Automation and Control Lab

IE 0906544

Experiment 3: Performance of First order and second order systems

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System's Response

- We can find the time response of dynamic systems for arbitrary initial conditions and inputs

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}[G(s)U(s)]$$

- Classifying the response of some standard systems to standard inputs can provide insight
  - Ex Systems: first order, second order
  - Ex Inputs: impulse, step, ramp, sinusoid

System's Order

The order of the system is given by the maximum power of s in the denominator polynomial, Q(s).

Here, $Q(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots\dots + a_{n-1} s + a_n$.

Now, n is the order of the system

When n = 0, the system is zero order system.

When n = 1, the system is first order system.

When n = 2, the system is second order system and so on.

The numerator and denominator polynomial of equation (2.10) can be expressed in the factorized form as shown in equation (2.11).

$$T(s) = \frac{P(s)}{Q(s)} = \frac{(s+z_1)(s+z_2)\dots\dots(s+z_m)}{(s+p_1)(s+p_2)\dots\dots(s+p_n)} \quad \dots\dots(2.11)$$

where, $z_1, z_2, \dots\dots z_m$ are zeros of the system.

$p_1, p_2, \dots\dots p_n$ are poles of the system.

Now, the value of **n** gives the number of poles in the transfer function. Hence the order is also given by the number of poles of the transfer function.

Transient Response and Steady-State Response.

The time response of a control system consists of two parts:

- the transient response
- and the steady-state response.

By transient response, we mean that which goes from the initial state to the final state.

By steady-state response, we mean the manner in which the system output behaves as t approaches infinity. Thus the system response c(t) may be written as

$$c(t) = c_{tr}(t) + c_{ss}(t)$$

Laplace Transform of Standard Functions

Step Function:

The unit step function is,

$$u(t) = 1 \quad \text{for } t \geq 0$$
$$= 0 \quad \text{for } t < 0$$

$$L\{u(t)\} = \int_0^\infty u(t) \cdot e^{-st}\, dt = \int_0^\infty 1 \cdot e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \left[\frac{e^{-\infty}}{-s} - \frac{e^{0}}{-s}\right] \quad \text{as } e^{-\infty} = 0$$

$$L\{u(t)\} = \frac{1}{s}$$

SR

Laplace Transform of Standard Functions

Ramp Function:

The unit ramp function is defined as,

$$r(t) = t \quad \text{for } t \geq 0$$
$$= 0 \quad \text{for } t < 0$$

$$L\{r(t)\} = \int_0^\infty r(t)\, e^{-st}\, dt = \int_0^\infty t e^{-st}\, dt$$

Integrating by parts,

$$= \left[\frac{t \,.\, e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} \,.\, 1\, dt = [0 - 0] + \frac{1}{s}\int_0^\infty e^{-st}\, dt$$

$$= \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}\left[\frac{e^{-\infty}}{-s} - \frac{e^0}{-s}\right] \quad \text{as } e^{-\infty} = 0$$

$$L\{r(t)\} = \frac{1}{s^2}$$

$$L\{t\, u(t)\} = \frac{1}{s^2} \qquad \text{as } r(t) = t\, u(t)$$

Laplace Transform of Standard Functions

Ramp Function:

The unit ramp function is defined as,

$$r(t) = t \quad \text{for } t \geq 0$$
$$= 0 \quad \text{for } t < 0$$
$$L\{r(t)\} = \int_0^\infty r(t)\, e^{-st}\, dt = \int_0^\infty t e^{-st}\, dt$$

Integrating by parts,

$$= \left[\frac{t\,.\,e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\,.\,1\, dt = [0 - 0] + \frac{1}{s}\int_0^\infty e^{-st}\, dt$$
$$= \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}\left[\frac{e^{-\infty}}{-s} - \frac{e^0}{-s}\right] \quad \text{as } e^{-\infty} = 0$$
$$L\{r(t)\} = \frac{1}{s^2}$$
$$L\{t\, u(t)\} = \frac{1}{s^2} \qquad \text{as } r(t) = t\, u(t)$$

Laplace Transform of Standard Functions

Impulse Function:

The unit impulse function is δ(t) and defined as,

$$\delta(t) = 1 \quad \text{for } t = 0$$
$$= 0 \quad \text{for } t \neq 0$$

We know the relation between unit step and unit impulse.

$$\delta(t) = \frac{d\,u(t)}{dt}$$

Taking Laplace transform of both sides,

$$L\{\delta(t)\} = L\left\{\frac{d\,u(t)}{dt}\right\}$$

$$L\left\{\frac{d\,f(t)}{dt}\right\} = s\,F(s) - f\left(0^-\right)$$

$$L\{\delta(t)\} = s\,.\,L\{u(t)\} - u(t)\big|_{t=0^-}$$

$$u(t)\big|_{t=0^-} = 0$$

$$L\{u(t)\} = \frac{1}{s}$$

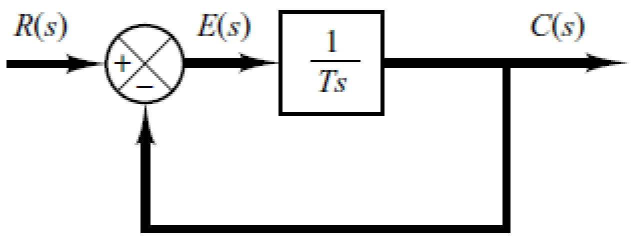
$$L\{\delta(t)\} = s\,.\,\frac{1}{s} - 0$$

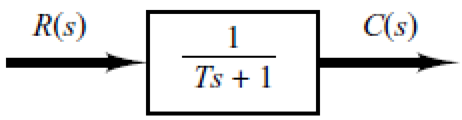
$$L\{\delta(t)\} = 1$$

FIRST-ORDER SYSTEMS

- Consider the first-order system shown in Figure 5–1(a). Physically, this system may represent an *RC* circuit, thermal system, or the like

$$\frac{C(s)}{R(s)} = \frac{1}{Ts + 1}$$





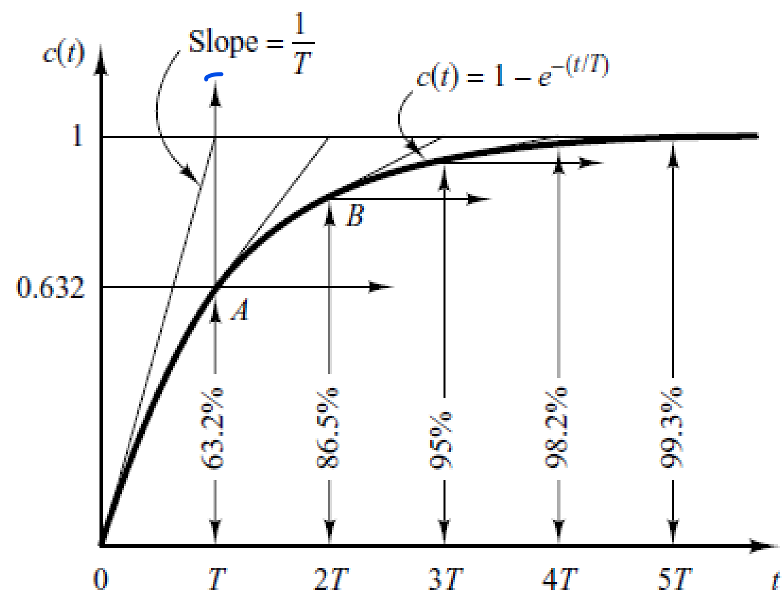
- **Unit-Step Response of First-Order Systems.** Since the Laplace transform of the unit-step function is 1/s, substituting R(s)=1/s into Equation, we obtain

