\ 0 & 1 & 0 & \frac{4}{21} & \frac{-5}{21} & \frac{4}{21} \\ 0 & 0 & 1 & \frac{1}{21} & \frac{4}{21} & \frac{1}{21} \end{array}\right)$$

ex: let $A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 4 & 5 \\ 0 & 1 & -2 \end{pmatrix}$

• Inverse method to solve a system.

Let $A\vec{x} = \vec{b}$ s.t $\det(A) \neq 0$

$\Rightarrow A^{-1} A \vec{x} = A^{-1} \vec{b}$

$I \vec{x} = A^{-1} \vec{b}$

$\underline{\vec{x} = A^{-1} \vec{b}} \longrightarrow$ solution

ex solve the system.
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ -x_2 + 4x_3 &= 5 \\ -x_1 + 2x_2 + 2x_3 &= 2 \end{aligned}$$

sol: $\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ -1 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{10}{21} & \frac{-2}{21} & \frac{-11}{21} \\ \frac{4}{21} & \frac{-5}{21} & \frac{4}{21} \\ \frac{1}{21} & \frac{4}{21} & \frac{1}{21} \end{pmatrix} \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$$

$$= \frac{1}{21}\underset{3\times3}{\begin{pmatrix} 10 & -2 & -11 \\ 4 & -5 & 4 \\ 1 & 4 & 1 \end{pmatrix}}\underset{3\times1}{\begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}} = \frac{1}{21}\begin{pmatrix} -22 \\ -13 \\ 23 \end{pmatrix} = \begin{pmatrix} \frac{-22}{21} \\ \frac{-13}{21} \\ \frac{23}{21} \end{pmatrix}$$

- Cramer's Role (3rd method to solve a system)

ex Solve

$\boxed{1}$ $2x + 3y = 1$

$-x + 4y = 5$

Sol

(1) $\begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

(2) $\det(A) = 11 \neq 0$

(3) $A_1 = \begin{pmatrix} 1 & 3 \\ 5 & 4 \end{pmatrix}$ $\quad A_2 \begin{pmatrix} 2 & 1 \\ -1 & 5 \end{pmatrix}$

(4) $\det(A_1) = -11$ $\quad \det(A_2) = 11$

(5) $x = \dfrac{\det A_1}{\det A} = \dfrac{-11}{11} = -1$

$y = \dfrac{\det A_2}{\det A} = \dfrac{11}{11} = 1$

$\boxed{2}$ $2x_1 + 3x_2 - 4x_3 = 1$ use cramer's Rule

$-x_1 + x_2 + x_3 = 2$

$x_1 + 2x_2 + 5x_3 = 3$

$$A = \begin{pmatrix} 2 & 3 & -4 \\ -1 & 1 & 1 \\ 1 & 2 & 5 \end{pmatrix} \qquad \det(A) = 6 - 3(-6) - 4(-3) = 36 \neq 0$$

$$A_1 = \begin{pmatrix} 1 & 3 & -4 \\ 2 & 1 & 1 \\ 3 & 2 & 5 \end{pmatrix} \qquad \det(A_1) = 3 - 3(7) - 4 = -22$$

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-22}{36}$$

$$A_2 = \begin{pmatrix} 2 & 1 & -4 \\ -1 & 2 & 1 \\ 1 & 3 & 5 \end{pmatrix} \qquad \det(A_2) = 14 + 6 + 20 = 40$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{40}{36}$$

$$A_3 = \begin{pmatrix} 2 & 3 & 1 \\ -1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \qquad \det(A_3) = 2(-1) + 3(-5) + 1(-3) = -20$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{-20}{36}$$

ليه

- Another method to find A^{-1} :

let A be 3×3 matrix s.t $\det(A) \neq 0$, then $A^{-1} = \frac{1}{\det(A)} \cdot C^T$

C: cofactor matrix

If $C = (C_{ij})$

then

$C_{ij} = (-1)^{i+j} \cdot M_{ij}$ where $M_{ij} = \begin{vmatrix} 2\times2 \\ \text{matrix} \end{vmatrix}$

↳ obtained by elemenating the ith row & the jth coloumn in A

ex let $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 4 & 3 & -2 \end{pmatrix}$ find if A^{-1} exists.

$$\det(A) = -3 - 1(-8) - 1 \cdot 6 = -1 \neq 0$$

let $C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$

$C_{11} = (-1)^2 \begin{vmatrix} 0 & 1 \\ 3 & -2 \end{vmatrix} = -3$ $\qquad$ $C_{12} = (-1)^3 \begin{vmatrix} 2 & 1 \\ 4 & -2 \end{vmatrix} = 8$

$C_{13} = (-1)^4 \begin{vmatrix} 2 & 0 \\ 4 & 3 \end{vmatrix} = 6$ $\qquad$ $C_{21} = (-1)^3 \begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix} = -1$

$C_{22} = (-1)^4 \begin{vmatrix} 1 & -1 \\ 4 & -2 \end{vmatrix} = 2$ $\qquad$ $C_{23} = (-1)^5 \begin{vmatrix} 1 & 1 \\ 4 & 3 \end{vmatrix} = 1$

$C_{31} = (-1)^4 \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$ $\qquad$ $C_{32} = (-1)^5 \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = -3$

$C_{33} = (-1)^6 \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -2$

$\Rightarrow C = \begin{pmatrix} -3 & 8 & 6 \\ -1 & 2 & 1 \\ 1 & -3 & -2 \end{pmatrix} \Rightarrow C^T = \begin{pmatrix} -3 & -1 & 1 \\ 8 & 2 & -3 \\ 6 & 1 & -2 \end{pmatrix}$

$$A^{-1} = \frac{1}{-1} \cdot \begin{pmatrix} -3 & -1 & 1 \\ 8 & 2 & -3 \\ 6 & 1 & -2 \end{pmatrix}$$

CH 4 : Systems of 1st order ODE

• General form: $y_1' = a_{11} y_1 + a_{12} y_2 + g_1(x)$ ①

$y_2' = a_{21} y_1 + a_{22} y_2 + g_2(x)$ ②

this System can be written as :

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix}$$

also it can be written as :

$$\vec{y}' = A\vec{y} + \vec{g}(x) \quad --- (*)$$

$\vec{y}' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}$, $\vec{y} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $A \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $\vec{g}(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix}$

• now, $(*)$ is equiuilant to $\vec{y}' - A\vec{y} = \vec{g}(x)$

→ If $\vec{g}(x) = 0$, $(*)$ is called homogenous.

→ If $g(x) \neq 0$, $(*)$ is called non-homogenous.

Def: the Solution of the system $(*)$ is a Pair of functions, $y_1(x)$ & $y_2(x)$ that satisfies both eq. ① & ②

ex let $y_1' = -3y_1 + y_2$

$y_2' = y_1 - 3y_2$

Show that $\vec{y} = \begin{pmatrix} e^{-2t} \\ e^{-2t} \end{pmatrix}$ is a solution of the system.

Sol: $\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} \overset{?}{=} \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$\vec{y}' \begin{pmatrix} -2e^{-2x} \\ -2e^{-2x} \end{pmatrix} \overset{??}{=} \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} e^{-2x} \\ e^{-2x} \end{pmatrix}$

$= \begin{pmatrix} -2e^{-2x} \\ -2e^{-2x} \end{pmatrix} \Rightarrow$ indeed $\vec{y} = \begin{pmatrix} e^{-2t} \\ e^{-2t} \end{pmatrix}$ is a solution

Def: let A be an $n \times n$ matrix. A constant $\lambda \in \mathbb{C}$ is called an eigenvalue of A if $\exists\ \underline{\vec{x} \neq 0}$ s.t $A\vec{x} = \lambda \vec{x}$

Such vector $\vec{x}$ is called the eigenvector corresponds to λ and denoted by $\vec{x}_\lambda$

ex let $A = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$, $\lambda = -2$, $\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ show that λ is an eigen value of A with $\vec{x}_\lambda = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Sol: $\begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda \vec{x}_\lambda$

recall: ① any system of linear eq. can be written as $A\vec{x} = \vec{b}$.

* ② If $\det(A) \neq 0$, then A^{-1} exists & the system has (by inverse method) the unique solution

$$\vec{x} = A^{-1} \vec{b}$$

- let λ be an eigen value of A with eigen vector $\vec{x}_\lambda$

then:

$$A\vec{x}_\lambda = \lambda\vec{x}_\lambda$$

$$A\vec{x}_\lambda - \lambda\vec{x}_\lambda = 0$$

$$(A-\lambda I)\vec{x}_\lambda = 0 \quad --- \quad (*)$$

we have two cases:

case ①: if $\det(A.\lambda I) \neq 0$ then $(*)$ has a unique solution

$$X_\lambda = (A.\lambda I)^{-1}.\vec{0} = \vec{0}$$

And this contradicts $\vec{x}_\lambda \neq 0$ (by def)

this case forces case 2

Case ②: $\det(A.\lambda I) = 0$ then $(*)$ has a non-trivial solution

$(*)$ has infinitly many solutions (values) for $\vec{x}_\lambda$

- to find the eigen values of $A_{2\times2}$ let $\det(A-\lambda I) = 0 \rightarrow$ char. eq

$\Rightarrow a\lambda^2 + b\lambda + c = 0 \rightarrow$ quadratic eq.

$D = b^2 - 4ac$

[1] if $D > 0 \rightarrow$ Real distinct

[2] $D = 0 \rightarrow$ Repeated

[3] $D < 0 \rightarrow$ complex.

ex: let $A = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$ find the eig. value.

sol: $\det(A - \lambda I) = 0$

$$\left| \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0 \qquad \left| \begin{pmatrix} 2-\lambda & 1 \\ 1 & -2-\lambda \end{pmatrix} \right| = 0$$

$(2-\lambda)(-2-\lambda) - 1 = 0$

$(\lambda - 2)(\lambda + 2) - 1 = 0 \qquad \lambda^2 - 5 = 0 \qquad \lambda = \pm\sqrt{5}$

ex $\begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$

$$\begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = 0 \Rightarrow (-3-\lambda)^2 - 1 = 0$$

$$(\lambda + 3)^2 = -1$$

$\lambda_1 = -2 \qquad \lambda_2 = -4$

ex: let $A = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} = 0$ find e.v of A

Char. eq: $|A - \lambda I| = 0 \qquad \begin{vmatrix} 1-\lambda & 1 \\ 3 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)^2 - 3 = 0 \qquad 1 - \lambda = \pm\sqrt{3} \qquad \lambda_1 = 1 - \sqrt{3}$

$\lambda_2 = 1 + \sqrt{3}$

Eigen Vector

To find the eigen vector $\vec{X}_\lambda$ coresponds to λ: $\Rightarrow$

① find λ

② plug the value of λ in the system $(A-\lambda I)\vec{X}_\lambda = 0 \rightarrow (*)$

③ let $\vec{X}_\lambda = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

then solve $(*)$ for x_1 & x_2 using the Guaissain elemenation method

ex let $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$ find the eigen values & the eigen vector

