Chapter 5
Logic and Fuzzy Systems

Dr. Wafa' H. AlAlaween
wafa.alalaween@gmail.com

Logic: Definition

- Logic is a small part of the human capacity to reason.
- Fuzzy logic is a method to formalize the human capacity of imprecise (approximate) reasoning.
- Reasoning represents the human ability to judge under uncertainty.
- Interpolative reasoning: the process of interpolating between the binary extremes of true and false is represented by the ability of fuzzy logic to encapsulate partial truths.
- Proposition is associated with the concepts of truth sets, tautologies, and contradictions.

Crisp Logic

- A proposition (P) is a linguistic (declarative) statement contained within a universe of elements (X) that can be identified as being a collection of elements in X, which are strictly true or strictly false.
- The veracity of an element in the proposition P can be assigned a binary (Boolean) truth value, T (P).
- Assume that U is the universe of all propositions, then one can consider T is a mapping of the elements (u):

$$T : u \in \mathrm{U} \longrightarrow (0, 1)$$

Continue…

- Truth set, T(P): all elements u in U that are true for proposition P.
- Falsity set: all elements u in U that are false for proposition P.
- So what are:

$$T(\mathrm{U}) = ?$$
$$T(\emptyset) = ?$$

Crisp Logic: Connectives

- Assume that P and Q are two propositions on the same universe of discourse, such propositions can be combined using the following connectives:

disjunction ($\vee$)
conjunction ($\wedge$)
negation ($-$)
implication ($\rightarrow$)
equivalence ($\leftrightarrow$)

- Equivalence comes from dual implication.

Continue…

- Let us define sets A and B on a universe X, and propositions P and Q measure the truth of the statement that an element is contained in sets A and B, respectively, or more conventionally:

$$\text{P : truth that } x \in \text{A} \quad \Rightarrow \quad \text{if } x \in \text{A}, \; T(\text{P}) = 1; \text{ otherwise, } T(\text{P}) = 0$$

$$\text{Q : truth that } x \in \text{B} \quad \Rightarrow \quad \text{if } x \in \text{B}, \; T(\text{Q}) = 1; \text{ otherwise, } T(\text{Q}) = 0$$

- Using a characteristic function: $\chi_{\text{A}}(x) = \begin{cases} 1, & x \in \text{A} \\ 0, & x \notin \text{A} \end{cases}$

Continue…

Disjunction

$$\begin{aligned}&P \vee Q : x \in A \text{ or } x \in B;\\&\text{hence, } T(P \vee Q) = \max(T(P), T(Q)).\end{aligned}$$

Conjunction

$$\begin{aligned}&P \wedge Q : x \in A \text{ and } x \in B;\\&\text{hence, } T(P \wedge Q) = \min(T(P), T(Q)).\end{aligned}$$

Negation

$$\text{If } T(P) = 1, \text{ then } T(\overline{P}) = 0; \text{ if } T(P) = 0, \text{ then } T(\overline{P}) = 1.$$

Continue…

Implication

$$(\mathrm{P} \longrightarrow \mathrm{Q}) : x \notin \mathrm{A} \text{ or } x \in \mathrm{B};$$
$$\text{hence, } T(\mathrm{P} \longrightarrow \mathrm{Q}) = T(\overline{\mathrm{P}} \cup \mathrm{Q}).$$

Equivalence

$$(\mathrm{P} \longleftrightarrow \mathrm{Q}) : T(\mathrm{P} \longleftrightarrow \mathrm{Q}) = \begin{cases} 1, & \text{for } T(\mathrm{P}) = T(\mathrm{Q}) \\ 0, & \text{for } T(\mathrm{P}) \neq T(\mathrm{Q}) \end{cases}.$$

- If $T(\mathrm{P}) \cap T(\mathrm{Q}) = \emptyset$ and the truth of P always implies the falsity of Q and vice versa, then P and Q are mutually exclusive propositions.

Continue…

- Truth table for various compound propositions:

P	**Q**	$\overline{\mathbf{P}}$	$\mathbf{P \lor Q}$	$\mathbf{P \land Q}$	$\mathbf{P \rightarrow Q}$	$\mathbf{P \leftrightarrow Q}$
T (1)	T (1)	F (0)	T (1)	T (1)	T (1)	T (1)
T (1)	F (0)	F (0)	T (1)	F (0)	F (0)	F (0)
F (0)	T (1)	T (1)	T (1)	F (0)	T (1)	F (0)
F (0)	F (0)	T (1)	F (0)	F (0)	T (1)	T (1)

Crisp Logic: Tautologies

- Tautologies are compound propositions that are always true irrespective of the truth values of the individual simple propositions.

- Tautologies are useful for reasoning, proving theorems, and making deductive inferences.

“All humans are mammals”

“Prime numbers are not divisible by 6”

- Assignment: Using the truth table, represent a tautology.

Continue…

- A valid argument is a list of premises from which the conclusion follows.
- Modus ponens: is a very common scheme used in forward-chaining rule-based expert systems. It is an operation to find the truth value of a consequent given the truth value of the antecedent in a rule.