$$C(s) = \frac{1}{Ts + 1}\frac{1}{s}$$

Expanding $C(s)$ into partial fractions gives

$$C(s) = \frac{1}{s} - \frac{T}{Ts + 1} = \frac{1}{s} - \frac{1}{s + (1/T)} \qquad c(t) = 1 - e^{-t/T}, \quad \text{for } t \geq 0$$

- $$c(t) = 1 - e^{-t/T}, \qquad \text{for } t \geq 0$$
- Equation states that initially the output c(t) is zero and finally it becomes unity.
- One important characteristic of such an exponential response curve c(t) is that at t=T the value of c(t) is 0.632, or the response c(t) has reached 63.2% of its total change
- This may be easily seen by substituting t=T in c(t).That is,
- $c(T) = 1 - e^{-1} = 0.632$
- The exponential response curve c(t) is shown.
- In one time constant, the exponential response curve has gone from 0 to 63.2%of the final value.
- In two time constants, reaches 86.5%.
- At t=3T, 4*T*, and 5T, the response
- reaches 95%, 98.2%, and 99.3%,
- Thus, for t > 4T, the response
- remains within 2%.
- As seen from Equation
- , the steady state is reached
- mathematically only after an
- infinite time.



- **Unit-Ramp Response of First-Order Systems.**
- Since the Laplace transform of the unit-ramp function is $1/s^2$, we obtain the output of the system of

$$C(s) = \frac{1}{Ts+1}\frac{1}{s^2}$$

Expanding $C(s)$ into partial fractions gives

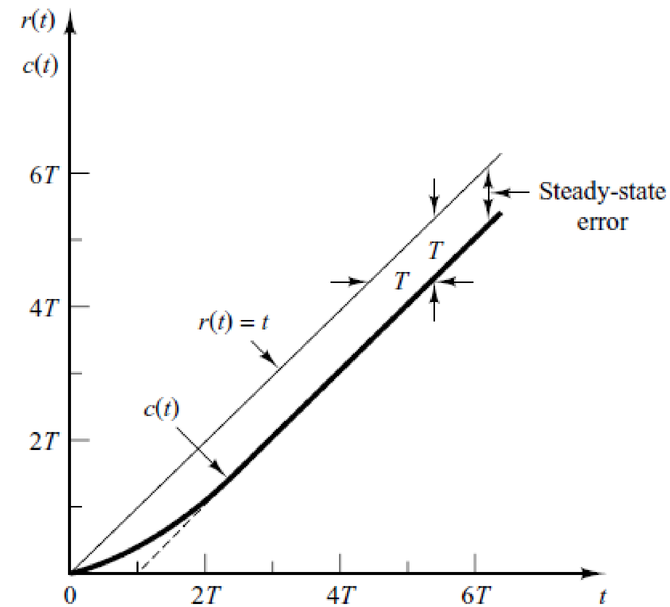
$$C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts+1}$$

Taking the inverse Laplace transform:

$$c(t) = t - T + Te^{-t/T}, \qquad \text{for } t \geq 0$$

The error signal $e(t)$ is then

$$e(t) = r(t) - c(t)$$
$$= T(1 - e^{-t/T})$$



Above Equation states that initially the output c(t) is zero and finally it becomes unity

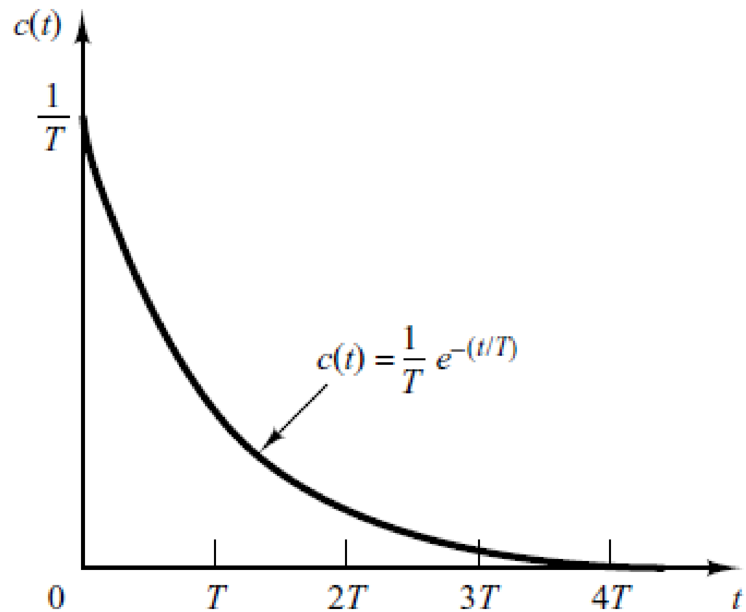
Unit-Impulse Response of First-Order Systems. For the unit-impulse input, $R(s) = 1$ and the output of the system of Figure 5–1(a) can be obtained as

$$C(s) = \frac{1}{Ts + 1} \tag{5–7}$$

The inverse Laplace transform of Equation (5–7) gives

$$c(t) = \frac{1}{T} e^{-t/T}, \qquad \text{for } t \geq 0 \tag{5–8}$$

The response curve given by Equation (5–8) is shown in Figure 5–4.



