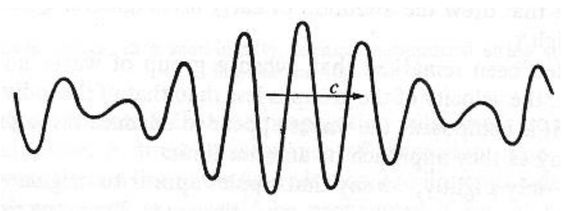
Introduction to Finite Element Methods

CAD/CAM

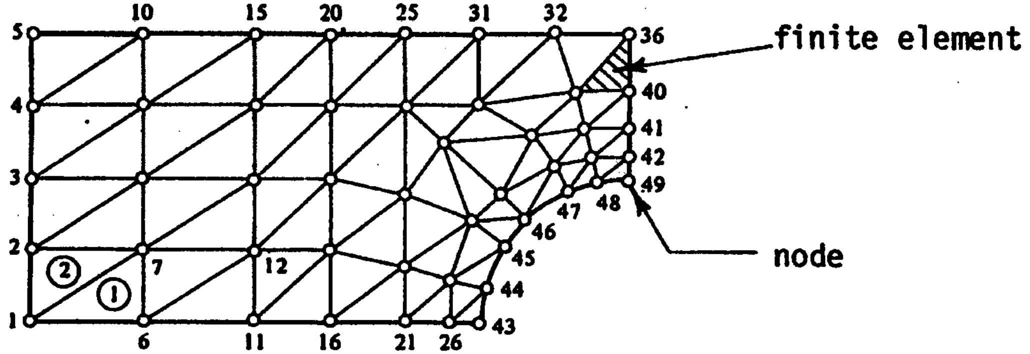


Need for Computational Methods

- **Solutions Using Either Strength of Materials or Theory of Elasticity Are Normally Accomplished for Regions and Loadings With Relatively Simple Geometry**
- **Many Applicaitons Involve Cases with Complex Shape, Boundary Conditions and Material Behavior**
- **Therefore a Gap Exists Between What Is Needed in Applications and What Can Be Solved by Analytical Closed-form Methods**
- **This Has Lead to the Development of Several Numerical/Computational Schemes Including: Finite Difference, Finite Element and Boundary Element Methods**

Introduction to Finite Element Analysis

The finite element method is a computational scheme to solve field problems in engineering and science. The technique has very wide application, and has been used on problems involving *stress analysis, fluid mechanics, heat transfer, diffusion, vibrations, electrical and magnetic fields*, etc. The fundamental concept involves dividing the body under study into a finite number of pieces (subdomains) called *elements* (see Figure). Particular assumptions are then made on the variation of the unknown dependent variable(s) across each element using so-called *interpolation or approximation functions*. This approximated variation is quantified in terms of solution values at special element locations called *nodes*. Through this discretization process, the method sets up an algebraic system of equations for unknown nodal values which approximate the continuous solution. Because element size, shape and approximating scheme can be varied to suit the problem, the method can accurately simulate solutions to problems of complex geometry and loading and thus this technique has become a very useful and practical tool.