Char eq: $\begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 - 4 = 0$

$1-\lambda = \pm 2$

$\lambda_1 = 3 \qquad \lambda_2 = -1$

- for $\lambda_1 = 3$, let $\vec{X}_{\lambda_1} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\begin{pmatrix} 1-3 & 1 \\ 4 & 1-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\left((A-\lambda I)\vec{X}_\lambda = 0 \right)$

Augmented matrix

$$\begin{array}{c} R_1 \\ R_2 \end{array}\left(\begin{array}{cc|c} -2 & 1 & 0 \\ 4 & -2 & 0 \end{array}\right) \xrightarrow{2R_1+R_2} \left(\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$-2x_1 + x_2 = 0 \Rightarrow \boxed{x_2 = 2x_1}$

x_1 is a free variable, so let $x_1 = t$, $t \in \mathbb{R}$

$x_2 = 2t$

$\vec{X}_{\lambda_1} = \begin{pmatrix} t \\ 2t \end{pmatrix} = t\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ let $t = 1$

$\Rightarrow \vec{X}_{\lambda_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

• for $\lambda_2 = -1$ let $X_{\lambda_2} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$2a+b=0 \Rightarrow b = -2a$ → free

$\Rightarrow \vec{X}_{\lambda_2} \begin{pmatrix} a \\ -2a \end{pmatrix} = a\begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad a \in R$

let $a=1$ $X_{\lambda_2} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

ex let $A = \begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix}$ find λ_1, λ_2, & X_{λ_1} & X_{λ_2}

$\begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = 0 \qquad (2-\lambda)(5-\lambda)=0 \qquad \lambda_1 = 2, \quad \lambda_2 = 5$

for $\lambda_1 = 2$ let $\vec{X}_{\lambda_1} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 3 & 0 \end{array}\right)$

$\xrightarrow{-3R_1 + R_2} \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\qquad 0x_1 + x_2 = 0$

$x_2 = 0$

x_1 is a free variable

this eq. is true for any value of x_1

Let $x_1 = t$, $t \in R$

$\vec{X}_{\lambda} = \begin{pmatrix} t \\ 0 \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$t=1 \rightarrow \vec{X}_{\lambda_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad \vec{X}_{\lambda_2}$: ex

• Systems of 1st order homogenous D.E:

Consider the system $\vec{y}' = A\vec{y}$ ...(**) A: coeffs matrix (2x2)

$\vec{y} : \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$

to solve (**) (find $y_1(t)$ & $y_2(t)$)

step 1: find the eigen values : λ_1, λ_2

step 2: find the eigen vectors : $\vec{X}_{\lambda_1}$, $\vec{X}_{\lambda_2}$

step 3: write the G.S of (**)

there are 3 cases for G.S :-

case 1: If $\lambda_1 \neq \lambda_2$ are real numbers:

$$G.S: \vec{y}(t) = C_1 \vec{X}_{\lambda_1} e^{\lambda_1 t} + C_2 \vec{X}_{\lambda_2} e^{\lambda_2 t}$$

ex: find the G.S of

$$y_1' = 2y_1 + y_2$$
$$y_2' = 9y_1 + 2y_2$$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 9 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

① eigen Values: $\begin{vmatrix} 2-\lambda & 1 \\ 9 & 2-\lambda \end{vmatrix} = 0$

$(2-\lambda)^2 - 9 = 0 \Rightarrow \lambda_1 = -1 , \lambda_2 = 5$

• for $\lambda_1 = -1$ let $x_{\lambda_1} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

free variable $\Rightarrow$ can be any value!

$$\begin{pmatrix} 3 & 1 \\ 9 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$3x_1 + x_2 = 0 \Rightarrow x_2 = -3x_1 \Rightarrow x_1 = 1 \quad x_2 = -3$

$$\boxed{\vec{x}_{\lambda_1} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}}$$

• for $\lambda_2 = 5$ let $x_{\lambda_2} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$$\begin{pmatrix} -3 & 1 \\ 9 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \left(\begin{array}{cc|c} -3 & 1 & 0 \\ 9 & -3 & 0 \end{array}\right)$$

$-3y_1 + y_2 = 0 \Rightarrow y_2 = 3y_1 \Rightarrow y_1 = 1 \quad y_2 = 3$

$$\boxed{\vec{y}_{\lambda_2} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}}$$

G.S: $\vec{y}(t) = C_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{5t}$

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \Rightarrow \begin{aligned} y_1(t) &= c_1 e^{-t} + C_2 e^{5t} \\ y_2(t) &= -3c_1 e^{-t} + 3C_2 e^{5t} \end{aligned}$$

Case ②: If $\lambda_1 = \lambda_2 = \lambda$, then

GS: $\vec{y} = C_1 \vec{x}_\lambda e^{\lambda t} + C_2 [t \vec{x}_\lambda + \vec{V} e^{\lambda t}]$

where $\vec{V}$ is the solution of the system $(A - \lambda I)\vec{V} = \vec{x}_\lambda$

ex: Solve the system

$y_1' = y_1 - y_2$

$y_2' = y_1 + 3y_2$

$$\vec{y}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \vec{y}$$

Char eq : $\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = 0$

$(1-\lambda)(3-\lambda) + 1 = 0 \rightsquigarrow (\lambda-2)(\lambda-2) = 0 \rightsquigarrow \lambda_1 = \lambda_2 = 2 = \lambda$

let $x_\lambda = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightsquigarrow (A - \lambda I)\vec{x}_\lambda = \vec{0}$

$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightsquigarrow -x_1 - x_2 = 0$

$x_2 = -x_1 = -t \quad t \in R$

$\vec{X}_\lambda = \begin{pmatrix} t \\ -t \end{pmatrix} = t \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad t=1 \Rightarrow \boxed{X_\lambda = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

Let $\vec{V} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow (A - 2I) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\left(\begin{array}{cc|c} -1 & -1 & 1 \\ 1 & 1 & -1 \end{array}\right) \xrightarrow{R_1 + R_2} \left(\begin{array}{cc|c} -1 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right)$

$-v_1 - v_2 = 1$

$\left.\begin{array}{l} v_2 = -1 - v_1 \\ \text{let } v_1 = t \\ v_2 = -1 - t \end{array}\right\}$

$\vec{V} = \begin{pmatrix} t \\ -1-t \end{pmatrix} = \begin{pmatrix} 0 + t \\ -1 + t \end{pmatrix} \quad -v_2 - v_1 = 1 \rightarrow$

$\vec{v} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ let $t = 0$

G.S: $\vec{y} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left[t \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t} \right]$

ex: $\vec{y}' = \begin{pmatrix} 1 & 9 \\ -1 & -5 \end{pmatrix} \vec{y}$, $\vec{y}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Char.eq: $\begin{vmatrix} 1-\lambda & 9 \\ -1 & -5-\lambda \end{vmatrix} = 0$

$(1-\lambda)(-5-\lambda) + 9 = 0$

$\lambda^2 + 4\lambda - 5 + 9 = 0$

$\lambda^2 + 4\lambda + 4 = 0$

$(\lambda+2)(\lambda+2) = 0 \Rightarrow \lambda_1 = \lambda_2 = -2$ repeated

let $\vec{x}_\lambda = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\begin{pmatrix} 3 & 9 \\ \cancel{-1} & \cancel{-3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \cancel{0} \end{pmatrix}$

$3x_1 + 9x_2 = 0$

$x_2 = -1/3\, x_1$

x_1 is a free variable

let $x_1 = 3$, $x_2 = -1$

$x_\lambda = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

let $\vec{V} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$\begin{pmatrix} 3 & 9 \\ \cancel{-1} & \cancel{-3} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 3 \\ \cancel{-1} \end{pmatrix}$

$3v_1 + 9v_2 = 3$

$v_2 = 1/3 - 1/3\, v_1$

let $v_1 = 0$

$\vec{V} = \begin{pmatrix} 0 \\ 1/3 \end{pmatrix}$

G.S: $\vec{y} = c_1 \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^{-2t} + c_2 \left[t \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^{-2t} + \begin{pmatrix} 0 \\ 1/3 \end{pmatrix} e^{-2t} \right]$

$$= \begin{pmatrix} 3c_1 e^{-2t} + 3t c_2 e^{-2t} \\ -c_1 e^{-2t} - t c_2 e^{-2t} + 1/3 c_2 e^{-2t} \end{pmatrix} = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

but $\vec{y}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3c_1 \\ -c_1 + 1/3 c_2 \end{pmatrix} \Rightarrow c_1 = \frac{1}{3}$

$c_2 = -2.$

case ③ if $\lambda_1 = \alpha + i\beta$ & $\lambda_2 = \alpha - i\beta$ are complex roots

then G.S: $\vec{y}(t) = c_1 \vec{u}(t) + c_2 \vec{v}(t)$

$$\vec{u}(t) = e^{\alpha t}\left[\vec{a}\cos(\beta t) - \vec{b}\sin(\beta t)\right]$$

$$\vec{v}(t) = e^{\alpha t}\left[\vec{a}\,\cancel{\sin}\cos(\beta t) + \vec{b}\cos(\beta t)\right]$$

where $a = Re\,\vec{X}_{\lambda_1} = Re\,\vec{X}_{\lambda_2}$

$b = Im\,\vec{X}_{\lambda_1} = Im\,\vec{X}_{\lambda_2}$

$Z = 2 + 3i$

$Re\,Z = 2$

$Im\,Z = 3$

$$\vec{X} = \begin{pmatrix} 2i \\ 1+3i \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\vec{a}} + i \underbrace{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}_{\vec{b}}$$

Let $z = a + ib \in \mathbb{C}$

1) $\bar{z} = a - ib$

2) $|z| = \sqrt{a^2 + b^2}$

3) $|z|^2 = z \cdot \bar{z}$

4) $(\alpha + i\beta)(a + ib) = (\alpha a - \beta b) + i(\alpha b + \beta a)$

5) $i^2 = -1$

6) $\frac{1}{z} = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{a - ib}{a^2 + b^2}$

$= \frac{a}{a^2+b^2} - \frac{ib}{a^2+b^2}$

$(2 - 3i)(1 + i) = 5 - i$

ex: $\frac{1}{1-i} = \frac{1}{1-i} \cdot \frac{1+i}{1+i} = \frac{1+i}{1^2+1^2} = 1/2 + 1/2\, i$

ex: Solve the systems :

-1- $\vec{y}' = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix} \vec{y}$

Sol: Char. eq $\begin{vmatrix} -1/2-\lambda & 1 \\ -1 & -1/2-\lambda \end{vmatrix} = 0$ $\qquad (\lambda + 1/2)^2 + 1 = 0$

$\lambda_1 = \frac{-1}{2} + i \qquad \lambda_2 = -1/2 - i$

let $\vec{X}_{\lambda_1} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, then solve

$\alpha = -1/2 \qquad \beta = 1$

$(A - \lambda_1 I)\vec{X}_{\lambda_1} = 0$

$\begin{pmatrix} -i & 1 \\ -1 & i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-i x_1 + x_2 = 0 \qquad x_2 = i x_1 \qquad$ let $x_1 = 1 \rightarrow x_2 = i$