- for the unit-ramp input the output c(t) is

$$c(t) = t - T + Te^{-t/T}, \qquad \text{for } t \geq 0$$

- For the unit-step input, which is the derivative of unit-ramp input, the output c(t) is

$$c(t) = 1 - e^{-t/T}, \qquad \text{for } t \geq 0$$

- Finally, for the unit-impulse input, which is the derivative of unit-step input, the output c(t) is

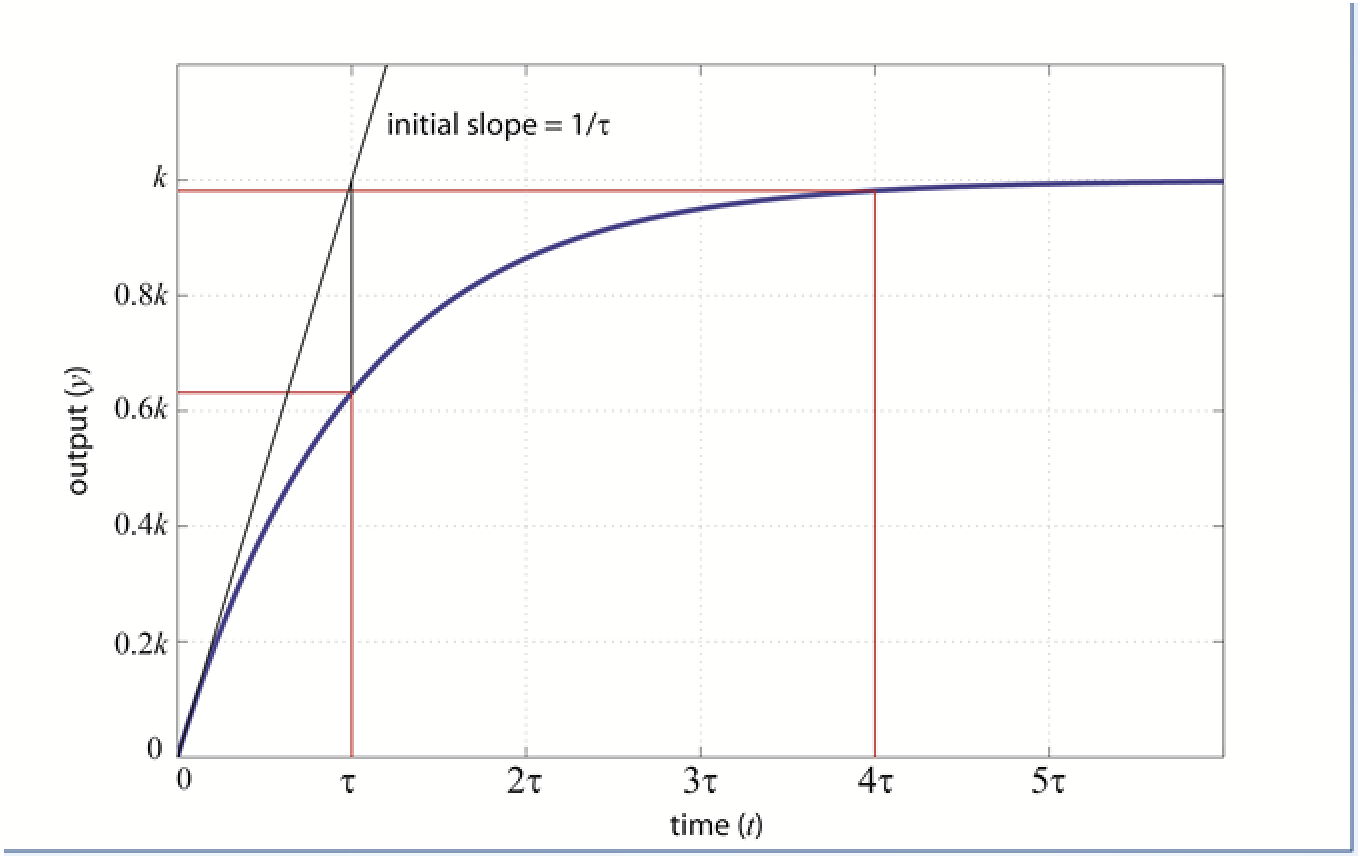
$$c(t) = \frac{1}{T}e^{-t/T}, \qquad \text{for } t \geq 0$$

Comparing the system responses to these three inputs clearly indicates that the response to the derivative of an input signal can be obtained by differentiating the response of the system to the original signal. It can also be seen that the response to the integral of the original signal can be obtained by integrating the response of the system to the original signal and by determining the integration constant from the zero-output initial condition. This is a property of linear time-invariant systems. Linear time-varying systems and nonlinear systems do not possess this property.

- **First-Order Systems:**

$$G(s) = \frac{Y(s)}{U(s)} = \frac{k}{\tau s + 1}$$

- **Step Response:**



Transient Response Specifications: Rise Time

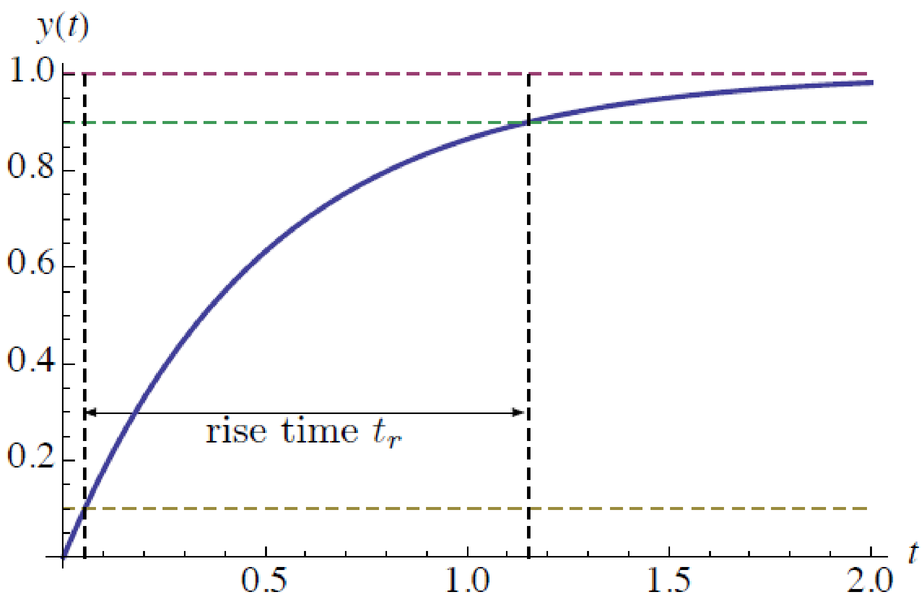
Let's first take a look at *1st-order step response*

$$H(s) = \frac{a}{s+a}, \qquad a > 0 \qquad \text{(stable pole)}$$

DC gain = 1 (by FVT)

Step response: $$Y(s) = \frac{H(s)}{s} = \frac{a}{s(s+a)} = \frac{1}{s} - \frac{1}{s+a}$$