Form: If A, then B.

A.

Therefore, B.

$$(A \land (A \longrightarrow B))$$

A	B	$A \rightarrow B$	$(A \land (A \rightarrow B))$
0	0	1	0
0	1	1	0
1	0	0	0
1	1	1	1

Continue…

- Modus Tollens: is a very common scheme used in backward-chaining expert systems.

Form:

If A, then B.

~B.

Therefore, ~A.

$(\overline{B} \wedge (A \longrightarrow B))$

A	B	$A \to B$	$(\overline{B} \wedge (A \to B))$
0	0	1	1
0	1	1	0
1	0	0	0
1	1	1	0

Crisp Logic: Contradictions and Equivalence

- Contradictions are compound propositions that are always false, regardless of the truth value of the individual propositions constituting the compound proposition.

“Prime numbers are a multiple of 4”

- Propositions P and Q are equivalent ($P \leftrightarrow Q$) when both P and Q are true or when both P and Q are false.

- Assignment: Using the truth table, represent a contradiction.
- Assignment: plot the Venn diagram for equivalence.

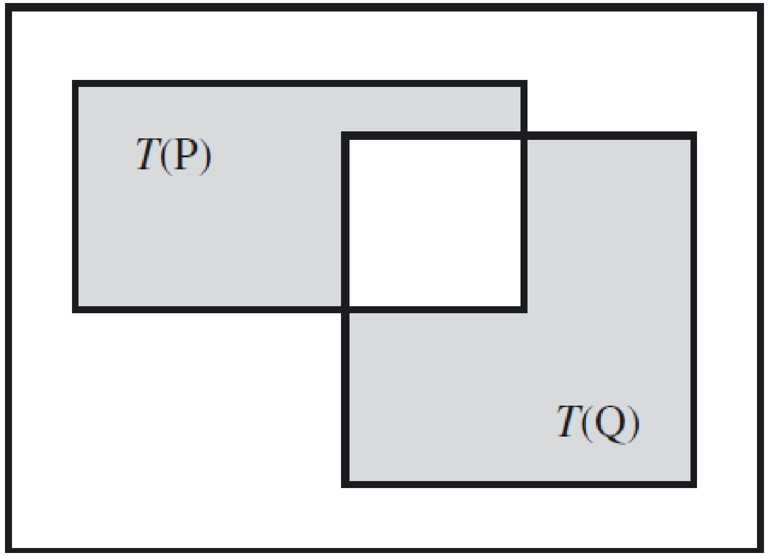
Crisp Logic: Example

Example Suppose we consider the universe of positive integers, X $= \{1 \le n \le 8\}$. Let P $=$ "n is an even number" and let Q $=$ "$(3 \le n \le 7) \wedge (n \ne 6)$." Then $T(P) = \{2, 4, 6, 8\}$ and $T(Q) = \{3, 4, 5, 7\}$. The equivalence P $\leftrightarrow$ Q has the truth set

$$T(\mathrm{P} \longleftrightarrow \mathrm{Q}) = (T(P) \cap T(Q)) \cup (\overline{T(P)} \cap \overline{T(Q)}) = \{4\} \cup \{1\} = \{1, 4\}$$

Crisp Logic: Exclusive or

- Exclusive or (XOR): it arises in many situations involving natural language and human reasoning.
- This situation involves the exclusive or; it does not involve the intersection.
- Assignment: Using the truth table and Venn diagram, represent the "Exclusive or



P	Q	P *XOR* Q
1	1	0
1	0	1
0	1	1
0	0	0

Crisp Logic: Exclusive nor

- Exclusive nor ($\overline{XOR}$) is the complement of the exclusive or.

- Assignment: Using the truth table and Venn diagram, represent the "Exclusive nor":

P	**Q**	$\overline{\mathbf{P}\ \mathit{XOR}\ \mathbf{Q}}$
1	1	1
1	0	0
0	1	0
0	0	1

Logical Proofs

- Inference: the process of making certain conclusions from some given hypotheses.
- How?
  1. The linguistic statement (compound proposition) is made.
  2. The statement is decomposed into its respective single propositions.
  3. The statement is expressed algebraically with logical connectives.
  4. A truth table is used to establish the veracity of the statement.

- Self-reading: Deductive inference.

Logical Proofs: Example

- Hypotheses: Engineers are mathematicians. Logical thinkers do not believe in magic. Mathematicians are logical thinkers.
- Conclusion: Engineers do not believe in magic.
- Decomposing the hypotheses:

  P : a person is an engineer.

  Q : a person is a mathematician.

  R : a person is a logical thinker.