$\vec{X}_{\lambda_1} = \begin{pmatrix} 1 \\ i \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{a} + i \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{b}$

$\vec{u}(t) = e^{-1/2 t}\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin t\right] = e^{-1/2 t}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$

$\vec{v}(t) = e^{-1/2 t}\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos t\right] = e^{-1/2 t}\begin{pmatrix} \sin t \\ -\sin t \end{pmatrix}$

-2- $\vec{y}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \vec{y}$

Sol: Char. eq $\begin{vmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix} = 0$

$(\lambda - 3)(\lambda + 1) + 8 = 0$

$\lambda^2 + 2\lambda - 3 + 8 = 0$

$\lambda^2 - 2\lambda + 5 = 0$

$D = -16 < 0$

$\lambda_1 = \frac{-(-2)+\sqrt{-16}}{2} = 1+2i$ $\qquad \lambda_2 = 1-2i$

$\alpha = 1 \qquad \beta = 2$

let $\vec{X}_{\lambda_1} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\qquad \begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 2(1-i) & -2 & 0 \\ 4 & 2(-1-i) & 0 \end{array}\right) \xrightarrow{\frac{-2}{1-i}R_1 + R_2} \left(\begin{array}{cc|c} 2(1-i) & -2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow \cancel{2}(1-i)(x_1) - \cancel{2}x_2 = 0$

$x_2 = (1-i)x_1$

let $x_1 = 1 \longrightarrow x_2 = (1-i)$

$\frac{-2}{1-i} \times -2 + (-2-i2) =$

$= \frac{4}{1-i} \cdot \frac{1+i}{1+i} + (-2-i2)$

$= \frac{4(1+i)}{\cancel{2}} + (-2-i2)$

$= 0$

$$\vec{X}_{\lambda_1} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\vec{a}} + i \underbrace{\begin{pmatrix} 0 \\ -1 \end{pmatrix}}_{\vec{b}}$$

$$\vec{u}(t) = e^{t}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos 2t - \begin{pmatrix} 0 \\ -1 \end{pmatrix}\sin 2t\right]$$

$$\vec{v}(t) = e^{t}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos 2t + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\sin 2t\right]$$

$\boxed{2}$ non homogenous Systems:

$y_1'(t) = a_{11}y_1 + a_{12}y_2 + g_1(t)$

$y_2'(t) = a_{21}y_1 + a_{22}y_2 + g_2(t)$

matrix form: $\vec{y}' = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\vec{y} + \vec{g}(t)$ Provided that $\vec{g}(t) \neq 0$

G.S: $\vec{y} = \vec{y}_h + \vec{y}_p$

$\vec{y}_h$: the G.S of $\vec{y}' = A\vec{y}$

$\vec{y}_p$: is the particular solution related to $\vec{g}(t)$. It can be found by:

[I] undetermined coeff. : the Same as in Ch.2 & Ch.3 except when we have repeated terms.

ex Solve:

① $y' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \vec{y} + \begin{pmatrix} -6 \\ 2 \end{pmatrix} e^{-3t}$

sol: $\vec{y}_h$: solve $\vec{y}' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \vec{y}$ eigen values : $\lambda_1 = -2$, $\lambda_2 = -4$

eigen vectors : $\vec{x}_{\lambda_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{x}_{\lambda_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\vec{y}_h = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t}$

$\vec{y}_p = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{-3t}$ Plug $\vec{y}_p$ in the system.

$\vec{y}_p'$ → $-3 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \cancel{e^{-3t}} = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \cancel{e^{-3t}} + \begin{pmatrix} -6 \\ 2 \end{pmatrix} \cancel{e^{-3t}}$

$\begin{pmatrix} -3v_1 \\ -3v_2 \end{pmatrix} - \begin{pmatrix} -3v_1 + v_2 \\ v_1 - 3v_2 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \end{pmatrix}$

$\begin{pmatrix} -v_2 \\ -v_1 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \end{pmatrix} \Rightarrow$ $v_1 = -2$
$v_2 = 6$

② $\vec{y}' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \vec{y} + \begin{pmatrix} -6 \\ 2 \end{pmatrix} e^{-2t}$

sol: $\vec{y}_h = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t}$

$\vec{y}_p = \underline{t}\, \vec{v} e^{-2t} + \underline{\vec{u} e^{-2t}}$

الجذر مكرر $\Leftarrow \times t$ $+ \vec{u} e^{-2t}$

$\vec{y}_p' = \vec{v} e^{-2t} + t \vec{v} \left(-2e^{-2t}\right) - 2\vec{u} e^{-2t}$

Plug $\vec{y}_p$ in the system

$\vec{v}\cancel{e^{-2t}} - 2t\vec{v}\cancel{e^{-2t}} - 2\vec{u}\cancel{e^{-2t}} = A\left(t\vec{v}\cancel{e^{-2t}} + \vec{u}\cancel{e^{-2t}}\right) + \begin{pmatrix} -6 \\ 2 \end{pmatrix}\cancel{e^{-2t}}$

$\vec{v} - \boxed{2t\vec{v}} - 2\vec{u} = \boxed{tA\vec{v}} + A\vec{u} + \begin{pmatrix} -6 \\ 2 \end{pmatrix}$

$\Rightarrow A\vec{v} = -2\vec{v} \quad \text{--}\triangleright ①$ $\to \vec{v}$ is eigen vector for -2.

$A\vec{u} + \begin{pmatrix} -6 \\ 2 \end{pmatrix} = \vec{v} - 2\vec{u} \quad \text{--}\triangleright ②$

since $A\vec{v} = -2\vec{v}$ & $\lambda_1 = -2$ is eigen value of A, then $\vec{v}$ is an eigen vector of $\lambda = -2$

but $\vec{X}_{\lambda_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\therefore \vec{v} = c \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

~~let c = 1~~ , then Plug $\vec{v}$ in 2

$A\vec{u} + \begin{pmatrix} -6 \\ 2 \end{pmatrix} = \begin{pmatrix} c \\ c \end{pmatrix} - 2\vec{u}$

$A\vec{u} + 2\vec{u} = \begin{pmatrix} c \\ c \end{pmatrix} - \begin{pmatrix} -6 \\ 2 \end{pmatrix} = \begin{pmatrix} c+6 \\ c-2 \end{pmatrix}$

$(A + 2I)(u) = \begin{pmatrix} c+6 \\ c-2 \end{pmatrix}$

2 على Diagonal

the only case included

$$\begin{array}{c} R_1 \\ R_2 \end{array}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} c+6 \\ c-2 \end{pmatrix}$$

$$\begin{cases} y_h = c_1 \vec{x}_{\lambda_1} e^{\lambda_1 t} + c_2 X_{\lambda_2} e^{\lambda_2 t} \\ y_p = \vec{V} e^{\lambda t} \quad \text{OR} \quad \vec{V} e^{\lambda_2 t} \\ \vec{y}_p = t\vec{V} e^{\lambda_1 t} + \vec{u} e^{\lambda_1 t} \end{cases}$$

$$R_1 + R_2 = \left(\begin{array}{cc|c} -1 & 1 & c+6 \\ 0 & 0 & 2c+4 \end{array}\right)$$

this system has solution only if $2c+4=0 \Rightarrow \boxed{c=-2}$

Plug $c=-2$ in the system

$$\left(\begin{array}{cc|c} -1 & 1 & 4 \\ 0 & 0 & 0 \end{array}\right)$$

$-u_1 + u_2 = 4$

$u_2 = 4 + u_1$

let $u_1 = 1 \rightsquigarrow u_2 = 5$

$\vec{u} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \vec{V} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$

$$\vec{y}_p = t\begin{pmatrix} -2 \\ -2 \end{pmatrix} e^{-2t} + \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{-2t}$$

ex write a suitable form for $\vec{y}_p$ if the method of undetermained coeff. is to be used

$\boxed{1}$ $y_1' = y_1 + y_2 + 10\cos t$

$y_2' = 3y_1 - y_2 - 10\sin t$

Sol: $\vec{y}' = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}\vec{y} + \begin{pmatrix} 10 \\ 0 \end{pmatrix}\cos t + \begin{pmatrix} 0 \\ -10 \end{pmatrix}\sin t$

step 1 : find $\vec{y}_h$

$$\begin{vmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{vmatrix} = 0 \rightarrow \lambda^2 - 4 = 0 \rightarrow \lambda_1 = -2 \quad \lambda_2 = 2$$

let $\vec{x}_{\lambda_1} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\qquad \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$3x_1 + x_2 = 0 \rightarrow x_2 = -3x_1$ let $x_1 = 1 \rightarrow x_2 = -3$

$\rightarrow X_{\lambda_1} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

let $\vec{X}_{\lambda_2} = \begin{pmatrix} a \\ b \end{pmatrix}$ $\begin{pmatrix} -1 & 1 \\ \cancel{3} & \cancel{-3} \end{pmatrix} \begin{pmatrix} a \\ \cancel{b} \end{pmatrix} = \begin{pmatrix} 0 \\ \cancel{0} \end{pmatrix}$

$-a+b=0 \Rightarrow a=b \rightarrow X_{\lambda_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\vec{y}_h = C_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-2t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$

$\vec{y}_p = \vec{A} \cos t + \vec{B} \sin t$

if $\vec{g}(t) = \begin{pmatrix} 10 \\ 0 \end{pmatrix} \cos^2 t + \begin{pmatrix} 0 \\ -10 \end{pmatrix} \sin t$

$= \begin{pmatrix} 10 \\ 0 \end{pmatrix} \left(1/2 + 1/2 \cos 2t \right) + \begin{pmatrix} 0 \\ -10 \end{pmatrix} \sin t$

$= \begin{pmatrix} 5 \\ 0 \end{pmatrix} + \begin{pmatrix} 5 \\ 0 \end{pmatrix} \cos 2t + \begin{pmatrix} 0 \\ -10 \end{pmatrix} \sin t$

$\vec{y}_p = y_{p_1} + y_{p_2} + y_{p_3}$

$y_{p_1} = \vec{A}$ $\quad y_{p_2} = B\cos 2t + C \sin 2t$ $\quad y_{p_3} = \vec{D} \cos t + \vec{E} \sin t$

$\boxed{2}$ $y_1' = y_1 + y_2 + t^2$

$y_2' = 3y_1 - y_2 - 3t$

Sol: $\vec{y}' = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \vec{y} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t^2 + \begin{pmatrix} 0 \\ -3 \end{pmatrix} t$

$\vec{y}_h$: the same

((Polynomial Deg 2.))