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = 1(t) - e^{-at}$$

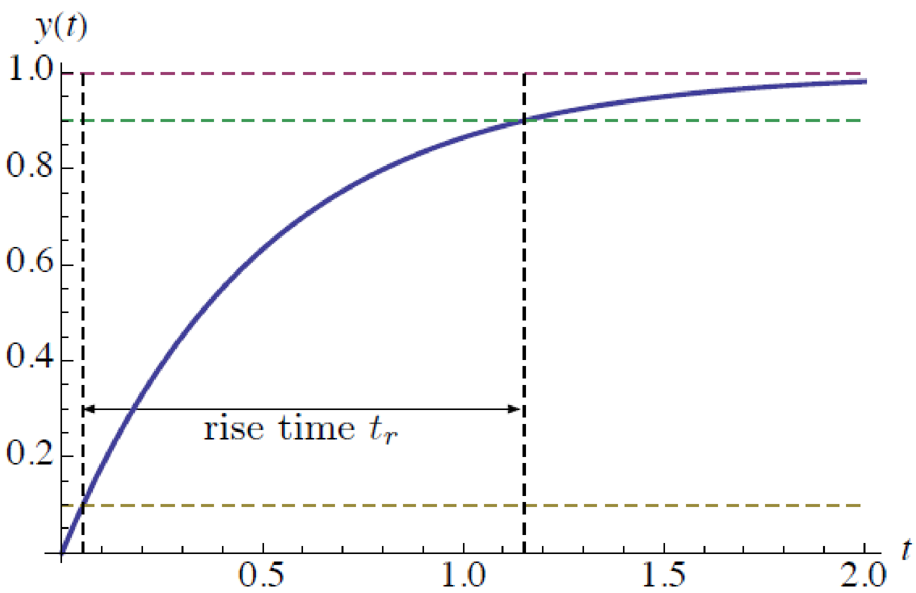


2.2τ

Rise time t_r: the time it takes to get from 10% of steady-state value to 90%

Rise Time

Step response: $y(t) = 1(t) - e^{-at}$



Rise time t_r: the time it takes to get from 10% of steady-state value to 90%

In this example, it is easy to compute t_r analytically:

$$1 - e^{-at_{0.1}} = 0.1 \qquad e^{-at_{0.1}} = 0.9 \qquad t_{0.1} = -\frac{\ln 0.9}{a}$$

$$1 - e^{-at_{0.9}} = 0.9 \qquad e^{-at_{0.9}} = 0.1 \qquad t_{0.9} = -\frac{\ln 0.1}{a}$$

$$t_r = t_{0.9} - t_{0.1} = \frac{\ln 0.9 - \ln 0.1}{a} = \frac{\ln 9}{a} \approx \frac{2.2}{a} = 2.2\tau$$

Transient Response Specs

Now let's consider the more interesting case: *2nd-order response*

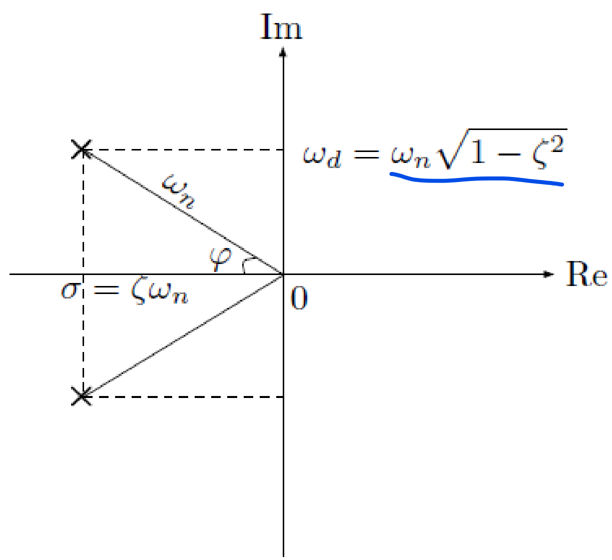
$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

where $\sigma = \zeta\omega_n \;\; \omega_d = \omega_n\sqrt{1-\zeta^2} \qquad (\zeta < 1)$