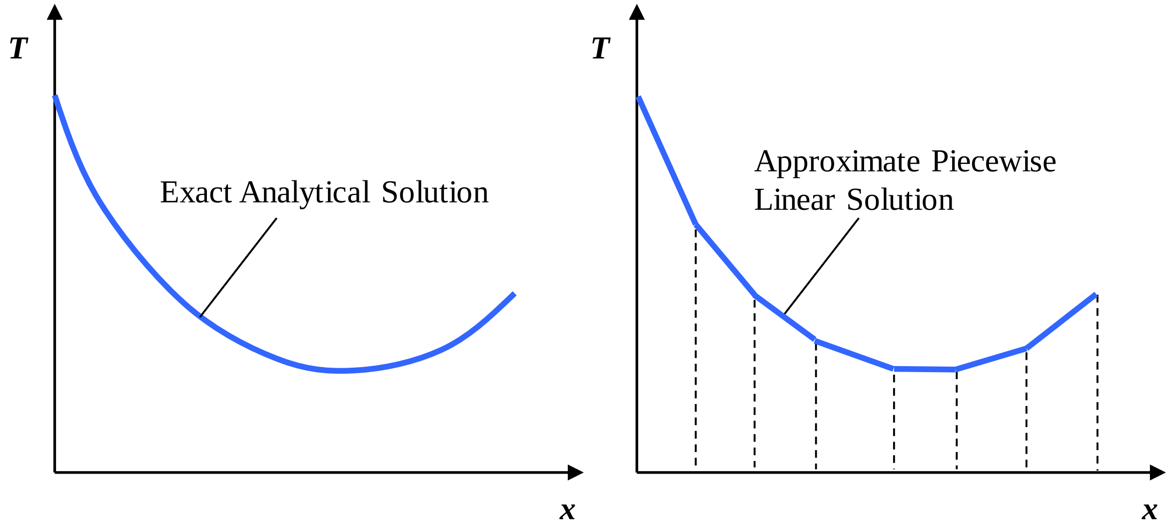
Advantages of Finite Element Analysis

- **Models Bodies of Complex Shape**
- **Can Handle General Loading/Boundary Conditions**
- **Models Bodies Composed of Composite and Multiphase Materials**
- **Model is Easily Refined for Improved Accuracy by Varying Element Size and Type (Approximation Scheme)**
- **Time Dependent and Dynamic Effects Can Be Included**
- **Can Handle a Variety Nonlinear Effects Including Material Behavior, Large Deformations, Boundary Conditions, Etc.**

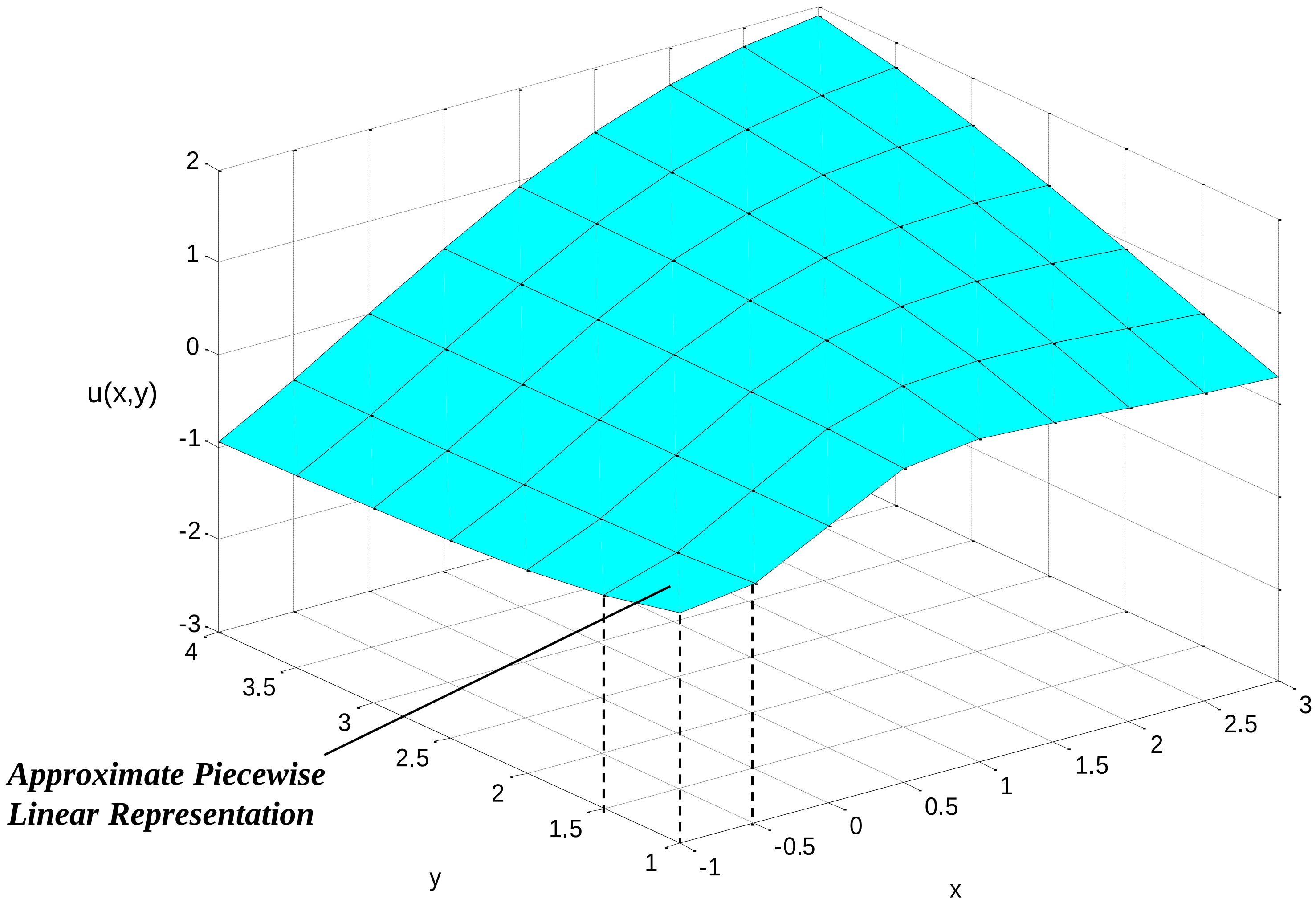
Basic Concept of the Finite Element Method