  S : a person believes in magic.

$$((P \longrightarrow Q) \wedge (R \longrightarrow \overline{S}) \wedge (Q \longrightarrow R)) \longrightarrow (P \longrightarrow \overline{S})$$

Fuzzy Logic

- A fuzzy logic proposition ($\underset{\sim}{P}$) is a statement involving some concept without clearly defined boundaries.
- The truth value assigned to $\underset{\sim}{P}$ can be any value on the interval [0, 1].
- Fuzzy propositions are assigned to fuzzy sets. Suppose proposition $\underset{\sim}{P}$ is assigned to fuzzy set $\underset{\sim}{A}$, then, the truth value of a proposition is given as follows:

$$T(\underset{\sim}{P}) = \mu_{\underset{\sim}{A}}(x), \quad \text{where } 0 \le \mu_{\underset{\sim}{A}} \le 1$$

Fuzzy Logic: Connectives

- The logical connectives:

Negation

$$T(\overline{\underset{\sim}{\mathrm{P}}}) = 1 - T(\underset{\sim}{\mathrm{P}}).$$

Disjunction

$$\underset{\sim}{\mathrm{P}} \vee \underset{\sim}{\mathrm{Q}} : x \text{ is } \underset{\sim}{\mathrm{A}} \text{ or } \underset{\sim}{\mathrm{B}} \quad T(\underset{\sim}{\mathrm{P}} \vee \underset{\sim}{\mathrm{Q}}) = \max(T(\underset{\sim}{\mathrm{P}}), T(\underset{\sim}{\mathrm{Q}})).$$

Conjunction

$$\underset{\sim}{\mathrm{P}} \wedge \underset{\sim}{\mathrm{Q}} : x \text{ is } \underset{\sim}{\mathrm{A}} \text{ and } \underset{\sim}{\mathrm{B}} \quad T(\underset{\sim}{\mathrm{P}} \wedge \underset{\sim}{\mathrm{Q}}) = \min(T(\underset{\sim}{\mathrm{P}}), T(\underset{\sim}{\mathrm{Q}})).$$

Continue...

Implication

$$\underset{\sim}{P} \rightarrow \underset{\sim}{Q} : x \text{ is } \underset{\sim}{A}, \text{ then } x \text{ is } \underset{\sim}{B}$$

$$T(\underset{\sim}{P} \rightarrow \underset{\sim}{Q}) = T(\overline{\underset{\sim}{P}} \vee \underset{\sim}{Q}) = \max(T(\overline{\underset{\sim}{P}}), T(\underset{\sim}{Q})).$$

- As in the crisp logic, the implication can be modelled in rule-based form:

$$\underset{\sim}{P} \rightarrow \underset{\sim}{Q} \text{ is IF } x \text{ is } \underset{\sim}{A}, \text{ THEN } y \text{ is } \underset{\sim}{B}$$

- Note: it is equivalent to $\underset{\sim}{R} = (\underset{\sim}{A} \times \underset{\sim}{B}) \cup (\overline{\underset{\sim}{A}} \times Y)$

Example

Example Suppose we are evaluating a new invention to determine its commercial potential. We will use two metrics to make our decisions regarding the innovation of the idea. Our metrics are the "uniqueness" of the invention, denoted by a universe of novelty scales, $X = \{1, 2, 3, 4\}$, and the "market size" of the invention's commercial market, denoted on a universe of scaled market sizes, $Y = \{1, 2, 3, 4, 5, 6\}$. In both universes, the lowest numbers are the "highest uniqueness" and the "largest market," respectively. A new invention in your group, say a compressible liquid of very useful temperature and viscosity conditions, has just received scores of "medium uniqueness," denoted by fuzzy set $\underset{\sim}{A}$, and "medium market size," denoted fuzzy set $\underset{\sim}{B}$. We wish to determine the implication of such a result, that is, IF $\underset{\sim}{A}$, THEN $\underset{\sim}{B}$. We assign the invention the following fuzzy sets to represent its ratings:

Continue…

$$\underset{\sim}{A} = \text{medium uniqueness} = \left\{ \frac{0.6}{2} + \frac{1}{3} + \frac{0.2}{4} \right\}.$$

$$\underset{\sim}{B} = \text{medium market size} = \left\{ \frac{0.4}{2} + \frac{1}{3} + \frac{0.8}{4} + \frac{0.3}{5} \right\}.$$

$$\underset{\sim}{C} = \text{diffuse market size} = \left\{ \frac{0.3}{1} + \frac{0.5}{2} + \frac{0.6}{3} + \frac{0.6}{4} + \frac{0.5}{5} + \frac{0.3}{6} \right\}.$$

Find $\underset{\sim}{R} = (\underset{\sim}{A} \times \underset{\sim}{B}) \cup (\overline{\underset{\sim}{A}} \times Y)$

Continue…

$$
\underset{\sim}{A} \times \underset{\sim}{B} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0.6 & 0.6 & 0.3 & 0 \\ 0 & 0.4 & 1 & 0.8 & 0.3 & 0 \\ 0 & 0.2 & 0.2 & 0.2 & 0.2 & 0 \end{bmatrix} \end{array}
$$

$$
\overline{\underset{\sim}{A}} \times Y = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.8 & 0.8 & 0.8 & 0.8 & 0.8 & 0.8 \end{bmatrix} \end{array}
$$

$$
\underset{\sim}{R} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0.4 & 0.4 & 0.6 & 0.6 & 0.4 & 0.4 \\ 0 & 0.4 & 1 & 0.8 & 0.3 & 0 \\ 0.8 & 0.8 & 0.8 & 0.8 & 0.8 & 0.8 \end{bmatrix} \end{array}
$$