$y_p = \vec{A} t^2 + \vec{B} t + \vec{C}$

[2] Variation of parameters:

consider the System

$\vec{y}' = A\vec{y} + \vec{g}(t)$ where $\vec{g}(t) \neq 0$

$\Rightarrow$ G.S: $\vec{y} = \vec{y}_h + \vec{y}_p$ where $\vec{y}_p = Y(t) \cdot u(t)$

$Y(t) = [\vec{y}_1(t) \quad \vec{y}_2(t)]$ is called the fundemental matrix

where $\vec{y}_1(t)$ & $\vec{y}_2(t)$ are linearly independent solutions for $\vec{y}' = A\vec{y}$

$$\left(\vec{y}_h = c_1 \underbrace{\vec{y}_1(t)}_{\vec{x}_{\lambda_1} e^{\lambda_1 t}} + c_2 \underbrace{\vec{y}_2(t)}_{\vec{x}_{\lambda_2} e^{\lambda_2 t}} \right)$$

$$\boxed{\vec{u}(t) = \int_0^t Y^{-1}(s) \cdot \vec{g}(s)\, ds}$$

ex use variation of parameters to solve

[I] $\vec{y}' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \vec{y} + \begin{pmatrix} -6 \\ 2 \end{pmatrix} e^{-3t}$

Step 1: find $\vec{y}_h$ $\qquad \vec{y}_h = c_1 \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}}_{y_1(t)} + c_2 \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t}}_{y_2(t)}$

Step 2: find the F.M $Y(t)$ $\qquad$ Fundemental Matrix

$$Y(t) = \begin{bmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{bmatrix}$$

Step 3 find $Y^{-1}(t)$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -y \\ -c & a \end{pmatrix}$

$\det(Y(t)) = -2e^{-6t}$

$$Y^{-1}(t) = \frac{1}{-2e^{-6t}}\begin{pmatrix} -e^{-4t} & -e^{-4t} \\ -e^{-2t} & e^{-2t} \end{pmatrix} = \begin{pmatrix} 1/2\, e^{2t} & 1/2\, e^{2t} \\ 1/2\, e^{4t} & -1/2\, e^{4t} \end{pmatrix}$$

Step 4 find $Y^{-1}(t) \cdot g(t)$

$$= \begin{pmatrix} 1/2 e^{2t} & 1/2 e^{2t} \\ 1/2 e^{4t} & -1/2 e^{4t} \end{pmatrix}\begin{pmatrix} -6e^{-3t} \\ 2e^{-3t} \end{pmatrix} = \begin{pmatrix} -3e^{-t} + e^{-t} \\ -3e^{t} - e^{t} \end{pmatrix} = \begin{pmatrix} -2e^{-t} \\ -4e^{t} \end{pmatrix}$$

step 5 $$\vec{u}(t) = \int_0^t \begin{pmatrix} -2e^{-s} \\ -4e^{s} \end{pmatrix} ds = \begin{pmatrix} 2e^{-s} \Big|_0^t \\ -4e^{s} \Big|_0^t \end{pmatrix} = \begin{pmatrix} 2e^{-t} - 2 \\ -4e^{t} + 4 \end{pmatrix}$$

step 6

$$\vec{y}_p = Y(t) \cdot \vec{u}(t)$$

$$= \begin{pmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{pmatrix}\begin{pmatrix} 2e^{-t} - 2 \\ -4e^{t} + 4 \end{pmatrix}$$

ex: Use variation of parameters to solve the system:-

$y_1' = 4y_1 - 3y_2 + t$

$y_2' = -2y_1 - y_2 - 2t$

Science direct

5:17 find

CH 7.1 CH16, CH17

CH Hydrostatics

Sol: $\vec{y}' = \begin{pmatrix} 4 & -3 \\ -2 & -1 \end{pmatrix} \vec{y} + \begin{pmatrix} t \\ 2t \end{pmatrix}$

Step 1: find $\vec{y}_h$

$\begin{vmatrix} 4-\lambda & -3 \\ -2 & -1-\lambda \end{vmatrix} = 0 \rightarrow (\lambda+1)(\lambda-4) - 6 = 0$

$\lambda^2 - 3\lambda - 4 - 6 = 0$

$\lambda^2 - 3\lambda - 10 = 0$

$(\lambda - 5)(\lambda + 2) = 0 \rightarrow \lambda_1 = 5, \ \lambda_2 = -2$

let $\vec{X}_{\lambda_1} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\begin{pmatrix} -1 & -3 \\ -2 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad -x_1 - 3x_2 = 0 \Rightarrow x_2 = -1/3\, x_1$

let $x_1 = 3$

$X_{\lambda_1} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

let $\vec{X}_{\lambda_2} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

$\begin{pmatrix} 6 & -3 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad 6z_1 - 3z_2 = 0 \Rightarrow z_2 = 2z_1$

let $z_1 = 1$

$z_2 = 2$

$\vec{X}_{\lambda_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\vec{y}_h = \underbrace{c_1 \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^{5t}}_{y_1 t} + \underbrace{c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t}}_{y_2 t}$

Step 2 Fundimantel matrix

$$Y(t) = \left[\vec{y}_1(t) \ \vec{y}_2(t)\right] = \begin{bmatrix} 3e^{5t} & e^{-2t} \\ -e^{5t} & 2e^{-2t} \end{bmatrix}$$

Step 3 $$Y^{-1}(t) = \frac{1}{7e^{3t}} \begin{bmatrix} 2e^{-2t} & -e^{-2t} \\ e^{5t} & 3e^{5t} \end{bmatrix} = \begin{bmatrix} \frac{2}{7}e^{-5t} & -1/7\, e^{-5t} \\ \frac{1}{7}e^{2t} & \frac{3}{7}e^{2t} \end{bmatrix}$$

Step 4 $$Y^{-1}(t)\vec{g}(t) = \begin{bmatrix} 2/7\, e^{-5t} & -1/7\, e^{-5t} \\ \frac{1}{7}e^{2t} & 3/7\, e^{2t} \end{bmatrix} \begin{bmatrix} t \\ -2t \end{bmatrix}$$

$$= \begin{bmatrix} 4/7\ t e^{-5t} \\ -5/7 t e^{2t} \end{bmatrix}$$

Step 5 $$\vec{u}(t) = \int_0^t Y^{-1}(s)\vec{g}(s)\, ds$$

by Parts

$$\vec{u}(t) = \int_0^t \begin{bmatrix} \frac{4}{7} s e^{-5s} \\ \frac{-5}{7} s e^{2s} \end{bmatrix} ds = \begin{bmatrix} 4/7 \left[-1/5 s e^{-5s} - \frac{1}{25} e^{-5s}\right]_0^t \\ -5/7 \left[\frac{s}{2} e^{2s} - \frac{1}{4} e^{2s}\right]_0^t \end{bmatrix}$$

$$= \begin{bmatrix} 4/7 \left[\left(-1/5 t e^{-5t} - \frac{1}{25} e^{-5t}\right) + \frac{1}{25}\right] \\ -5/7 \left[\left(\frac{t}{2} e^{2t} - \frac{1}{4} e^{2t}\right) + \frac{1}{4}\right] \end{bmatrix}$$

Step 6 $$Y_P = Y(t) \cdot u(t)$$

mid

G.S. : $\vec{y} = \vec{y}_h + \vec{y}_p$.

بند ٢

Repeated eigen values. $\underline{\underline{no}}$ { undetermined
→ suitable form
→ repeated in Y_n
→ the solved ex

Chapter 5 : Series Solution

- ordinary point
- Singular point → regular, → irregular

$P(x)y'' + q(x)y' + r(x)y = 0$

let $y = \sum_{n=0}^{\infty} a_n x^n$

$\Downarrow$

$a_n = \frac{1}{n!}$ $\quad y = \sum \frac{x^n}{x!} \Rightarrow e^x$

- Series Solution near ordinary point
- Frobenious method
  Indicial equation

Review Calc. 2 :-

def :

- A power series is a series of the form $\underbrace{\sum_{n=0}^{\infty} a_n x^n}_{\text{maclaurian Series}}$ oR $\underbrace{\sum_{n=0}^{\infty} a_n (x-c)^n}_{\text{taylor series}}$

maclaurian Series centered at $x=0$ — taylor series centered @ $x=c$

Theorem :

$\boxed{1}$ if $f^{(n)}(0)$ exists for all n, then f has convergent Mac. series given by $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

$\boxed{2}$ if $f^{(n)}(c)$ exists $\forall n$, then $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$

Series representation of some famous functions:

$f(x)$	series representation	interval of convergence.
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$x \in (-\infty, \infty)$
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$x \in (-1, 1)$
$\sin x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$x \in (-\infty, \infty)$
$\cos x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	

* Diff. Power Series:-

let $f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$

$$\Rightarrow f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

* Integration :-

$$\int f(x)\,dx = \int \sum_{n=0}^{\infty} a_n x^n dx = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}$$

* Shifting indicies:

$\boxed{1}$ $\sum_{n=k}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_{n+k} x^{n+k}$

$\boxed{2}$ $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=k}^{\infty} a_{n-k} x^{n-k}$

Rmk:

[1] to add or subtract two or more Power series, they all should have the same power of x & the same initial value for the index.

ex $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} + \sum_{n=1}^{\infty} \frac{2^n}{n} x^n$

$$= \sum_{n=1}^{\infty} \frac{x^n}{n-1!} + \sum_{n=1}^{\infty} \frac{2^n}{n} x^n = \sum_{n=1}^{\infty} \left(\frac{1}{n-1!} + \frac{2^n}{n} \right) x^n$$

ex $\sum_{n=0}^{\infty} \frac{x^n}{n} + \sum_{n=1}^{\infty} \frac{(n+1)x^{n-2}}{2^n+1}$

* وحد القوى

* على الأقل

$$= \sum_{n=2}^{\infty} \frac{x^{n-2}}{n-2} + \sum_{n=1}^{\infty} \frac{(n+1)x^{n-2}}{2^n+1} = \sum_{n=2}^{\infty} \frac{x^{n-2}}{n-2} + \frac{2}{3}x^{-1} + \sum_{n=2}^{\infty} \frac{(n+1)x^{n-2}}{2^n+1}$$

$$= \frac{2}{3}x^{-1} \sum_{n=2}^{\infty} \left(\frac{1}{n-2} + \frac{n+1}{2^n+1} \right) x^{n-2}$$

[2] If $\sum_{n=k}^{\infty} a_n x^n = 0$ for all x, then $\boxed{a_n = 0 \text{ for all } n \geq k}$

$ax^2 + bx + c = x^2 + 1$ for all x

$a = 1 \quad b = 0 \quad c = 1$

$ax^2 + bx + c = 0 \quad \forall x$

$a = 0 \quad b = 0 \quad c = 0$

Def:

let $p(x) y'' + q(x) y' + r(x) y = 0 \dashrightarrow (*)$

if $p(x)$, $q(x)$ & $r(x)$ are cts. at x_0

then

① if $p(x_0) \neq 0$, then x_0 is called ordinary point

② if $p(x_0) = 0$, then x_0 is called singular point.