Any continuous solution field such as stress, displacement, temperature, pressure, etc. can be approximated by a discrete model composed of a set of piecewise continuous functions defined over a finite number of subdomains.

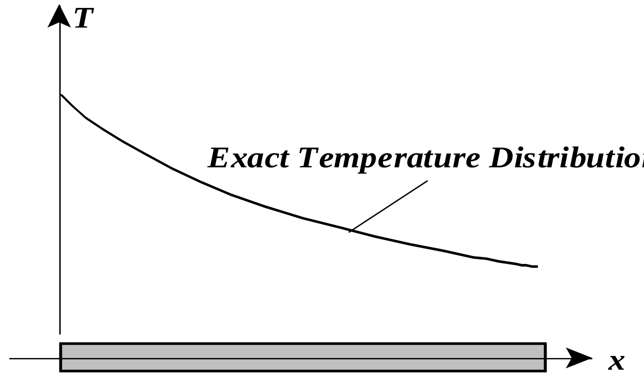
<u>One-Dimensional Temperature Distribution</u>



Two-Dimensional Discretization

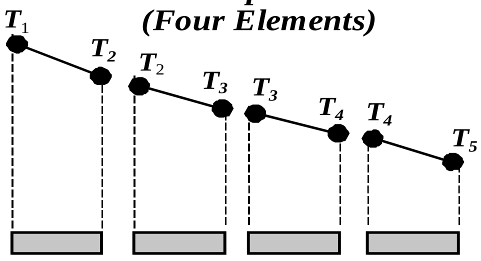


Discretization Concepts

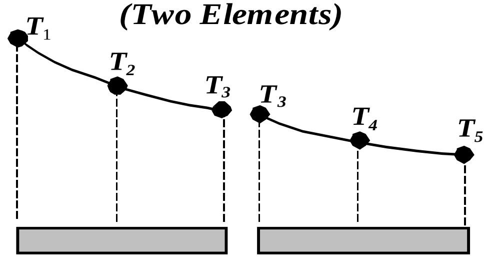


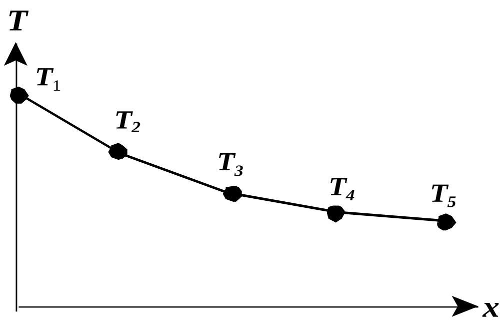
Finite Element Discretization

Linear Interpolation Model (Four Elements)



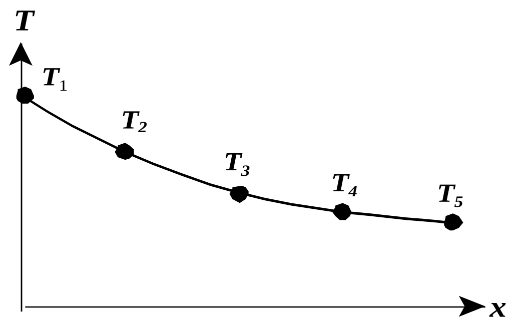
Quadratic Interpolation Model (Two Elements)