$\delta\omega \pm j\omega_n\sqrt{1-\xi^2}$

Real

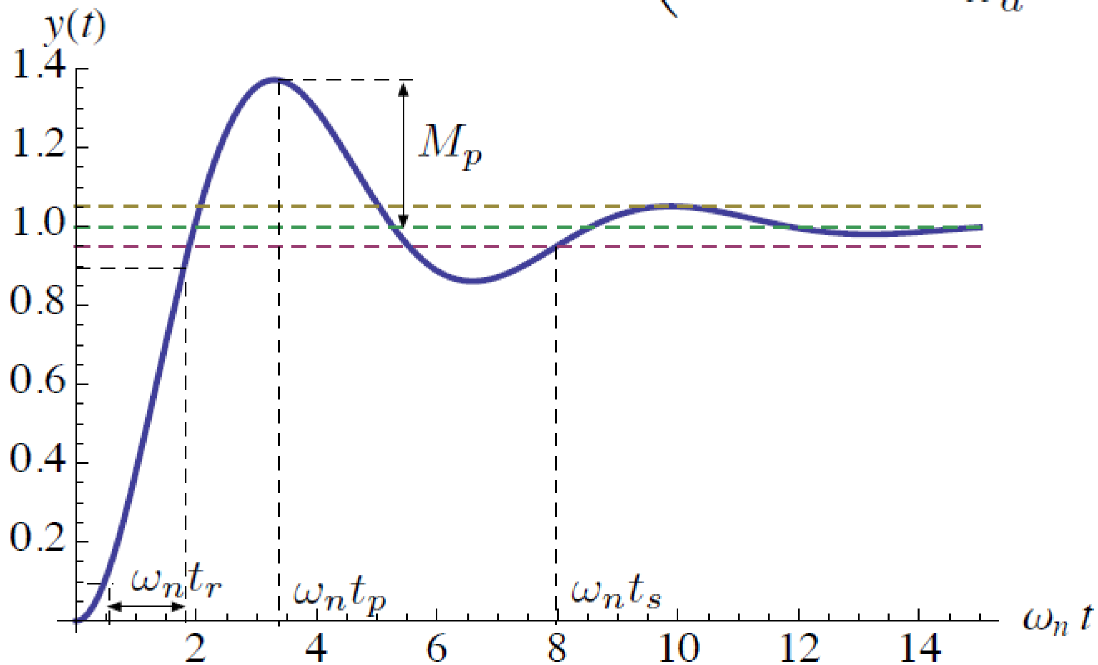
imaginary



Step response: $$y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

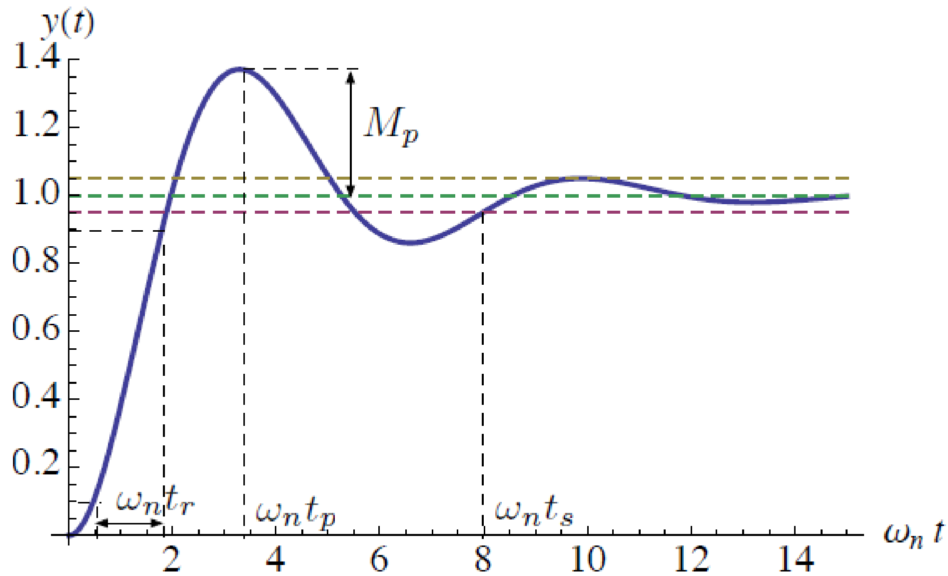
Transient-Response Specs

Step response: $$y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$



- rise time t_r — time to get from $0.1y(\infty)$ to $0.9y(\infty)$
- overshoot M_p and peak time t_p
- settling time t_s — first time for transients to decay to within a specified small percentage of $y(\infty)$ and stay in that range (we will usually worry about 5% settling time)

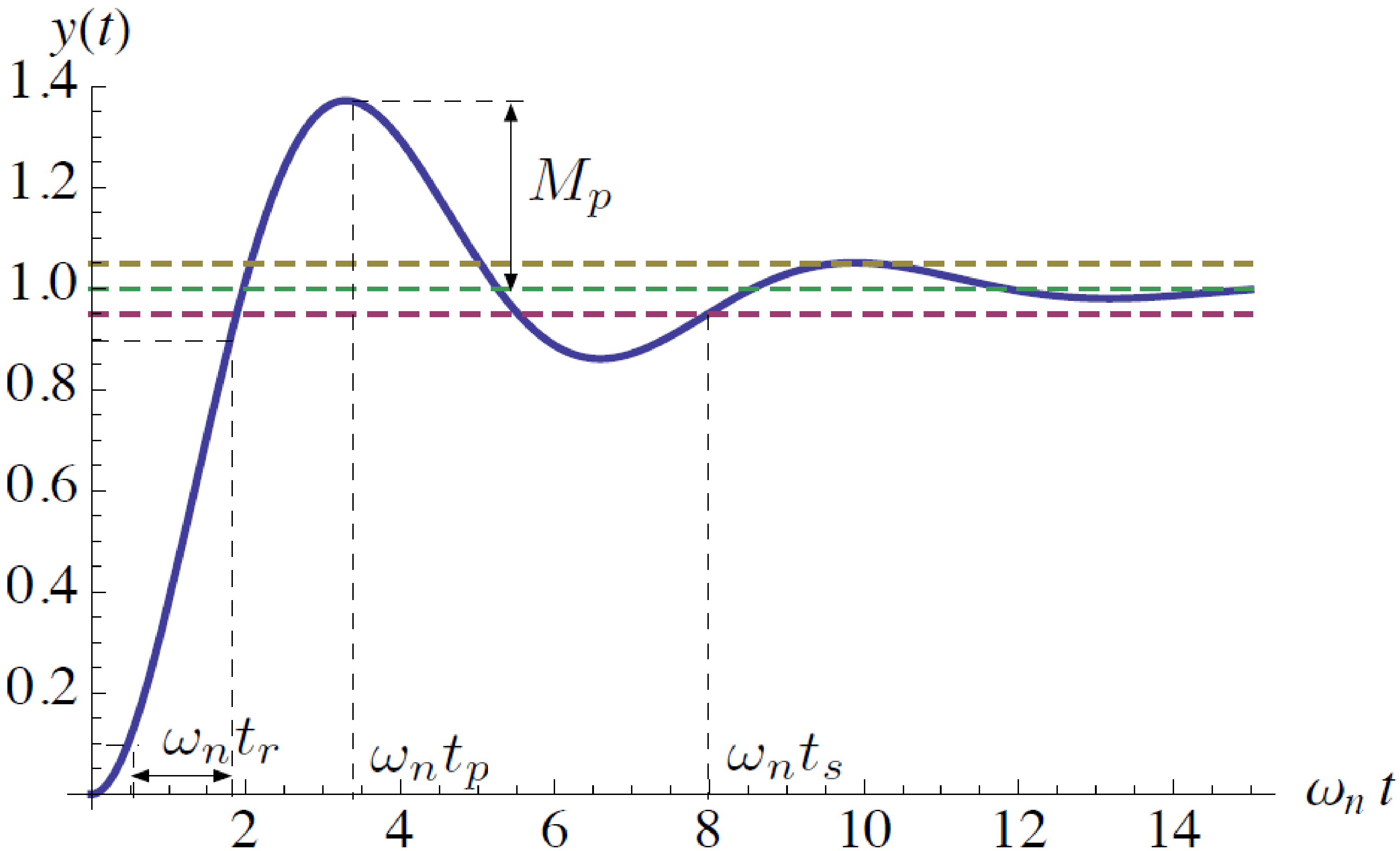
Transient-Response (or Time-Domain) Specs



Do we want these quantities to be large or small?

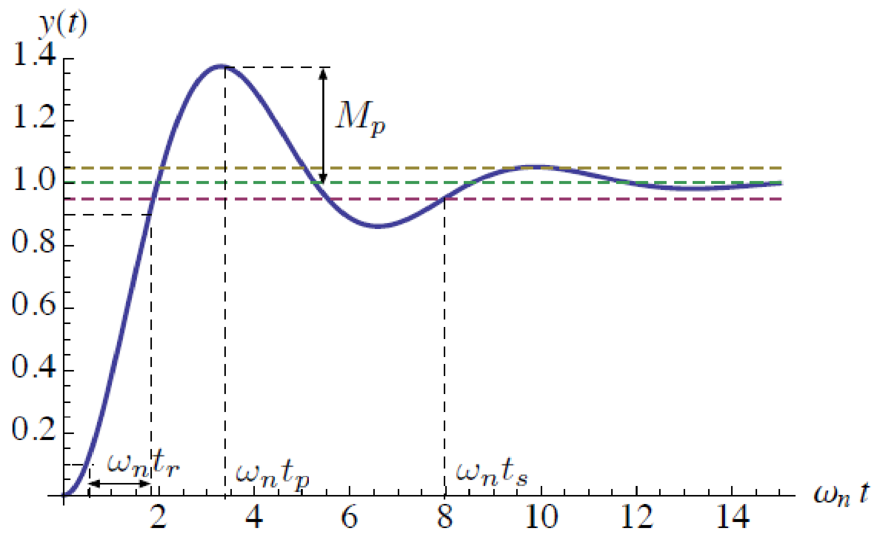
- t_r small
- M_p small
- t_p small
- t_s small