* Power Series Solution near ordinary point :-

Theorem: if x_0 is an ordinary point for $(*)$ then $(*)$ has a power series solution centered at x_0 & is given by
$y(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0\, y_1(x) + a_1\, y_2(x)$

where y_1 & y_2 are linearly. ind. Solutions of $(*)$

ex

[1] find a p.s.s for $y'' + y = 0$ near $x_0 = 0$

sol: $P(x) = 1 \Rightarrow P(x_0) = 1 \neq 0$ $x_0 = 0$ is an ordinary pt.
$\therefore$ by thm $(*)$ has pss near $x_0 = 0$ of the form
$y(x) = \sum_{n=0}^{\infty} a_n (x-0)^n$

$y' = \sum_{n=1}^{\infty} n\, a_n\, x^{n-1} \Rightarrow y'' = \sum_{n=2}^{\infty} n(n-1) a_n\, x^{n-2}$

Since $y'' + y = 0 \Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n\, x^{n-2} + \sum_{n=0}^{\infty} a_n\, x^n = 0$

$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n + \sum_{n=0}^{\infty} a_n\, x^n = 0$

$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_n\right] x^n = 0$ for all x

$\Rightarrow \boxed{(n+2)(n+1) a_{n+2} + a_n = 0 \text{ for all } n \geq 0}$

is called recursion formula.

$a_{n+2} = \dfrac{-a_n}{(n+2)(n+1)} \quad \forall\, n \geq 0$

No Gain without Pain

علايقا!

$n=0 \qquad a_2 = \dfrac{-a_0}{2\cdot 1} = \dfrac{-a_0}{2!}$

$n=1 \qquad a_3 = \dfrac{-a_1}{3\cdot 2} = \dfrac{-a_1}{3!}$

$n=2 \qquad a_4 = \dfrac{-a_2}{3\cdot 4} = \dfrac{(-1)^2\, a_0}{4\cdot 3\cdot 2!} = \dfrac{(-1)^2\, a_0}{4!}$

$n=3 \qquad a_5 = \dfrac{-a_3}{5\cdot 4} = \dfrac{(-1)^2\, a_1}{5!}$

$n=4 \qquad a_6 = \dfrac{-a_4}{6\cdot 5} = \dfrac{(-1)^3\, a_0}{6!}$

$n=5 \qquad a_7 = \dfrac{(-1)^3\, a_1}{7!}$

$$a_n = \begin{cases} a_{2k} \\ a_{2k+1} \end{cases} \qquad k = 0,1,2,3,\ldots$$

$$= \begin{cases} \dfrac{(-1)^k \cdot a_0}{(2k)!}, & k = 0,1,2,\ldots \\[2ex] \dfrac{(-1)^k \cdot a_1}{(2k+1)!}, & k = 0,1,2,\ldots \end{cases}$$

G.S :

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \cdots$$

$$= (a_0 + a_2 x^2 + a_4 x^4 + \cdots) + (a_1 x + a_3 x^3 + \cdots)$$

$$= \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k a_0 x^{2k}}{2k!} + \sum_{k=0}^{\infty} \frac{(-1)^k a_1 x^{2k+1}}{(2k+1)!}$$

$$= a_0 \cos x + a_1 \sin x$$

[2] $y'' + y = 0$ near $x_0 = 1$

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n \Rightarrow a_0 \cos(x-1) + a_1 \sin(x-1)$$

[3] $y'' - xy' - y = 0$ near $x_0 = \phi$ $x_0 = \phi$, is an ordinary Point

$\Rightarrow$ it has power series solution centered at $x_0 = 1$

& of the form $y = \sum_{n=0}^{\infty} a_n (x-1)^n$

$$y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$ Plug y, y', y'' in the D.E

$$\sum_{n=2}^{\infty} n(n-1) a_n \cancel{x^{n-2}} (x-1)^{n-2} - \underbrace{x \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}}_{((x-1)+1)} - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - \sum_{n=1}^{\infty} n a_n (x-1)^{n} - \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

توحيد القوى:

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - \sum_{n=1}^{\infty} n a_n (x-1)^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

حل الحدود

$$\Rightarrow 2a_2 + \sum_{n=1}^{\infty} \mathrm{I} - \sum_{n=1}^{\infty} \mathrm{II} - a_1 \sum_{n=1}^{\infty} \mathrm{III} - a_0 - \sum_{n=1}^{\infty} \mathrm{IV} = 0$$

$$2a_2 - a_1 - a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - n a_n - (n+1) a_{n+1} - a_n \right] (x-1)^n = 0$$ for all x

$\Rightarrow$ ① $2a_2 - a_1 - a_0 = 0 \Rightarrow a_2 = \dfrac{a_1 + a_0}{2}$

② $(n+2)\cancel{(n+1)} a_{n+2} - \cancel{(n+1)} a_n - \cancel{(n+1)} a_{n+1} = 0$

$a_{n+2} = \dfrac{a_n + a_{n+1}}{n+2}$ for all $n \geq 1$

$n=1 \rightarrow a_3 = \frac{a_2 + a_1}{3} = \frac{\frac{a_1+a_0}{2} + a_1}{3} = \frac{3a_1 + a_0}{6}$

$n=2 \rightarrow a_4 = \frac{a_3 + a_2}{3} = \frac{\frac{3a_1+a_0}{6} + \frac{a_1+a_0}{2}}{4} = \frac{6a_1 + 4a_0}{24} = \frac{a_1}{4} + \frac{a_0}{6}$

$\vdots$

the G.S

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4 + \cdots$$

$$= a_0 + a_1(x-1) + \frac{a_0}{2}(x-1)^2 + \frac{a_1}{2}(x-1)^2 + \frac{a_0}{6}(x-1)^3 + \frac{a_1}{2}(x-1)^3 + \frac{a_0}{6}(x-1)^4 + \frac{a_1}{4}(x-1)^4 + \cdots$$

$$y(x) = a_0\left(1 + 1/2\,(x-1)^2 + 1/6\,(x-1)^3 + 1/6\,(x-1)^4 + \cdots\right) + a_1\left((x-1) + 1/2\,(x-1)^2 + 1/2\,(x-1)^3 + 1/4\,(x-1)^4 + \cdots\right)$$

H.w $y'' - 4y = 0$ near $x_0 = 0$

ans: $a_0 \sum_{n=0}^{\infty} \frac{2^n x^n}{n!} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{n!}$

$= a_0 e^{2x} + a_1 e^{-2x}$

Singular Points

Let $P(x)y'' + Q(x)y' + R(x)y = 0 \rightarrow (*)$

1) x_0 is called singular point for $(*)$ if $P(x_0) = 0$

2) classification:

ⓐ x_0 is called rigular singular if Both of the following limits exists:

1. $\lim_{x \to x_0} (x - x_0)\frac{Q(x)}{P(x)} = \text{finite}$ 2. $\lim_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)} = \text{finite}.$

ⓑ x_0 is called irrigular singular if it's not regular.

example Find all singular points & determine whether each one is regular or irregular.

$\boxed{1}$ $xy'' + (1-x)y' + xy = 0$ $\quad P(x) = x \quad Q(x) = 1-x \quad R(x) = x$

sing. pts: let $P(x) = 0 \Rightarrow x = 0$

1) $\lim_{x\to 0} \cancel{(x-0)} \cdot \frac{1-x}{\cancel{x}} = 1$

2) $\lim_{x\to 0} (x-0)^2 \cdot \frac{x}{x} = 0$

$x=0$ is regular singular.

$\boxed{2}$ $x(1-x^2)^3 y'' - 2xy' + (1-x)^2 y = 0$ $\quad P(x) = x(1-x^2)^3 \quad Q(x) = 2x \quad R(x) = (1-x)$

sing pts: $x(1-x^2)^3 = 0 \Rightarrow x = 0 \quad x = 1 \quad x = -1$

at $x=0$

1) $\lim_{x\to 0} \frac{(x-0)(-2\cancel{x})}{\cancel{x}(1-x^2)^3} = 0$ $\quad$ 2) $\lim_{x\to 0} \frac{(x-0)^2 \cancel{(1-x^2)}}{x(1-x^2)^{3\,2}} = 0$

$x=0$ is reg. sing. pt.

at $x=1$

1) $\lim_{x\to 1} \frac{(x-1)(-2\cancel{x})}{\cancel{x}(1-x^2)^3} = -2 \lim_{x\to 1} \frac{\cancel{(x-1)}}{(1-x)^{3\,2}(1+x)^3} = \infty$ $\quad x=1$ is irregular sing. pt.

at $x=-1$

1) $\lim_{x\to -1} \frac{(x-1)(-2\cancel{x})}{\cancel{x}(1-x^2)^3} = -2 \lim \frac{(x-1)}{(1-x)^3(1+x)^3} = \infty$ $\quad x=-1$ is irregular.

$\boxed{3}$ $(\sin x)\, y'' + x y' + 4y = 0$

$\cos(n\pi) = (-1)^n$

let $P(x) = 0$

$\sin x = 0 \Rightarrow x = n\pi, \; n \in \mathbb{Z}$

Case ① $n = 0 \rightarrow x = 0$

1) $\lim\limits_{x \to 0} \dfrac{(x-0) \cdot x}{\sin x} = 0$ 2) $\lim \dfrac{(x-0)^2 * 4}{\sin x} = 0$

$x = 0$ is Regular singular.

Case ② $n \neq 0 \rightarrow x = n\pi$

1) $\lim\limits_{x \to n\pi} \dfrac{(x - n\pi)(x)}{\sin x} \overset{l'H}{=} \lim\limits_{x \to n\pi} = \dfrac{2x - n\pi}{\cos x} = \dfrac{n\pi}{(-1)^n}$

$\frac{0}{0}$

2) $\lim\limits_{x \to n\pi} \dfrac{(x - n\pi)^2 * 4}{\sin x} \overset{l'H}{=} 4 \lim\limits_{x \to n\pi} \dfrac{2(x - n\pi)}{\cos n\pi} = 0$

$x = n\pi$ is Regular singular.