Piecewise Linear Approximation

Temperature Continuous but with Discontinuous Temperature Gradients



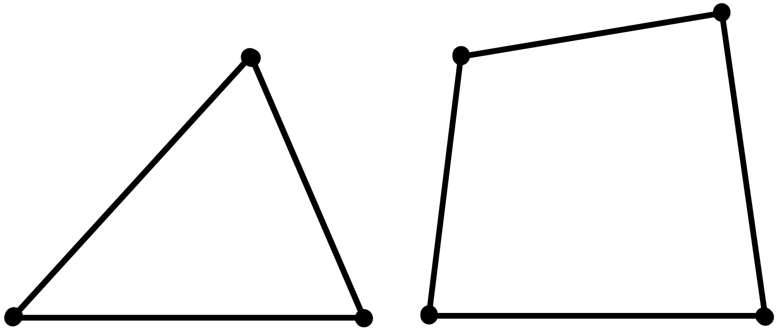
Piecewise Quadratic Approximation

Temperature and Temperature Gradients Continuous

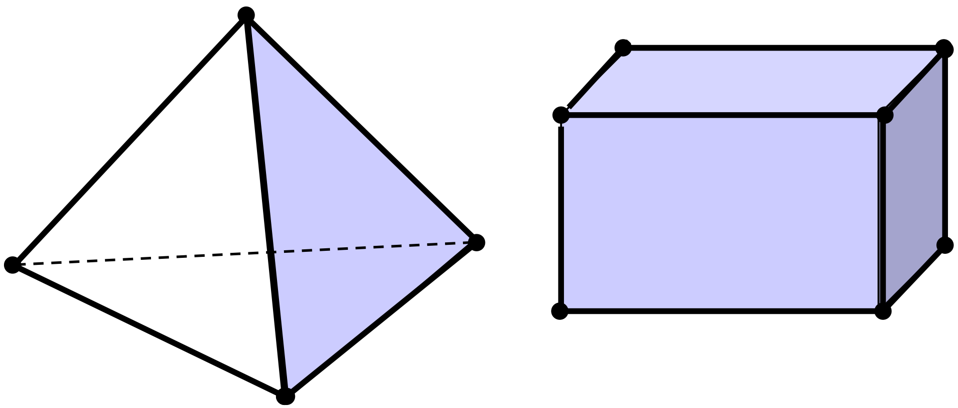
Common Types of Elements

One-Dimensional Elements
Line
Rods, Beams, Trusses, Frames

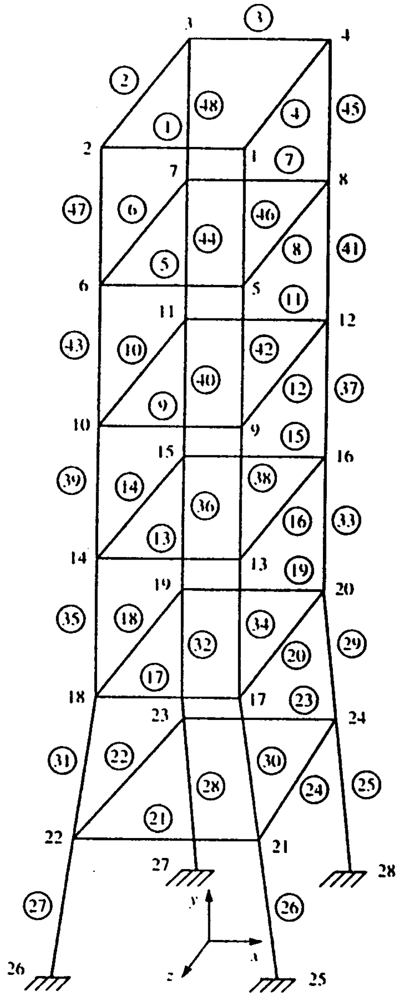
Two-Dimensional Elements
Triangular, Quadrilateral
Plates, Shells, 2-D Continua



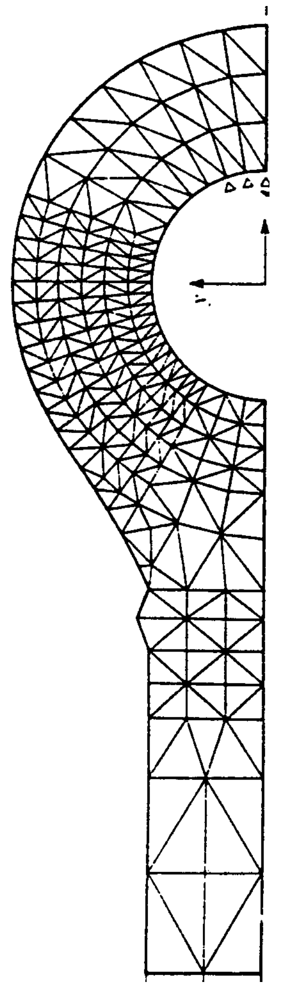
Three-Dimensional Elements
Tetrahedral, Rectangular Prism (Brick)
3-D Continua