Trade-offs among specs: decrease t_r $\longrightarrow$ increase M_p, etc.



y(t)
1.4
1.2
1.0
0.8
0.6
0.4
0.2
M_p
ω_n t_r
ω_n t_p
ω_n t_s
2
4
6
8
10
12
14
ω_n t

Formulas for TD Specs: Rise Time

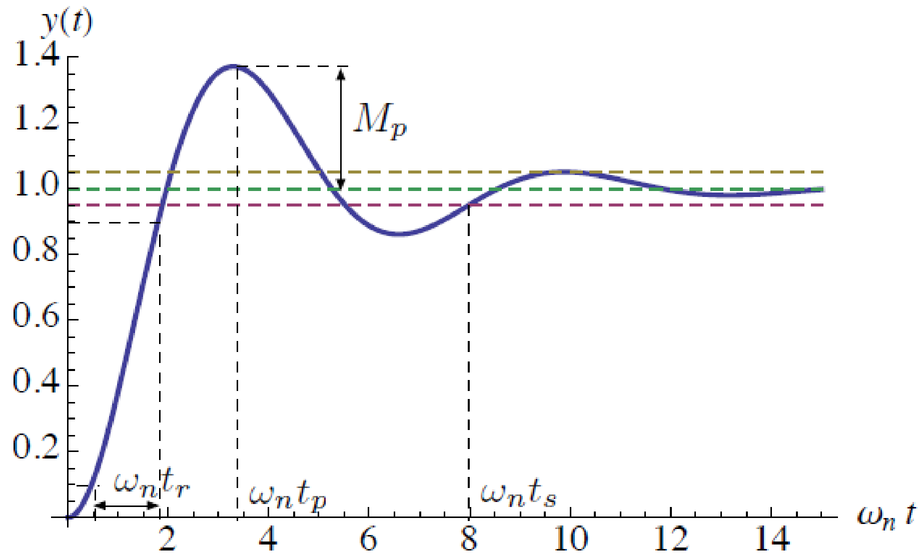


Rise time t_r — hard to calculate analytically.
Empirically, on the normalized time scale ($t \to \omega_n t$), rise times are *approximately* the same

$$w_n t_r \approx 1.8 \qquad \text{(exact for } \zeta = 0.5\text{)}$$

So, we will work with $t_r \approx \frac{1.8}{\omega_n}$ (good approx. when $\zeta \approx 0.5$)

Formulas for TD Specs: Overshoot & Peak Time



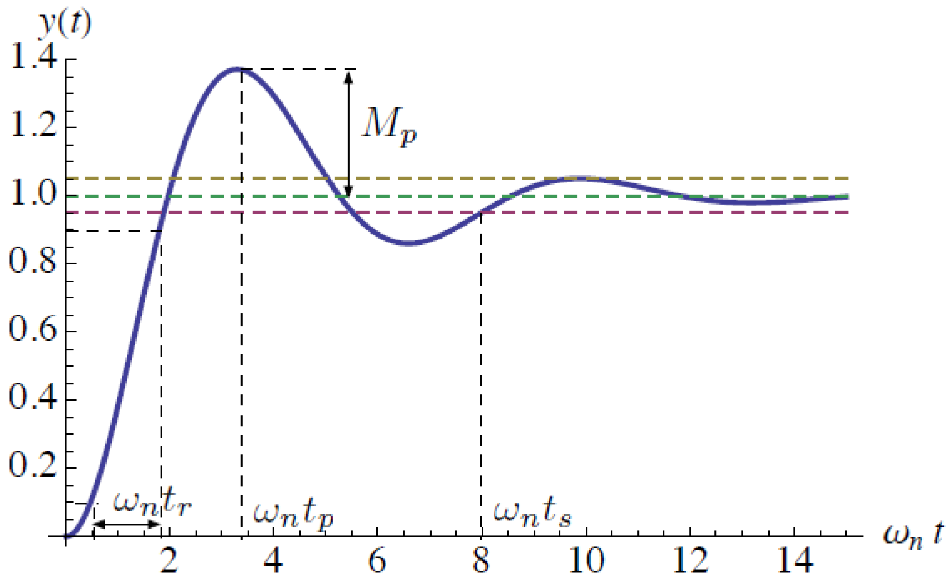
t_p is the *first time* $t > 0$ when $y'(t) = 0$

$$y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

$$y'(t) = \left(\frac{\sigma^2}{\omega_d} + \omega_d\right) e^{-\sigma t}\sin(\omega_d t) = 0 \text{ when } \omega_d t = 0, \pi, 2\pi, \ldots$$

so $t_p = \dfrac{\pi}{\omega_d}$ → $\omega_n\sqrt{1-\xi^2}$

Formulas for TD Specs: Overshoot & Peak Time



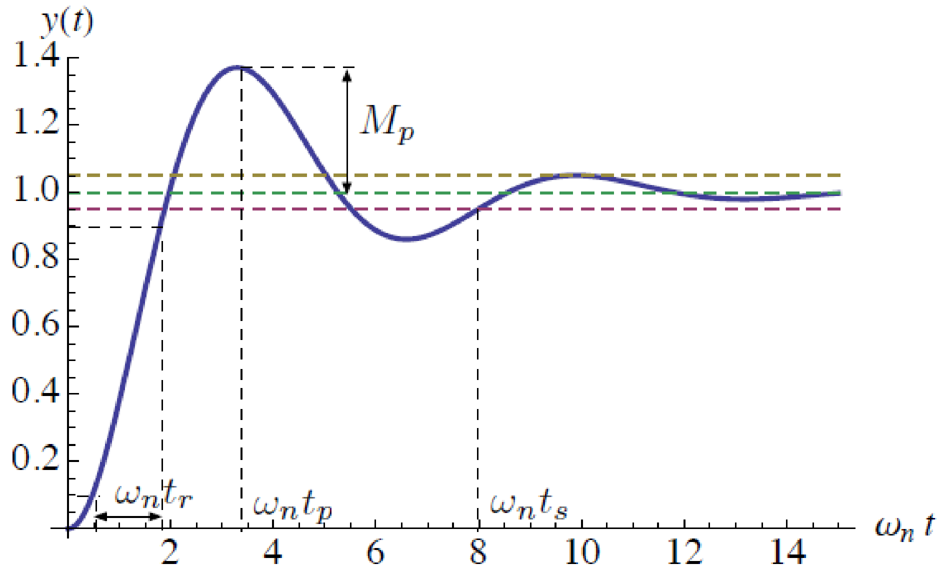
We have just computed $t_p = \dfrac{\pi}{\omega_d}$