Continue…

- When the logical conditional implication is of the compound form:

$$\text{IF } x \text{ is } \underset{\sim}{A}, \text{ THEN } y \text{ is } \underset{\sim}{B}, \text{ ELSE } y \text{ is } \underset{\sim}{C},$$

- Then, the fuzzy relation can be presented as

$$\underset{\sim}{R} = (\underset{\sim}{A} \times \underset{\sim}{B}) \cup (\overline{\underset{\sim}{A}} \times \underset{\sim}{C}).$$

$$\begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0.3 & 0.5 & 0.6 & 0.6 & 0.5 & 0.3 \\ 0.3 & 0.4 & 0.6 & 0.6 & 0.4 & 0.3 \\ 0 & 0.4 & 1 & 0.8 & 0.3 & 0 \\ 0.3 & 0.5 & 0.6 & 0.6 & 0.5 & 0.3 \end{bmatrix} \end{array}$$

Approximate Reasoning

- Approximate reasoning is about imprecise propositions.
- It deals with partial truth.
- Question: suppose we have a rule expressed as follows:

$$\text{IF } x \text{ is } \underset{\sim}{A}, \text{ THEN } y \text{ is } \underset{\sim}{B}$$

If we introduce a new antecedent, is it possible to derive the consequent?

$$\text{IF } x \text{ is } \underset{\sim}{A}', \text{ THEN } y \text{ is } \underset{\sim}{B}'$$

- By using the composition operation ($\underset{\sim}{B}' = \underset{\sim}{A}' \circ \underset{\sim}{R}$), the answer is YES.

Continue…

- For the previous example, what market size would be associated with a uniqueness score of "almost high uniqueness"

$$\underset{\sim}{A}' = \text{almost high uniqueness} = \left\{ \frac{0.5}{1} + \frac{1}{2} + \frac{0.3}{3} + \frac{0}{4} \right\}$$

- By using the max-min composition:

$$\underset{\sim}{B}' = \underset{\sim}{A}' \circ \underset{\sim}{R} = \left\{ \frac{0.5}{1} + \frac{0.5}{2} + \frac{0.6}{3} + \frac{0.6}{4} + \frac{0.5}{5} + \frac{0.5}{6} \right\}$$

Fuzzy Implication Operations

- There are other techniques one can use to obtaining the fuzzy relation ($\underset{\sim}{R}$) based on a fuzzy rule.
- The membership function values of $\underset{\sim}{R}$ can be presented as follows:

$$\mu_{\underset{\sim}{R}}(x, y) = \max[\mu_{\underset{\sim}{B}}(y), 1 - \mu_{\underset{\sim}{A}}(x)]$$

$$\mu_{\underset{\sim}{R}}(x, y) = \min[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y)]$$

$$\mu_{\underset{\sim}{R}}(x, y) = \min\{1, [1 - \mu_{\underset{\sim}{A}}(x) + \mu_{\underset{\sim}{B}}(y)]\}$$

Continue…

$$\mu_{\underset{\sim}{R}}(x, y) = \mu_{\underset{\sim}{A}}(x) \cdot \mu_{\underset{\sim}{B}}(y)$$

$$\mu_{\underset{\sim}{R}}(x, y) = \begin{cases} 1, & \text{for } \mu_{\underset{\sim}{A}}(x) \leq \mu_{\underset{\sim}{B}}(y); \\ \mu_{\underset{\sim}{B}}(y), & \text{otherwise.} \end{cases}$$

Fuzzy System (FS)

- Natural language: it vague and ambiguous, however, one can understand it.
- Example: “young” is a term that can be linguistically interpreted in terms of age.

$$\mu_{\underset{\sim}{M}}(\text{young},\ y) = \begin{cases} \left[1 + \left(\frac{y-25}{5}\right)^2\right]^{-1}, & y > 25 \text{ years;} \\ 1, & y \leq 25 \text{ years.} \end{cases}$$

Continue…

- A composite is a collection (set) of terms combined by various linguistic connectives such as and, or, and not.
- For the two terms α and β, the interpretation of the composite can be defined by using theoretic operations as follows:

$$\alpha \text{ or } \beta: \mu_{\alpha \text{ or } \beta}(y) = \max(\mu_\alpha(y), \mu_\beta(y)),$$
$$\alpha \text{ and } \beta: \mu_{\alpha \text{ and } \beta}(y) = \min(\mu_\alpha(y), \mu_\beta(y)),$$
$$\text{Not } \alpha = \overline{\alpha} : \mu_{\overline{\alpha}}(y) = 1 - \mu_\alpha(y).$$

FS: If-Then Rule-Based System

- Knowledge is usually represented using If-Then rule-based form.