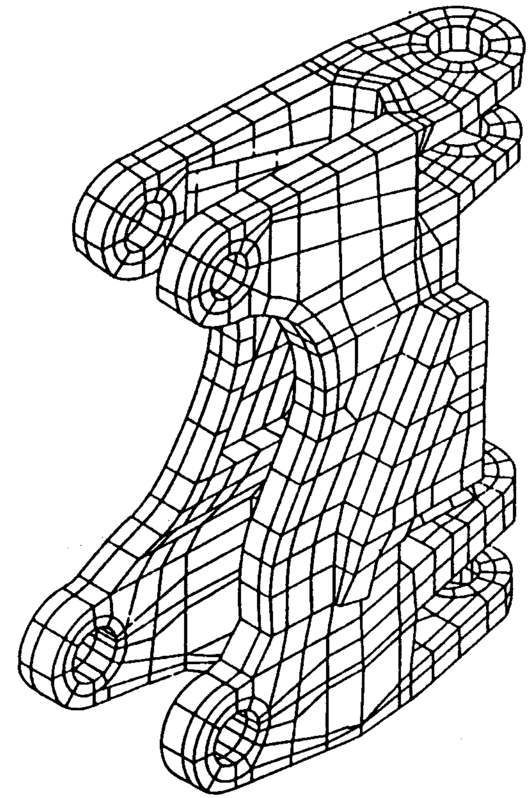
Discretization Examples



One-Dimensional Frame Elements



Two-Dimensional Triangular Elements



Three-Dimensional Brick Elements

Basic Steps in the Finite Element Method Time Independent Problems

- **Domain Discretization**
- **Select Element Type (Shape and Approximation)**
- **Derive Element Equations (Variational and Energy Methods)**
- **Assemble Element Equations to Form Global System**

$$[K]\{U\} = \{F\}$$

[K] = Stiffness or Property Matrix
{U} = Nodal Displacement Vector
{F} = Nodal Force Vector

- **Incorporate Boundary and Initial Conditions**
- **Solve Assembled System of Equations for Unknown Nodal Displacements and Secondary Unknowns of Stress and Strain Values**

Common Sources of Error in FEA

- **Domain Approximation**
- **Element Interpolation/Approximation**
- **Numerical Integration Errors (Including Spatial and Time Integration)**
- **Computer Errors (Round-Off, Etc.,)**

Measures of Accuracy in FEA

Accuracy

Error = |(Exact Solution)-(FEM Solution)|

Convergence

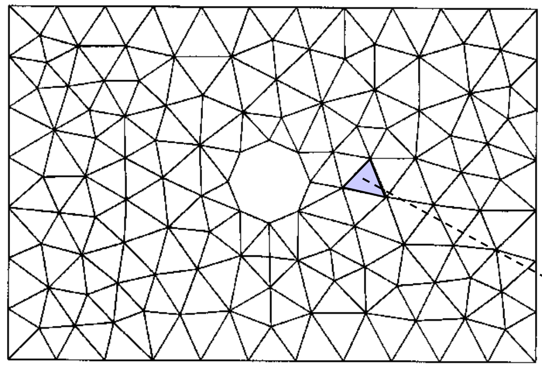
Limit of Error as:

Number of Elements (*h-convergence*)
or
Approximation Order (*p-convergence*)