To find M_p, plug this value into $y(t)$:

$$M_p = y(t_p) - 1 = -e^{-\frac{\sigma\pi}{\omega_d}} \left(\cos\left(\omega_d \frac{\pi}{\omega_d}\right) + \frac{\sigma}{\omega_d} \sin\left(\omega_d \frac{\pi}{\omega_d}\right)\right)$$

$$= \exp\left(-\frac{\sigma\pi}{\omega_d}\right) = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) \quad \text{— exact formula}$$

Formulas for TD Specs: Settling Time



$$t_s = \min\left\{t > 0 : \frac{|y(t') - y(\infty)|}{y(\infty)} \leq 0.05 \text{ for all } t' \geq t\right\}$$ (here, $y(\infty) = 1$)

$$|y(t) - 1| = e^{-\sigma t}\left|\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right|$$

here, $e^{-\sigma t}$ is what matters (sin and cos are bounded between ± 1), so $e^{-\sigma t_s} \leq 0.05$ this gives $t_s = -\frac{\ln 0.05}{\sigma} \approx \frac{3}{\sigma} = \frac{3}{\zeta \omega_n}$

Formulas for TD Specs

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\sigma^2 + \omega_d^2}{(s+\sigma)^2 + \omega_d^2}$$

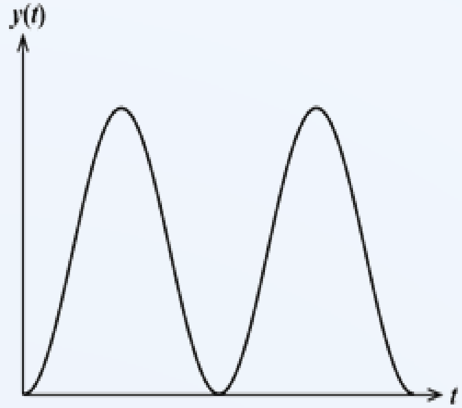
$$t_r \approx \frac{1.8}{\omega_n}$$

$$t_p = \frac{\pi}{\omega_d}$$

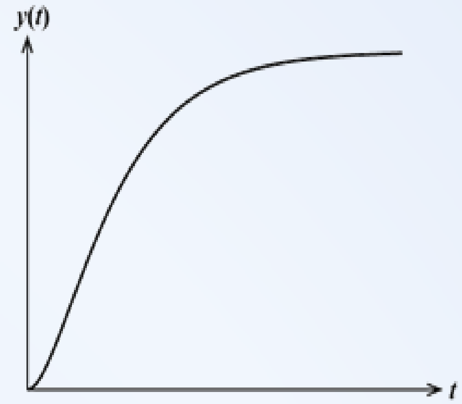
$$M_p = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

$$t_s \approx \frac{3}{\sigma}$$

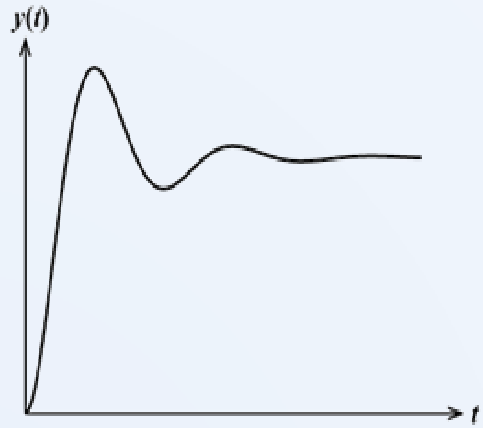
Second-Order Systems



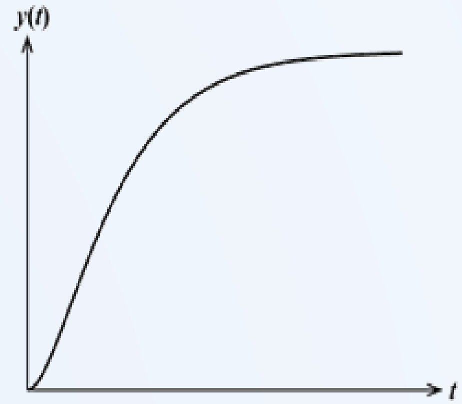
($\zeta = 0$) undamped



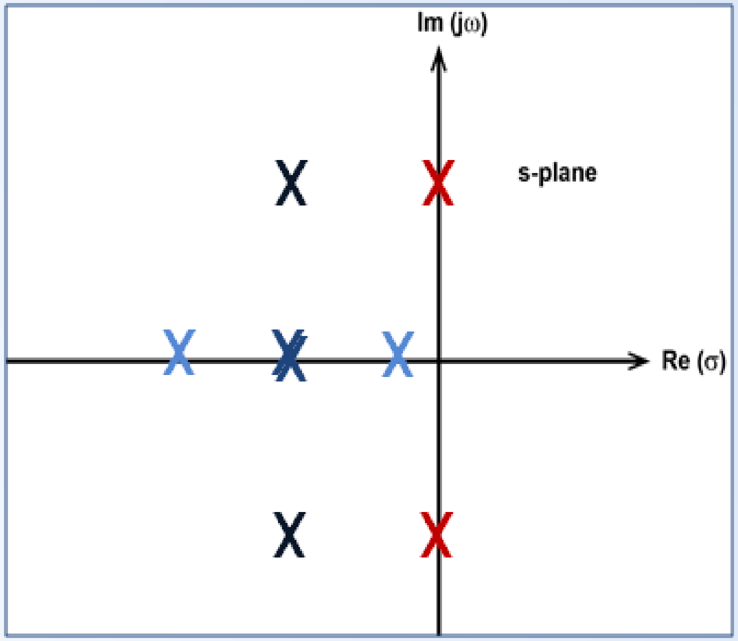
($\zeta = 1$) crit damped



($0 < \zeta < 1$) underdamped



($\zeta > 1$) overdamped



TD Specs in Frequency Domain

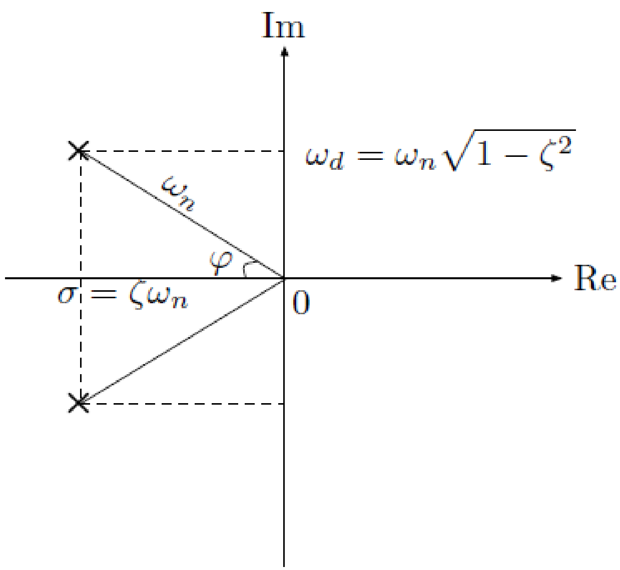
We want to *visualize* time-domain specs in terms of *admissible pole locations* for the 2nd-order system