IF premise (antecedent), THEN conclusion (consequent)

- The fuzzy rule-based system is useful in modelling complex systems that can be observed by humans, thus linguistic variables can be used to describe the antecedents and consequents.
- The linguistic variables can be naturally represented by fuzzy sets and logical connectives of these sets.

FS: Multiple Conjunctive Antecedents

- Suppose that the rule is as follows:

$$\text{IF } x \text{ is } \underset{\sim}{A}^1 \text{ and } \underset{\sim}{A}^2 \ldots \text{ and } \underset{\sim}{A}^L \text{ THEN } y \text{ is } \underset{\sim}{B}^s$$

- A new fuzzy set can be defined as follows:

$$\underset{\sim}{A}^s = \underset{\sim}{A}^1 \cap \underset{\sim}{A}^2 \cap \cdots \cap \underset{\sim}{A}^L \qquad \mu_{\underset{\sim}{A}^s}(x) = \min[\mu_{\underset{\sim}{A}^1}(x), \mu_{\underset{\sim}{A}^2}(x), \ldots, \mu_{\underset{\sim}{A}^L}(x)]$$

- Then the compound rule can be represented as:

$$\text{IF } \underset{\sim}{A}^s \text{ THEN } \underset{\sim}{B}^s$$

FS: Multiple Disjunctive Antecedents

- Suppose that the rule is as follows:

$$\text{IF } x \text{ is } \underset{\sim}{\mathrm{A}}^1 \text{ OR } x \text{ is } \underset{\sim}{\mathrm{A}}^2 \ldots \text{OR } x \text{ is } \underset{\sim}{\mathrm{A}}^L \text{ THEN } y \text{ is } \underset{\sim}{\mathrm{B}}^s$$

- A new fuzzy set can be defined as follows:

$$\underset{\sim}{\mathrm{A}}^s = \underset{\sim}{\mathrm{A}}^1 \cup \underset{\sim}{\mathrm{A}}^2 \cup \cdots \cup \underset{\sim}{\mathrm{A}}^L \qquad \mu_{\underset{\sim}{\mathrm{A}}^s}(x) = \max\left[\mu_{\underset{\sim}{\mathrm{A}}^1}(x), \mu_{\underset{\sim}{\mathrm{A}}^2}(x), \ldots, \mu_{\underset{\sim}{\mathrm{A}}^L}(x)\right]$$

- Then the compound rule can be represented as:

$$\text{IF } \underset{\sim}{\mathrm{A}}^s \text{ THEN } \underset{\sim}{\mathrm{B}}^s$$

FS: Aggregation of Fuzzy Rules

- Aggregation: The process of obtaining the overall consequent from the individual consequents contributed by each rule.

- Two aggregation strategies:
  1. Conjunctive system of rules
  2. Disjunctive system of rules

Continue…

- Conjunctive system of rules

$$y = y^1 \text{ and } y^2 \text{and} \ldots \text{and } y^r \qquad y = y^1 \cap y^2 \cap \cdots \cap y^r$$

$$\mu_y(y) = \min(\mu_{y^1}(y), \mu_{y^2}(y), \ldots, \mu_{y^r}(y)), \text{ for } y \in Y$$

- Disjunctive system of rules

$$y = y^1 \text{ or } y^2 \text{ or} \ldots \text{or } y^r \qquad y = y^1 \cup y^2 \cup \cdots \cup y^r$$

$$\mu_y(y) = \max(\mu_{y^1}(y), \mu_{y^2}(y), \ldots, \mu_{y^r}(y)), \text{for} y \in Y$$

Inference: Graphical Techniques

- Have you read “Deductive inference”?

- Graphical methods usually make the manual computations of the inference easy and straightforward (with a few rules).
- Three common methods for fuzzy systems based on linguistic rules:
  1. Mamdani systems
  2. Takagi Sugeno systems
  3. Tsukamoto systems