$\boxed{4}$ $x^2(1-x)y'' + (x-2)y' - 3xy = 0$

$\boxed{5}$ $xy'' + e^x y' + (3\cos)y = 0$

*$\boxed{6}$ $(x \sin x)\, y'' + 3y' + xy = 0$

* PSS near regular Singular Pts. ($x=0$)

analytic

let $x^2y'' + x\,b(x)\,y' + c(x)\,y = 0$ --> (*)
where $b(x)$ & $c(x)$ are diff at $x=0$ (of all orders)
then (*) has a p.S.S centered at $x=0$ ~~& given by~~:

Def: let $x^2 y'' + x\,b(x)\,y' + c(x)\,y = 0$
where $b(x)$ & $c(x)$ are diff at $x=0$
then the indicial equation is
a quadratic eq. of the form: $\boxed{r(r-1) + b_0(r) + C_0 = 0}$
where $b_0 = b(0)$ & $C_0 = C(0)$

- Frobernous method

Let r_1 & r_2 be the roots of the indicial equation of (*)
then the basis for the G.S of (*) has the following forms:

case [1] $r_1 - r_2 \neq$ integer

$$y_1(x) = x^{r_1}(a_0 + a_1x + a_2x^2 + \dots)$$
$$= x^{r_1}\sum_{n=0}^{\infty} a_n x_n$$
$$= \sum_{n=0}^{\infty} a_n\, x^{n+r_1}$$

& $y_2(x) = \sum_{n=0}^{\infty} b_n\, x^{n+r_2}$

G.S: $y = C_1 y_1 + C_2 y_2$

case $\boxed{2}$ If $r_1 = r_2 = r$

$$y_1(x) = \sum_{n=0}^{\infty} a_n X^{n+r}$$

$$y_2(x) = y_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n+r}$$

case $\boxed{3}$ If $r_1 - r_2 =$ Integer (Positive Integer)

$$y_1(x) = \sum_{n=0}^{\infty} a_n X^{n+r_1}$$

$$y_2(x) = k y_1 \ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

ex: find the ind. eq & sutable form for the G.S :-

$\boxed{1}$ $x(x-1)y'' + (3x-1)y' + y = 0$

$$x^2 y'' + \frac{(3x-1)}{x-1} y' + \frac{x}{x-1} y = 0$$

$$b(x) = \frac{3x-1}{x-1} \qquad c(x) = \frac{x}{x-1}$$

Since $b(x)$ & $c(x)$ are diff. at $x=0$ of all orders, then $\exists$ a p.s.s near $x=0$

$b_0 = b(0) = 1$

$c_0 = c(0) = 0$

ind. eq : $r(r-1) + r + 0 = 0 \Rightarrow r^2 - r + r = 0 \Rightarrow r^2 = 0 \Rightarrow r_1 = r_2 = 0$

case 2

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n \qquad y_2(x) = y_1(x) \ln(x) + \sum_{n=0}^{\infty} b_n x^n$$

$$y = C_1 y_1 + C_2 y_2$$

$\boxed{2}$ $(x+1)^2 y'' + (x+1) y' - y = 0$

$$\Rightarrow x^2 y'' + \frac{x^2}{x+1} y' - \frac{x^2}{(x+1)^2} y = 0 \Rightarrow x^2 y'' + x \cdot \underbrace{\frac{x}{x+1}}_{b} y' - \underbrace{\frac{x^2}{(x+1)^2}}_{c} y = 0$$

$b_0 = b(0) = 0$

$c_0 = c(0) = 0$

ind. eq : $r(r-1) + 0 + 0 = 0$ $\quad r_1 = 1, \ r_2 = 0$ $\quad$ case $\underline{3}$

<u>ex</u> Find the G.S of $x(x-1)y'' + (3x-1)y' + y = 0$ near $x=0$

التعويض بعد الاشتقاق فقط

Step $\underline{1}$: Standard form : $x^2 y'' + x \dfrac{(3x-1)}{x-1} y' + \dfrac{x}{x-1} y = 0$

Step $\underline{2}$: Show that $x=0$ is a reg. singular pt.

Step $\underline{3}$: find the ind. eq & solve it : $r^2 = 0$ $\quad r_1 = r_2 = 0$

Step $\underline{4}$: find $y_1(x)$

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+0}$$

$$y_1'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y_1''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1}$$

Plug in the original eq.

$$(x^2-x)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + (3x+1)\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} 3n a_n x^n - \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$// \quad - \sum_{n=1}^{\infty} n(n+1)a_{n+1} x^n + \quad // \quad - \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n + \quad // \quad = 0$$ وسيد قوى

$$// \quad -2a_2 - \sum_{n=2}^{\infty} (n+1)n a_{n+1} x^n + 3a_1 x + \sum_{n=2}^{\infty} 3n a_n x^n - a_1 - 2a_2 x - \sum_{n=2}^{\infty} (n+1)a_{n+1} x^n + a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n = 0$$

$$-4a_2 x + 4a_1 x + a_0 - a_1 + \sum_{n=2}^{\infty} \left[n(n-1)a_n - n(n+1)a_{n+1} + 3n a_n - (n+1)a_{n+1} + a_n \right] x^n = 0$$

$$n^2 a_n - n a_n - n^2 a_{n+1} - n a_{n+1} + 3n a_n - n a_{n+1} - a_{n+1} + a_n$$

① $a_0 - a_1 = 0 \Rightarrow a_0 = a_1$

② $-4a_2 + 4a_1 = 0 \Rightarrow a_2 = a_1$

③ $a_{n+1}(-n^2 - 2n - 1) + a_n(n^2 + 2n + 1) = 0$

$\Rightarrow a_{n+1} = a_n$ for all $n \geq 2$

$a_0 = a_1 = a_2 = \cdots$

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 \sum_{n=0}^{\infty} x^n = a_0 \frac{1}{1-x}$$

to find $y_2(x)$ use reduction of order $y_2(x) = u\, y_1(x)$ where

$$u(x) = \int \frac{1}{y_1^2} \cdot e^{-\int p(x)\,dx}\, dx$$

$$y'' + p(x)\, y' + q(x)\, y = 0$$

Ch.6 Laplace Transformation.

$f(t) \rightarrow \boxed{\mathcal{L}} \rightarrow F(s)$

Def: assume $f(t)$ is defined on $(0,\infty)$
then laplace transformation for $f(t)$
is defined as

$$\boxed{\begin{matrix}\text{D.E} \\ xy'' + y' + x^2 y = 0\end{matrix}} \rightarrow \boxed{\mathcal{L}} \rightarrow \text{Algebraic eq. in terms of } \mathcal{L}(y(t)) = Y(s)$$

$Y(s) = h(s)$

$y(t) = \mathcal{L}^{-1}\{h(s)\}$

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt \quad \text{where } s > k$$

for some $k \in \mathbb{R}$

ex let $f(t) = e^{3t}$ find $\mathcal{L}\{f(t)\}$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} e^{3t}\,dt = \lim_{u\to\infty} \int_0^u e^{(3-s)t}\,dt = \lim_{u\to\infty} \left. \frac{e^{(3-s)t}}{3-s} \right|_0^u$$

$$= \lim_{u\to\infty} \frac{e^{(3-s)u}}{3-s} - \frac{1}{3-s}$$

$$= \begin{cases} \dfrac{-1}{3-s}, & s > 3 \\ \text{div}, & s \leq 3 \end{cases}$$

HW: $\mathcal{L}\{t\}$

$$\mathcal{L}\{e^{3t}\} = \frac{1}{s-3}, \quad s > 3$$

	$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
(1)	$a \rightarrow$ constant	$\frac{a}{s}$
(2)	t	$\frac{1}{s^2}$
(3)	t^2	$\frac{2!}{s^3}$
(4)	t^n	$\frac{n!}{s^{n+1}}$

$\left.\right\} s > 0$

(5)	e^{at}	$\frac{1}{s-a}$, $s > a$
(6)	$\cos(\alpha t)$	$\frac{s}{s^2+\alpha^2}$
(7)	$\sin(\alpha t)$	$\frac{\alpha}{s^2+\alpha^2}$
(8)	$\cosh(\alpha t)$	$\frac{s}{s^2-\alpha^2}$
(9)	$\sinh(\alpha t)$	$\frac{\alpha}{s^2-\alpha^2}$

examples find $\mathcal{L}\{f(t)\}$ if

Rmk: laplace transform is a linear operator

$\boxed{1}$ $\mathcal{L}\{f_1(t)\} + \mathcal{L}\{f_2(t)\} = \mathcal{L}\{f_1(t)+f_2(t)\}$

$\boxed{2}$ $\mathcal{L}\{cf(t)\} = c\mathcal{L}\{f(t)\}$

$\boxed{1}$ $f(t) = \cos^2 t$
$= 1/2 + 1/2 \cos 2t$

$\therefore \mathcal{L}\{f(t)\} = \mathcal{L}\{1/2 + 1/2\cos 2t\}$

$= \mathcal{L}\{1/2\} + 1/2\,\mathcal{L}\{\cos 2t\} = \frac{1/2}{s} + 1/2\,\frac{s}{s^2+4}$

$\boxed{2}$ $f(t) = (t+2)^2$
$= t^2 + 4t + 4$

$\therefore \mathcal{L}\{f(t)\} = \frac{2!}{s^3} + 4 \cdot \frac{1}{s^2} + \frac{4}{s}$

Def:
If $F(s)$ is the laplase transform for some function f, then the inverse laplase transform is defined as

$\mathcal{L}^{-1}\{F(s)\} = f(t)$ $\quad (\mathcal{L}\{f(t)\} = F(s)$ iff $f(t) = \mathcal{L}\{F(s)\})$

ex find $\mathcal{L}^{-1}\{F(s)\}$ if

[1] $F(s) = \frac{1}{s^7}$

$\mathcal{L}^{-1}\{\frac{1}{s^7}\} = \mathcal{L}^{-1}\{\frac{1}{s^{6+1}}\} = \frac{1}{6!}\mathcal{L}^{-1}\{\frac{6!}{s^{6+1}}\} = \frac{1}{6!}t^6$

[2] $F(s) = \frac{s-3}{s^2+9} = \frac{s}{s^2+9} - \frac{3}{s^2+9}$

$\mathcal{L}^{-1}\{F(s)\} = \cos(3t) - \sin 3t$

[3] $F(s) = \frac{s+2}{s^2+5} = \frac{s}{s^2+5} + \frac{2}{\sqrt5}\cdot\frac{\sqrt5}{s^2+5}$

$\mathcal{L}^{-1}\{F(s)\} = \cos(\sqrt5 t) + \frac{2}{\sqrt5}\sin(\sqrt5 t)$

[4] $F(s) = \frac{1}{s^2-3s+2} = \frac{1}{(s-1)(s-2)} = \frac{a}{s-1} + \frac{b}{s-2}$

$\frac{a(s-2)+b(s-1)}{(s-1)(s-2)} \Rightarrow a=-1$
$b=1$

$F(s) = \frac{-1}{s-1} + \frac{1}{s-2}$ $\quad \mathcal{L}^{-1}\{F(s)\} = -e^t + e^{2t}$

* first shifting theorem :-

If $F(s) = \mathcal{L}\{f(t)\}$, then :

① $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$. shifted a units to the right

② $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$

ex

$$\mathcal{L}\{e^{2t}\cos 3t\} = \frac{s-2}{(s-2)^2+9}$$

$$\mathcal{L}\{e^{3t}\, t^5\} = \frac{5!}{(s-3)^6}$$

$\Rightarrow$ another solution: $F(s) = \dfrac{1}{s^2-3s+2}$

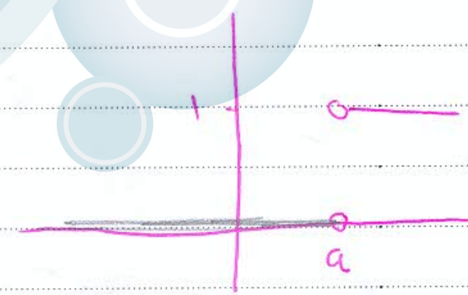
$$F(s) = \frac{1}{s^2-3s+\frac{9}{4}-\frac{9}{4}+2} = \frac{1}{(s-\frac{3}{2})^2-\frac{1}{4}} = 2\,\frac{1/2}{(s-\frac{3}{2})^2-1/4}$$

$$\mathcal{L}^{-1}\{F(s)\} = e^{3/2t}\,\sinh(1/2\,t) * 2$$

* Unit step function :-

Def: let $a \geq 0$ be a real number then $u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$

is called unit step function



theorem: $\mathcal{L}\{u(t-a)\} = \dfrac{e^{-as}}{s}$

Pf: $\mathcal{L}\{u(t-a)\} = \int_0^{t} e^{-st}\, u(t-a)\, dt$

$$= \int_a^{\infty} e^{-st}\cdot 1\, dt = \frac{e^{-st}}{-s}\Big|_a^{\infty} = 0 - \frac{e^{-as}}{-s}$$