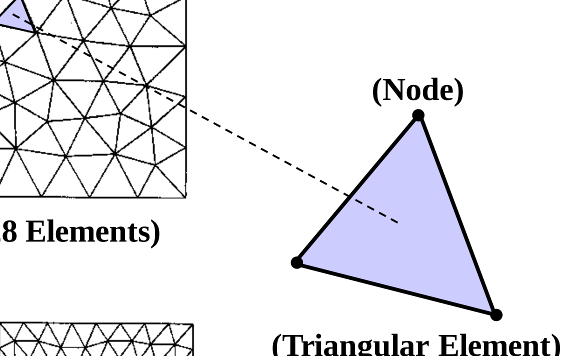
Increases

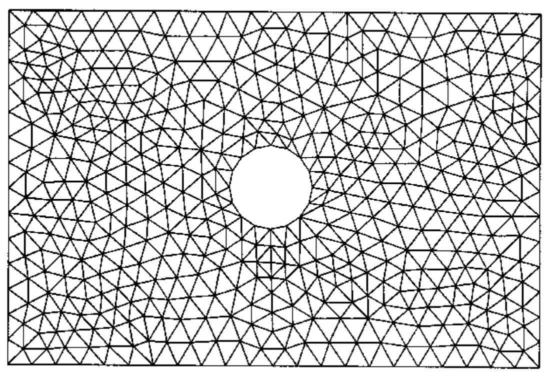
Ideally, Error $\rightarrow$ 0 as Number of Elements or Approximation Order $\rightarrow \infty$

Two-Dimensional Discretization Refinement



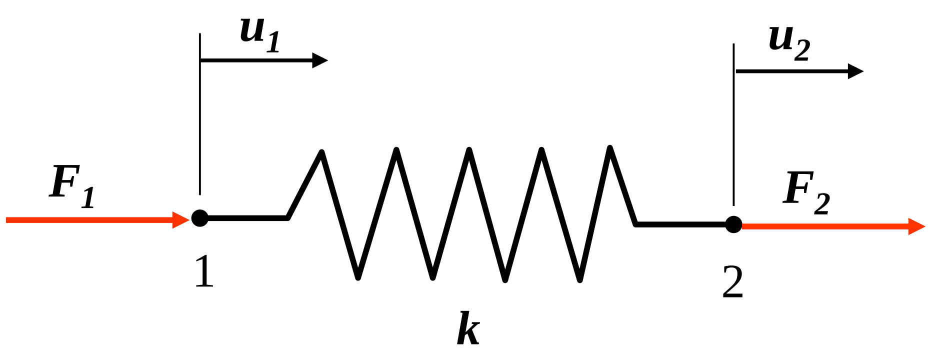
(Discretization with 228 Elements)





(Discretization with 912 Elements)

Simple Element Equation Example
Direct Stiffness Derivation



$$\text{Equilibrium at Node 1} \Rightarrow \quad F_1 = ku_1 - ku_2$$

$$\text{Equilibrium at Node 2} \Rightarrow \quad F_2 = -ku_1 + ku_2$$

or in Matrix Form

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

Stiffness Matrix

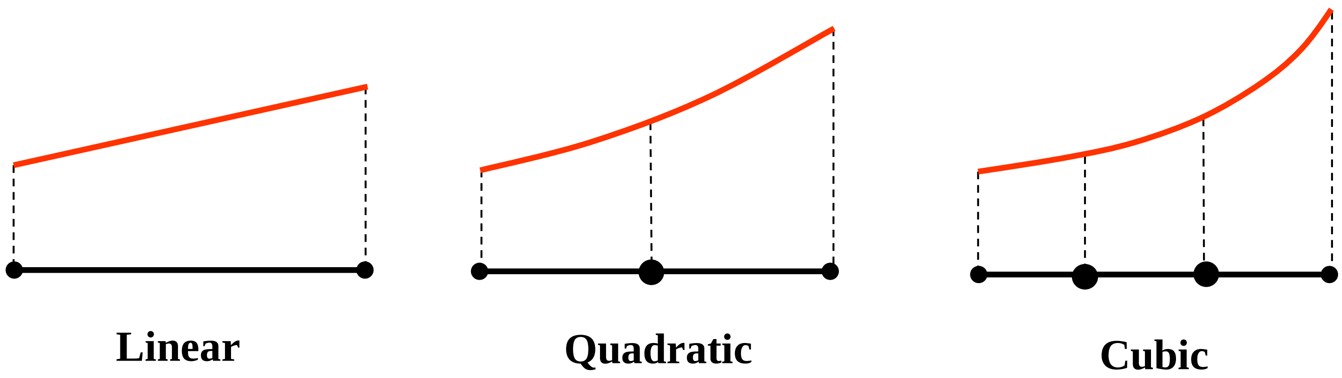
$$[K]\{u\} = \{F\}$$

Nodal Force Vector

Common Approximation Schemes One-Dimensional Examples

<u>Polynomial Approximation</u>

Most often polynomials are used to construct approximation functions for each element. Depending on the order of approximation, different numbers of element parameters are needed to construct the appropriate function.



<u>Special Approximation</u>

For some cases (e.g. infinite elements, crack or other singular elements) the approximation function is chosen to have special properties as determined from theoretical considerations