Inference: Mamdani Systems

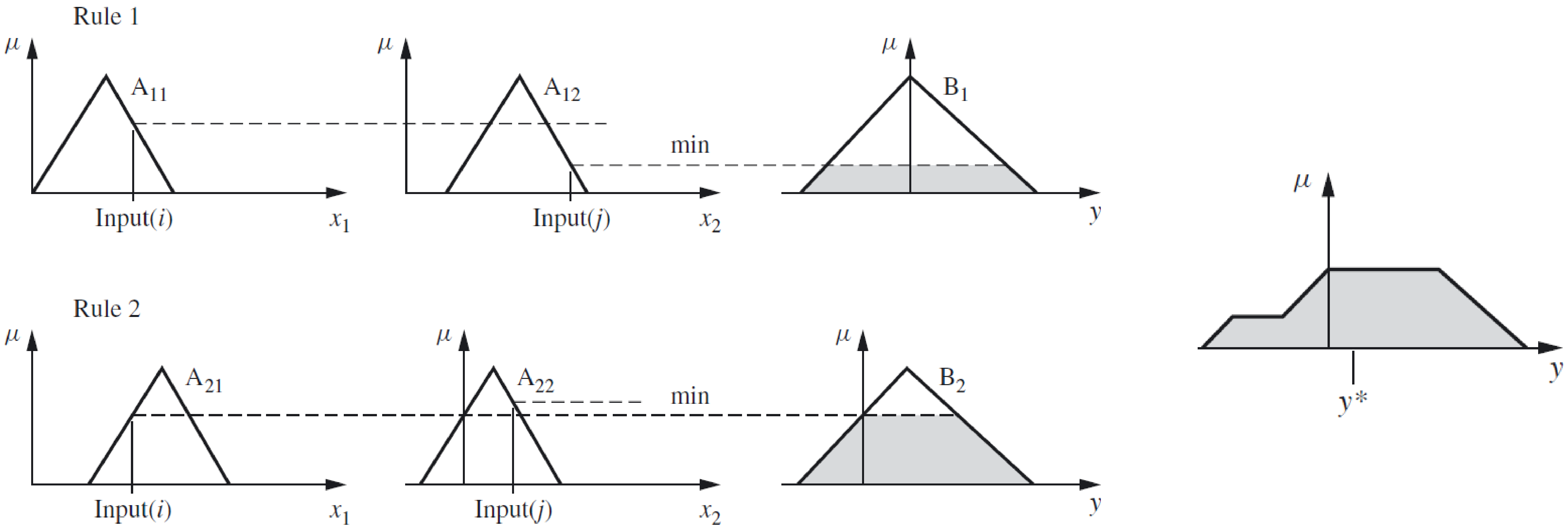
- Let us consider a simple two-rule system where each rule comprises of two antecedents and one consequent, the Mamdani form is given as:

$$\text{IF } x_1 \text{ is } \underset{\sim}{A}_1^k \text{ and } x_2 \text{ is } \underset{\sim}{A}_2^k \text{ THEN } y^k \text{ is } \underset{\sim}{B}^k, \qquad \text{for } k = 1, 2, \ldots, r,$$

- For the Mamdani system, two cases can be considered:
  - A max–min inference method
  - A max–product inference method

Continue…

A max–min inference method:



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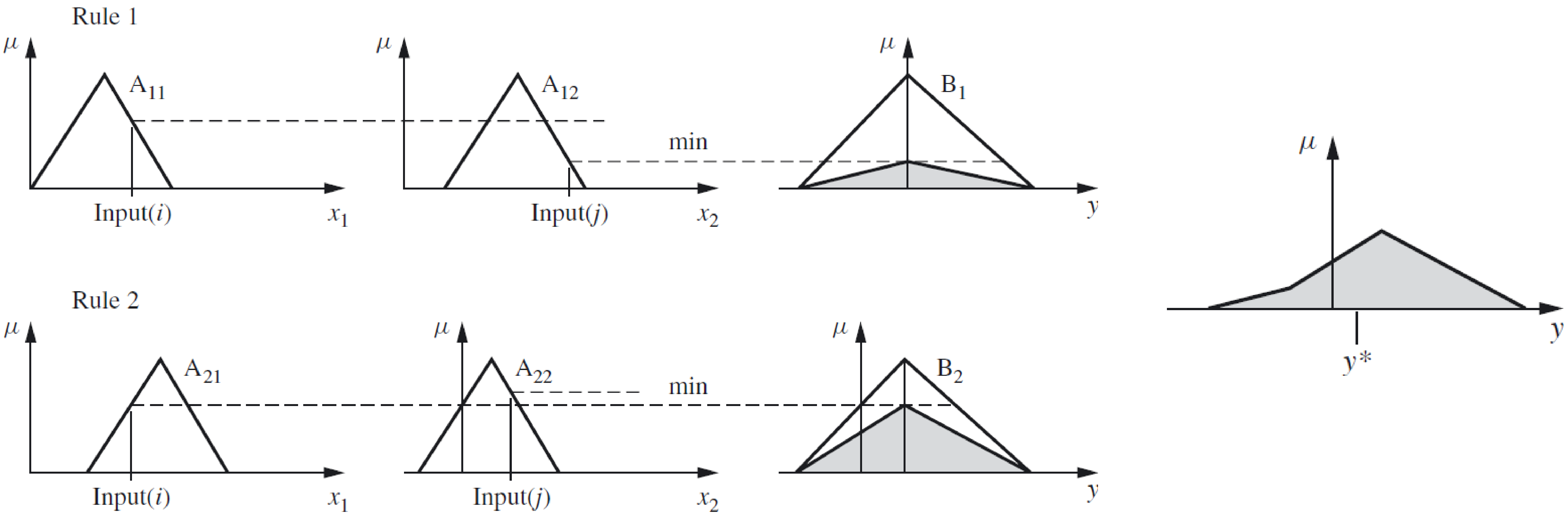
A max–product inference method