* 2^{nd} Shifting theorem :

Let $F(s) = \mathcal{L}\{f(t)\}$ then

① $\mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-as} F(s)$

② $\mathcal{L}^{-1}\{e^{-as} F(s)\} = f(t-a)\, u(t-a)$

ex find L.T for

$\boxed{1}$ $f(t) = t^2 \cdot u(t-3)$ $\quad t^2 = (t-3+3)^2 = (t-3)^2 + 6(t-3) + 9$

$f(t) = (t-3)^2\, U(t-3) + 6(t-3)\, U(t-3) + 9\, u(t-3)$

$\mathcal{L}\{f(t)\} = e^{-3s} \frac{2!}{s^3} + 6 e^{-3s} \frac{1}{s^2} + 9 \frac{e^{-3s}}{s}$

$\boxed{2}$ $\mathcal{L}\{\cos t \; U(t - \frac{\pi}{2})\}$

$\cos t = \cos \underbrace{t - \frac{\pi}{2}}_{\alpha} + \underbrace{\frac{\pi}{2}}_{\beta} = \cos(t - \frac{\pi}{2}) \cancel{\cos\frac{\pi}{2}}^{0} - \sin(t - \frac{\pi}{2}) \cancel{\sin\frac{\pi}{2}}^{1}$

$= -\sin(t - \frac{\pi}{2})$

$\Rightarrow -\mathcal{L}\{\sin(t - \frac{\pi}{2})\, u(t - \frac{\pi}{2})\} = -e^{-\pi/2\, s} \frac{1}{s^2+1}$

$\boxed{3}$ $\mathcal{L}\{e^{3t}\, u(t-2)\}$

$= \mathcal{L}\{e^{3(t-2)+6}\, u(t-2)\} = e^6 \mathcal{L}\{e^{3(t-2)}\, u(t-2)\}$

$= e^6 \cdot e^{-2s} \cdot \frac{1}{s-3}$

$\boxed{4}$ $f(t) = \begin{cases} 0, & t<1 \\ t, & t>1 \end{cases}$

$= t \begin{cases} 0, & t<1 \\ 1, & t>1 \end{cases}$ $\quad f(t) = t\,u(t-1)$

$\boxed{5}$ $f(t) = \begin{cases} t, & t<1 \\ 0, & t>1 \end{cases}$

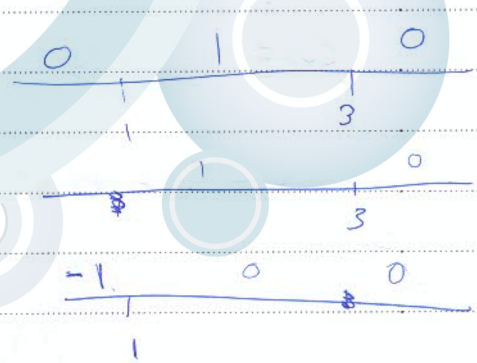
$= t \begin{cases} 1, & t<1 \\ 0, & t>1 \end{cases} = t \cdot \left[1 - \begin{cases} 0, & t<1 \\ 1, & t>1 \end{cases} \right]$

$= t \cdot [1 - u(t-1)] = t - t\,u(t-1)$

$\boxed{6}$ $f(t) = \begin{cases} t, & t < \frac{\pi}{2} \\ \cos t, & t > \frac{\pi}{2} \end{cases}$

$= \begin{cases} t, & t > \pi/2 \\ 0, & t < \pi/2 \end{cases} + \begin{cases} 0, & t < \frac{\pi}{2} \\ \cos t, & t > \frac{\pi}{2} \end{cases} = t\left[1 - u(t-\tfrac{\pi}{2})\right] + \cos t\left[\cancel{1} - u(t-\tfrac{\pi}{2})\right]$

$\boxed{7}$ $f(t) = \begin{cases} 0, & t<1 \\ 1, & 1<t<3 \\ 0, & 3<t \end{cases}$



$= f(t) = \begin{cases} 1, & t<3 \\ 0, & t>3 \end{cases} - \begin{cases} 1, & t<1 \\ 0, & t>1 \end{cases}$

$= 1 - u(t-3) - [1 - u(t-1)]$

$= 1 - u(t-3) - 1 + u(t-1)$

$= u(t-1) - u(t-3)$

* In general If $f(t) = \begin{cases} g(t) & , t \in (a,b) \\ 0 & , t \notin (a,b) \end{cases}$

then $f(t) = \big(u(t-a) - u(t-b)\big) \cdot g(t)$

$\boxed{8}$ $f(t) = \begin{cases} \sin t & , t \in (0,\pi) \\ 0 & , t \notin (0,\pi) \end{cases}$

$$f(t) = \sin t \; u(t-0) - u(t-\pi)$$

$$= \sin t \cdot u(t-0) - \sin t \cdot \big(u(t-\pi)\big)$$

examples exer. 2-11

$$\sin t = \sin(t-\pi+\pi) = -\sin(t-\pi)$$

$$f(t) = \sin t \cdot u(t) + \sin(t-\pi)\, u(t-\pi)$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2+1} + e^{-\pi s} \cdot \frac{1}{s^2+1}$$

$\boxed{9}$ $f(t) = \begin{cases} 2 & , t < 1 \\ 1/2\, t^2 & , 1 < t < \pi/2 \\ \cos t & , t > \pi/2 \end{cases}$

$$= 2 \cdot \begin{cases} 1 & , t < 1 \\ 0 & , t > 1 \end{cases} + 1/2\, t^2 \begin{cases} 0 & , t < 1 \\ 1 & , 1 < t < \pi/2 \\ 0 & , t < \pi/2 \end{cases} + \cos t \begin{cases} 0 & , t < \frac{\pi}{2} \\ 1 & , t > \frac{\pi}{2} \end{cases}$$

$$= 2\big[1 - u(t-1)\big] + 1/2\, t^2 \big[u(t-1) - u(t-\pi/2)\big] + \cos t \; u(t-\pi/2)$$

* Laplace transformation for derivative:

if $F(t)$ is twice diff on $[0,\infty)$ and $F(s) = \mathcal{L}\{f(t)\}$, then

① $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

② $\mathcal{L}\{f''(t)\} = s^2F(s) - sF(0) - f'(0)$

ex solve $y'' - 2y' + 3y = 0$ $\quad y(0) = 0 \quad y'(0) = 0$

sol: let $\mathcal{L}\{y(t)\} = Y(s)$

Step 1: take L.T for both sides

$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = \mathcal{L}\{0\} = 0$

$s^2Y(s) - sy(0) - y'(0) - 2(sY(s) - y(0)) + 3Y(s) = 0$

$s^2Y(s) - 1 - 2sY(s) + 3Y(s) = 0$

step 2: solve the above equation for $Y(s)$

$Y(s) = [s^2 - 2s + 3] = 1 \qquad Y(s) = \dfrac{1}{s^2 - 2s + 3}$

step 3: the Solution of the I.V.P is $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$= \mathcal{L}^{-1}\left\{\dfrac{1}{(s-1)^2+2}\right\} = \dfrac{1}{\sqrt{2}}\mathcal{L}^{-1}\left\{\dfrac{\sqrt{2}}{(s-1)^2+2}\right\} = \dfrac{1}{\sqrt{2}}e^{t}.\sin(\sqrt{2}t)$

ex Solve the I.V.P

$y'' - y = t \qquad y(0) = y'(0) = 1$

sol: let $Y(s) = \mathcal{L}\{y(t)\}$

$\mathcal{L}\{y''\} - \mathcal{L}\{y\} = \mathcal{L}\{t\}$

$s^2 Y(s) - s y(0) - y'(0) - Y(s) = \frac{1}{s^2}$

$Y(s)[s^2 - 1] = \frac{1}{s^2} + s + 1$

$Y(s) = \frac{1}{s^2(s^2-1)} + \frac{s}{s^2-1} + \frac{1}{s^2-1}$

$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2-1)}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\}$

$= f(t) + \cosh t + \sinh t$

$\frac{1}{s^2(s^2-1)} = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s-1} + \frac{d}{s+1}$

$1 = a(s)(s-1) + b(s^2-1) + c\, s^2(s+1) + d\, s^2(s-1)$

$a = a, \quad b = -1, \quad c = \frac{1}{2}, \quad d\ \frac{1}{2}$

$\Rightarrow \frac{1}{s^2(s^2-1)} = \frac{1}{s^2-1} - \frac{1}{s^2} \Rightarrow f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2-1)}\right\} = \sinh t - t$

GS = ..

ex Solve $y'' + 4y = \begin{cases} 4\cos t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$ $\quad y(0) = 0$, $y'(0) = 0$

$$= 4\cos t\,[1 - u(t-\pi)] \qquad \cos t = \cos(t - \pi + \pi) = -\cos(t-\pi)$$

$\Rightarrow y'' + 4y = 4\cos t + 4\cos(t-\pi)\,u(t-\pi)$

$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = 4 \cdot \dfrac{s}{s^2+1} + 4 \cdot e^{-\pi s} \cdot \dfrac{s}{s^2+1}$

$s^2 Y(s) - s y(0) - y'(0) + 4Y(s) = (e^{-\pi s} + 1) \cdot \dfrac{4s}{s^2+1}$

$\Rightarrow Y(s) = (e^{-\pi s} + 1)\, \dfrac{4s}{(s^2+1)(s^2+4)}$

$\Rightarrow \dfrac{4s}{(s^2+1)(s^2+4)} = \dfrac{as+b}{s^2+1} + \dfrac{cs+d}{s^2+4}$ $\qquad a = 4/3,\ b = 0,\ c = -4/3,\ d = 0$

$\therefore Y(s) = e^{-\pi s}\left[\dfrac{4/3\, s}{s^2+1} - \dfrac{4/3\, s}{s^2+4}\right] + \left[\dfrac{4/3\, s}{s^2+1} - \dfrac{4/3\, s}{s^2+4}\right]$

$4/3 \cos(t-\pi)$ $\quad$ $4/3 \cos(2(t-\pi))$ $\quad$ $4/3 \cos(t)$ $\quad$ $4/3 \cos 2t$

$y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$= \dfrac{4}{3} u(t-\pi) \cdot \cos(t-\pi) - \dfrac{4}{3} u(t-\pi) \cos 2(t-\pi) + \dfrac{4}{3}\cos t - \dfrac{4}{3}\cos 2t$

ex Let $f(t) = t\sin t$ use $\mathcal{L}.T$ for Derivatives to find $\mathcal{L}\{f(t)\}$