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\sigma^2 + \omega_d^2}{(s+\sigma)^2 + \omega_d^2}$$

where $\sigma = \zeta\omega_n$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Step response: $y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$



$$\omega_n^2 = \sigma^2 + \omega_d^2$$

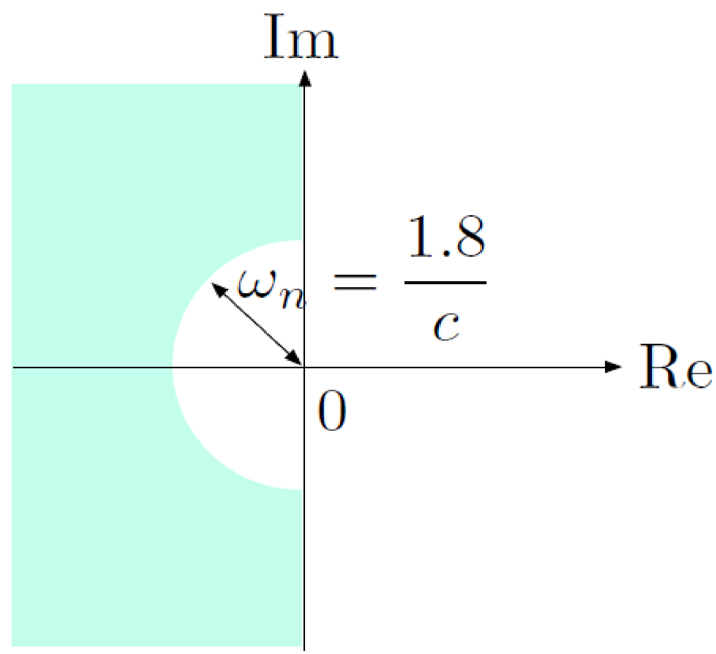
$$\zeta = \cos\varphi$$

Rise Time in Frequency Domain

Suppose we want $t_r \leq c$ (c is some desired given value)

$$t_r \approx \frac{1.8}{\omega_n} \leq c \qquad \Longrightarrow \qquad \omega_n \geq \frac{1.8}{c}$$

Geometrically, we want poles to lie in the shaded region:



(recall that ω_n is the *magnitude of the poles*)

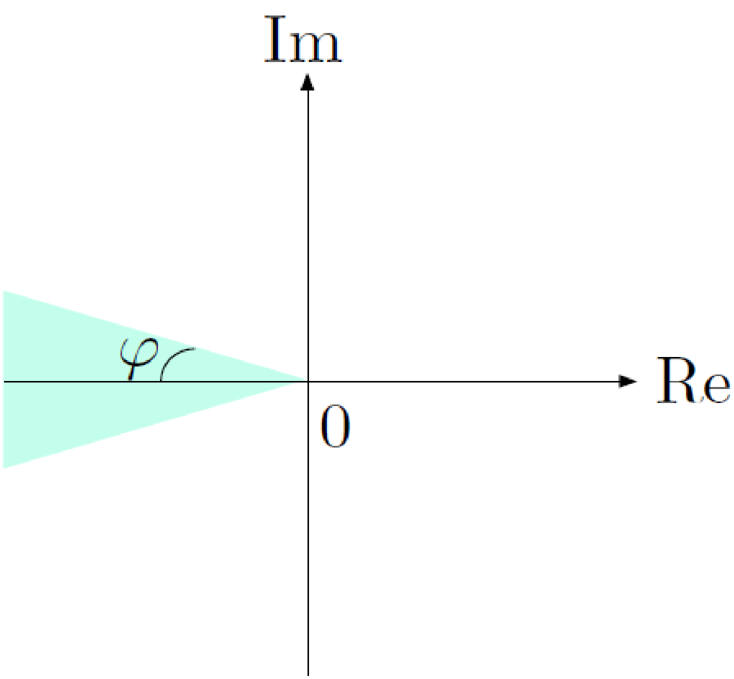
Overshoot in Frequency Domain

Suppose we want $M_p \leq c$

$$M_p = \underbrace{\exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)}_{\text{decreasing function}} \leq c$$

— need large damping ratio

Geometrically, we want poles to lie in the shaded region:



$$\frac{\zeta}{\sqrt{1-\zeta^2}} = \frac{\omega_n \zeta}{\omega_n \sqrt{1-\zeta^2}} = \frac{\sigma}{\omega_d} = \cot\varphi$$

— need φ to be small

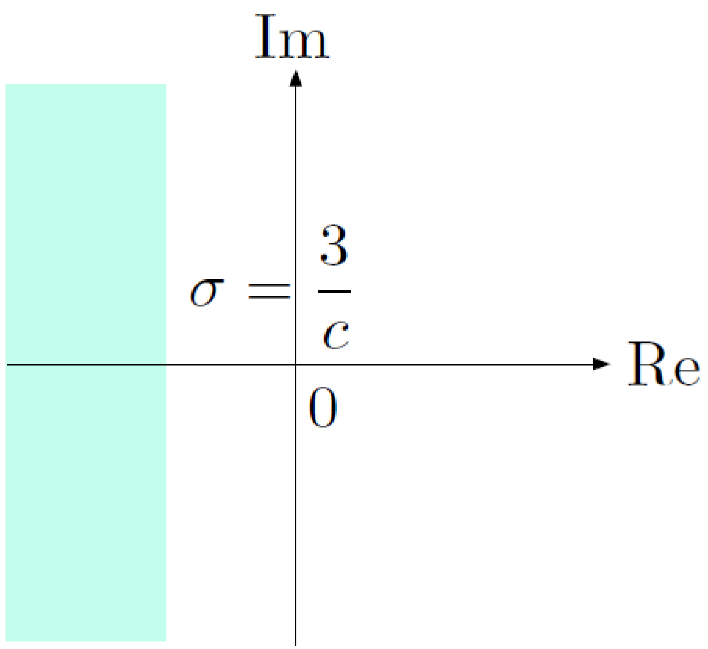
Intuition: good damping → good decay in 1/2 period

Settling Time in Frequency Domain

Suppose we want $t_s \leq c$

$$t_s \approx \frac{3}{\sigma} \leq c \qquad \Longrightarrow \qquad \sigma \geq \frac{3}{c}$$

Want poles to be sufficiently fast (large enough magnitude of real part):



Intuition: poles far to the left → transients decay faster → smaller t_s