$f'(t) = \sin t + t\cos t$

$f''(t) = \cos t + \cos t - t\sin t = 2\cos t - f(t)$

$\mathcal{L}\{f''(t)\} = 2\mathcal{L}\{\cos t\} - \mathcal{L}\{f(t)\}$

$s^2\mathcal{L}\{f(t)\} - \cancel{sf(0) - f'(0)} = 2\frac{s}{s^2+1} - \mathcal{L}\{f(t)\}$ $\quad f(0)=0$, $f'(0)=0$

$(s^2+1)\mathcal{L}\{f(t)\} = \frac{2s}{s^2+1}$

$\therefore \mathcal{L}\{f(t)\} = \frac{2s}{(s^2+1)^2}$

$\mathcal{L}.T$ for integrals :-

Let $F(s) = \mathcal{L}\{f(t)\}$ where $f(t)$ is integrable on $[0, \infty)$

① $\mathcal{L}\left\{\int_0^t f(z)\,dz\right\} = \frac{1}{s}F(s) = \frac{1}{s}\mathcal{L}\{f(t)\}$

ex $\mathcal{L}\left\{\int_0^t \sin x\,dx\right\} = \frac{1}{s}\cdot\frac{1}{s^2+1}$

② $\mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\} = \int_0^t f(z)\,dz$

ex Find $\mathcal{L}^{-1}\left\{\frac{1}{s^4+9s^2}\right\}$

Partial fractions ...

sol: $\frac{1}{s^4+9s^2} = \frac{1}{s^2(s^2+9)} = \frac{1}{s}\cdot\frac{1}{s(s^2+9)}$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s(s^2+9)}\right\} = \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+9)}\right\}(z)\,dz$$

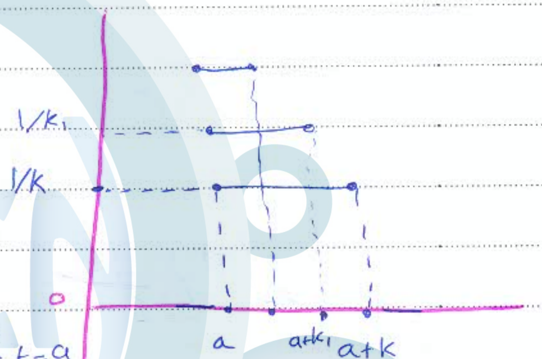
$$\mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+9}\right\} = \int_0^t \frac{1}{3}\sin 3z\,dz = \frac{-\cos 3z}{9}\Big|_0^t = \frac{1}{9}-\frac{1}{9}\cos 3t$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s(s^2+9)}\right\} = \int_0^t \frac{1}{9}-\frac{1}{9}\cos(3z)\,dz.$$

Impulses & Dirac delta function

Let $k>0$ be small enough & let $a>0$, then

$$f_k(t-a) = \begin{cases} \frac{1}{k}, & a \le t \le a+k \\ 0, & \text{otherwise} \end{cases}$$



the Dirac delta function is defined as

$$\delta(t-a) = \lim_{k\to 0} f_k(t-a) = \begin{cases} \infty, & \text{if } t=a \\ 0, & \text{if } t\ne a \end{cases}$$

when $k \to 0$ $f_k(t-a) \to +\infty$

Important Facts:

[1] $\int_0^{+\infty} \delta(t-a)\,dt = 1$

P.f: $\int_0^{\infty} \lim_{k\to 0} f_k(t-a)\,dt$

$\lim_{k\to 0}\int_0^{\infty} f_k(t-a)\,dt = \lim_{k\to 0}\int_a^{a+k} f_k(t-a)\,dt$ ($1/k$)

$= \lim_{k\to 0} 1 = 1$

[2] $\mathcal{L}\{\delta(t-a)\} = e^{-as}$

[3] $\int_0^{\infty} g(t)\,\delta(t-a)\,dt = g(a)$

ex $\int_0^{\infty} \cos^2(t)\,\delta(t-\frac{\pi}{4})\,dt = \cos^2(\frac{\pi}{4}) = 1/2$

$\boxed{4}$ $\mathcal{L}\{g(t)\,\delta(t-a)\} = \int_0^{\infty} e^{-st}\, g(t).\,\delta(t-a)\,dt = e^{-as} \cdot g(a)$

ex Solve $y'' - 6y' + 9y = t^2\delta(t-3)$ $\quad y(0)=0 \quad y'(0)=0$

sol: $\mathcal{L}\{y''-6y'+9y\} = \mathcal{L}\{t^2\delta(t-3)\}$

$s^2Y(s) - \cancel{sy(0)}^{0} - \cancel{y'(0)}^{0} - 6[sY(s) - \cancel{y(0)}^{0}] + 9Y(s) = 9e^{-3s}$

$[\underbrace{s^2-6s+9}_{(s-3)^2}]\,Y(s) = 9e^{-3s}$

$\therefore Y(s) = \dfrac{9e^{-3s}}{(s-3)^2}$

$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 9.\,u(t-3)\cdot f(t-3)$

$\boxed{f(t-3) = \mathcal{L}^{-1}\left\{\dfrac{1}{(s-3)^2}\right\}_{t-3} = e^{t-3}.(t-3)}$

$= 9u(t-3)\,e^{t-3}(t-3$

ex find

$\boxed{1}$ $\mathcal{L}\{e^{3t}\cos t\} = \dfrac{s-3}{(s-3)^2+1}$

$\boxed{2}$ $\mathcal{L}^{-1}\left\{\dfrac{s+4-4}{(s-4)^2+25}\right\} = \mathcal{L}^{-1}\left\{\dfrac{s-4}{(s-4)^2+25}\right\} + \dfrac{4}{5}\mathcal{L}^{-1}\left\{\dfrac{1\times5}{(s-4)^2+25}\right\}$

$= e^{4t}\cos 5t + \dfrac{4}{5}e^{4t}\,\text{sine}\,5t)$

means ⇒ unitstep function / /

$\boxed{3}$ $\mathcal{L}^{-1}\left\{e^{-2s}\cdot\frac{s}{(s-1)^2+9}\right\} = \mathcal{L}^{-1}\left\{e^{-2s}F(s)\right\}$

$\frac{s}{(s-1)^2+9}$

$= \frac{s-1}{(s-1)^2+9} + 1/3\ \frac{3}{(s-1)^2+9}$

$\mathcal{L}^{-1}\left\{\frac{s}{(s-1)^2+9}\right\} = e^{t}\cos(3t) + \frac{1}{3}e^{t}\sin 3t$

$\Rightarrow \mathcal{L}^{-1}\left\{e^{-2t}\frac{s}{(s-1)^2+9}\right\} = u(t-2)\left[\cancel{e^{t-2}\cos 3(t-2) + 1/3\ e^{t-2}\sin 3(t-2)}\right]$

Differentiation of Laplace transform

let $F(s) = \mathcal{L}\{f(t)\}$

$\boxed{1}$ $\mathcal{L}\{t f(t)\} = -F'(s)$ $\ast\ \mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{d s^n}$

$\boxed{2}$ $\mathcal{L}^{-1}\{F'(s)\} = -t f(t)$

examples :

① find $\mathcal{L}\{t\sin t\}$

$= -\left(\frac{1}{s^2+1}\right)' = -\frac{-2s}{(s^2+1)^2} = \frac{2s}{(s^2+1)^2}$

② find $\mathcal{L}^{-1}\left\{\frac{s}{(s^2-4)^2}\right\} = \frac{-1}{2}\mathcal{L}^{-1}\left\{\left(\frac{1}{(s^2-4)}\right)'\right\}$

$\rightarrow \left(\frac{1}{s^2-4}\right)' = \frac{-2s}{(s^2-4)^2} \rightarrow \therefore \frac{s}{(s^2-4)^2} = \frac{-1}{2}\cdot\left(\frac{1}{s^2-4}\right)'$

$= \frac{-1}{2} \cdot -t \cdot \mathcal{L}^{-1}\left\{\frac{2}{s^2-4}\right\} \cdot 1/2$

$= 1/2 \; t \cdot \sinh 2t \cdot 1/2$

(3) $\mathcal{L}^{-1}\left\{\ln\left(1+\frac{4}{s^2}\right)\right\}$

let $F(s) = \ln\left(\frac{s^2+4}{s^2}\right)$

$= \ln(s^2+4) - \ln s^2$

$F'(s) = \frac{2s}{s^2+4} - \frac{2}{s}$

want: $f(t) = \mathcal{L}^{-1}\{F(s)\}$

$\mathcal{L}^{-1}\{F'(s)\} = 2 \cdot \cos 2t - 2$

but $\mathcal{L}^{-1}\{F'(s)\} = -t f(t)$

$\therefore -t f(t) = 2\cos 2t - 2$

$\therefore f(t) = \frac{2 - 2\cos 2t}{t} = \mathcal{L}^{-1}\left\{\ln\left(1+\frac{4}{s^2}\right)\right\}$

ex find $\mathcal{L}^{-1}\left\{\ln\left(2+\frac{3}{s}\right)\right\}$

$F(s) = \ln\left(\frac{2s+3}{s}\right)$

$= \ln 2s+3 - \ln s$

$F'(s) = \frac{2}{2s+3} - \frac{1}{s}$

$= \frac{1}{s+\frac{3}{2}} - \frac{1}{s}$

$\mathcal{L}^{-1}\{F'(s)\} = e^{-3/2t} - 1$

$= -t f(t)$

$\therefore f(t) = \frac{1 - e^{-3/2t}}{t}$

:: Integration of $\mathcal{L}$.T.

$\boxed{1}$ $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{+\infty} F(\cancel{x})\,dx$ where $F(s) = \mathcal{L}\{f(t)\}$

$\boxed{2}$ $\mathcal{L}^{-1}\left\{\int_s^{\infty} F(x)\,dx\right\} = \frac{f(t)}{t}$

ex find $\mathcal{L}\left\{\frac{\sin t}{t}\right\}$

$$= \int_s^{\infty} \frac{1}{x^2+1}\,dx = \tan^{-1} x \Big|_s^{\infty} = \frac{\pi}{2} - \tan^{-1}(s)$$

ex find $\mathcal{L}^{-1}\left\{\int_s^{+\infty} \frac{x}{x^2+4}\,dx\right\}$

$$= \frac{\cos 2t}{t}$$

ex find $\mathcal{L}^{-1}\left\{\int_s^{-\infty} \frac{3x}{(x-1)^2+4}\,dx\right\}$

$$= \frac{f(t)}{t}$$

$$= \frac{3e^t \cos 2t + 3/2\, e^t \cdot \sin 2t}{t}$$

$F(s) = \frac{3s}{(s-1)^2+4}$

$f(t) = \mathcal{L}^{-1}\{F(s)\}$

$F(s) = \frac{3(s-1)}{(s-1)^2+4} + \frac{3}{(s-1)^2+4}$

$= 3 \cdot \frac{s-1}{(s-1)^2+4} + \frac{3}{2} \frac{2}{(s-1)^2+4}$

$f(t) = 3e^t \cos 2t + \frac{3}{2} e^t \cdot \sin 2t$