Example

Example In mechanics, the energy of a moving body is called kinetic energy. If an object of mass m (kilograms) is moving with a velocity v (meters per second), then the kinetic energy k (in joules) is given by the equation $k = \frac{1}{2}mv^2$. Suppose we model the mass and velocity as inputs to a system (moving body) and the energy as output, then observe the system for a while and deduce the following two disjunctive rules of inference based on our observations:

$$\text{Rule 1: IF } x_1 \text{ is } \underset{\sim}{A}_1^1 \text{ (small mass) } \textit{and } x_2 \text{ is } \underset{\sim}{A}_2^1 \text{ (high velocity),}$$

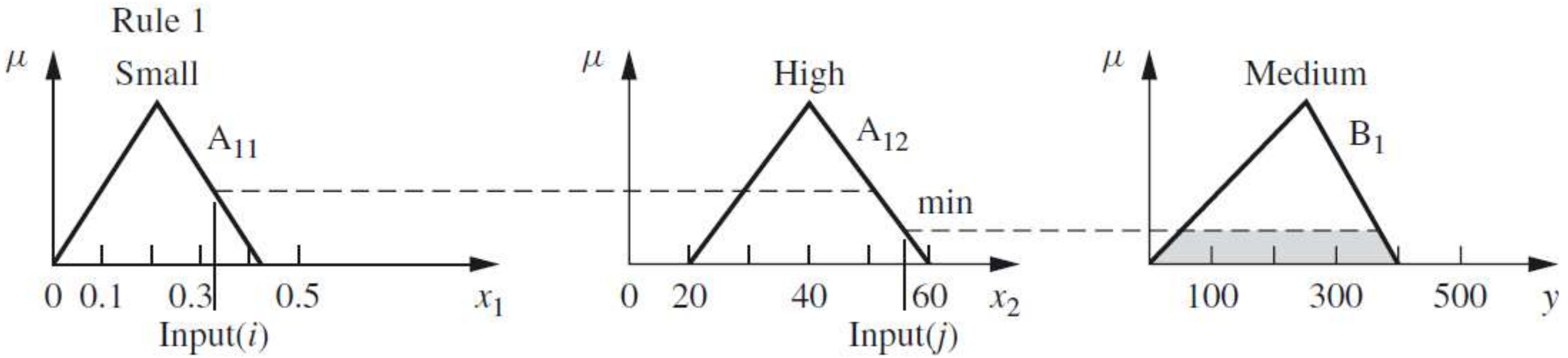
$$\text{THEN } y \text{ is } \underset{\sim}{B}^1 \text{(medium energy).}$$

$$\text{Rule 2: IF} x_1 \text{ is } \underset{\sim}{A}_1^2 \text{ (large mass) } \textit{or } x_2 \text{ is } \underset{\sim}{A}_2^1 \text{ (high velocity),}$$

$$\text{THEN } y \text{ is } \underset{\sim}{B}^2 \text{ (high energy).}$$

Continue…

- Assume that mass=0.35 kg and velocity=55m/s.
- By using the max–min inference:



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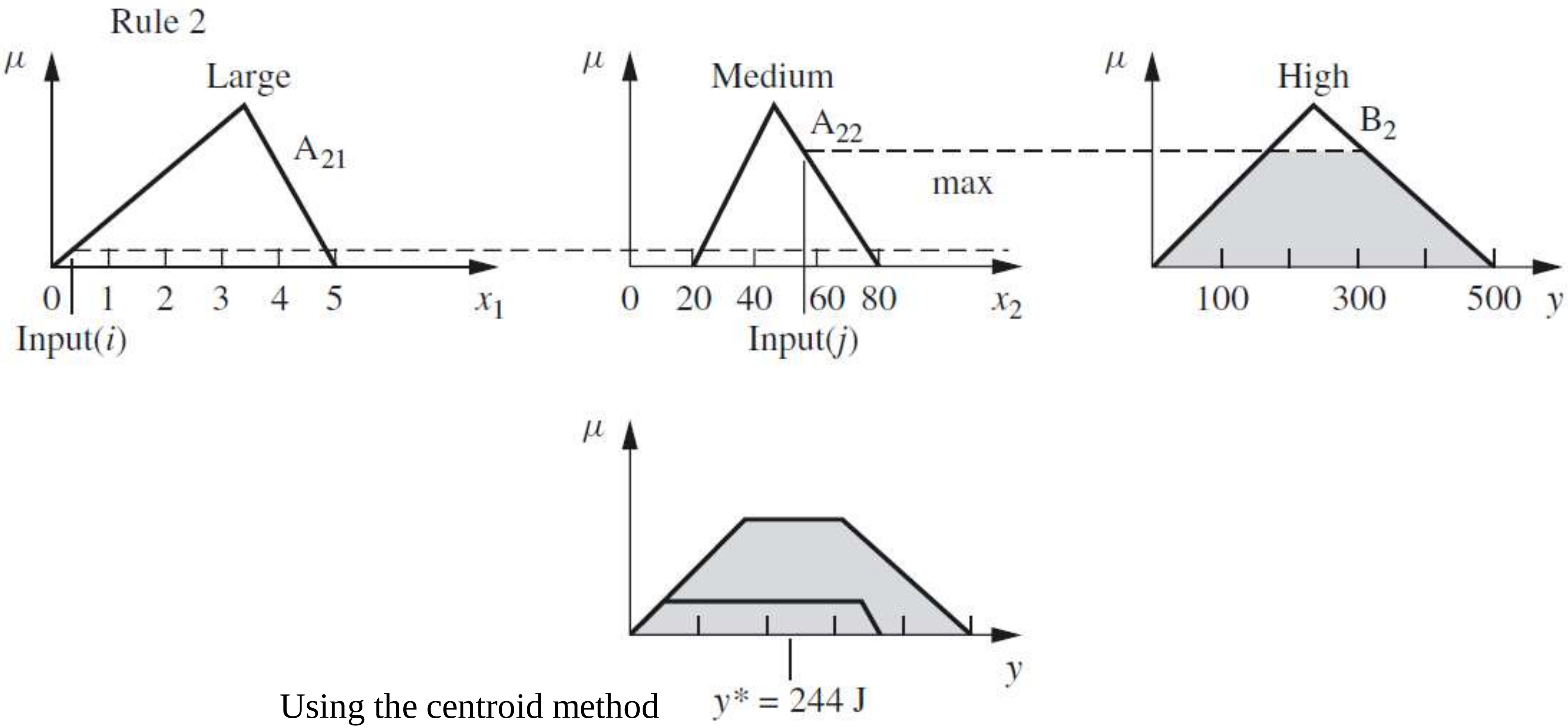
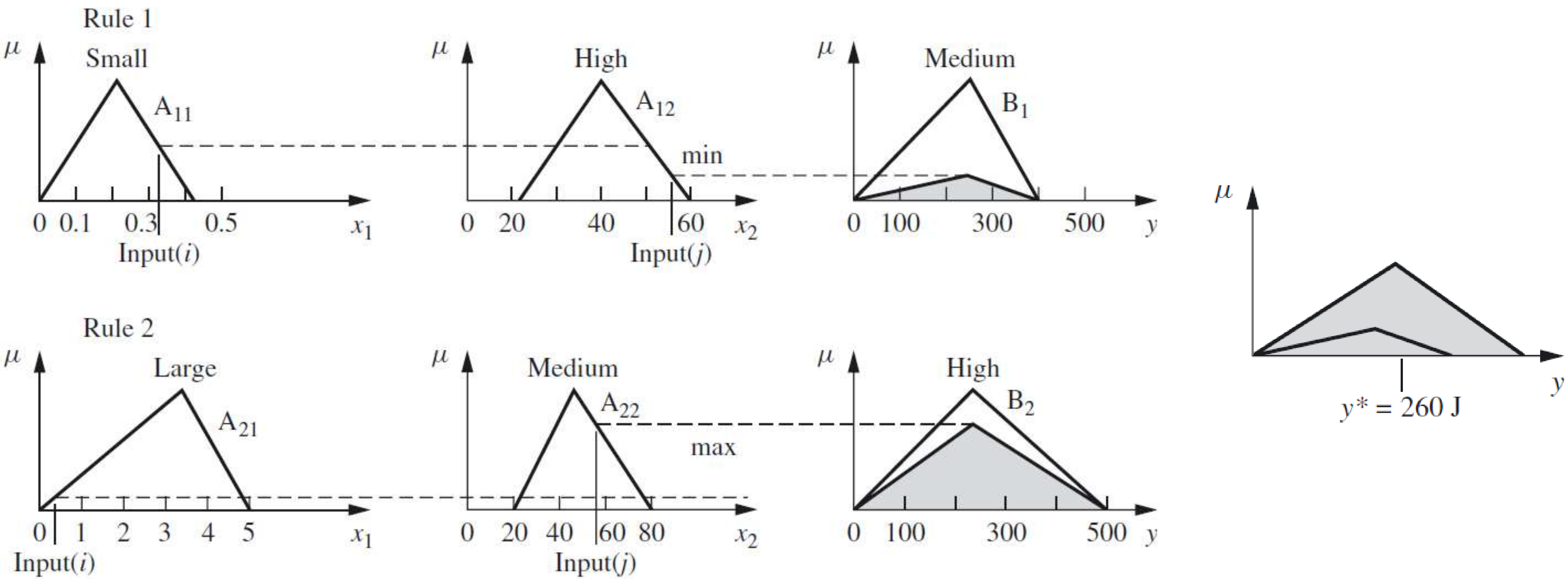


Using the centroid method

Continue…

- By using the max–min inference:



Inference: Takagi Sugeno Systems

- The Takagi Sugeno rule, which has two inputs x and y and output z, is given as:

$$\text{IF } x \text{ is } \underset{\sim}{A} \textit{ and } y \text{ is } \underset{\sim}{B}, \text{ THEN } z \text{ is } z = f(x, y)$$

- The f (x, y) can be any function that describes the output of the system. A polynomial function is common.
- A zero-order system (special case of Mamdani system): f (x, y) is constant.
- A first-order system: f (x, y) is a linear function.

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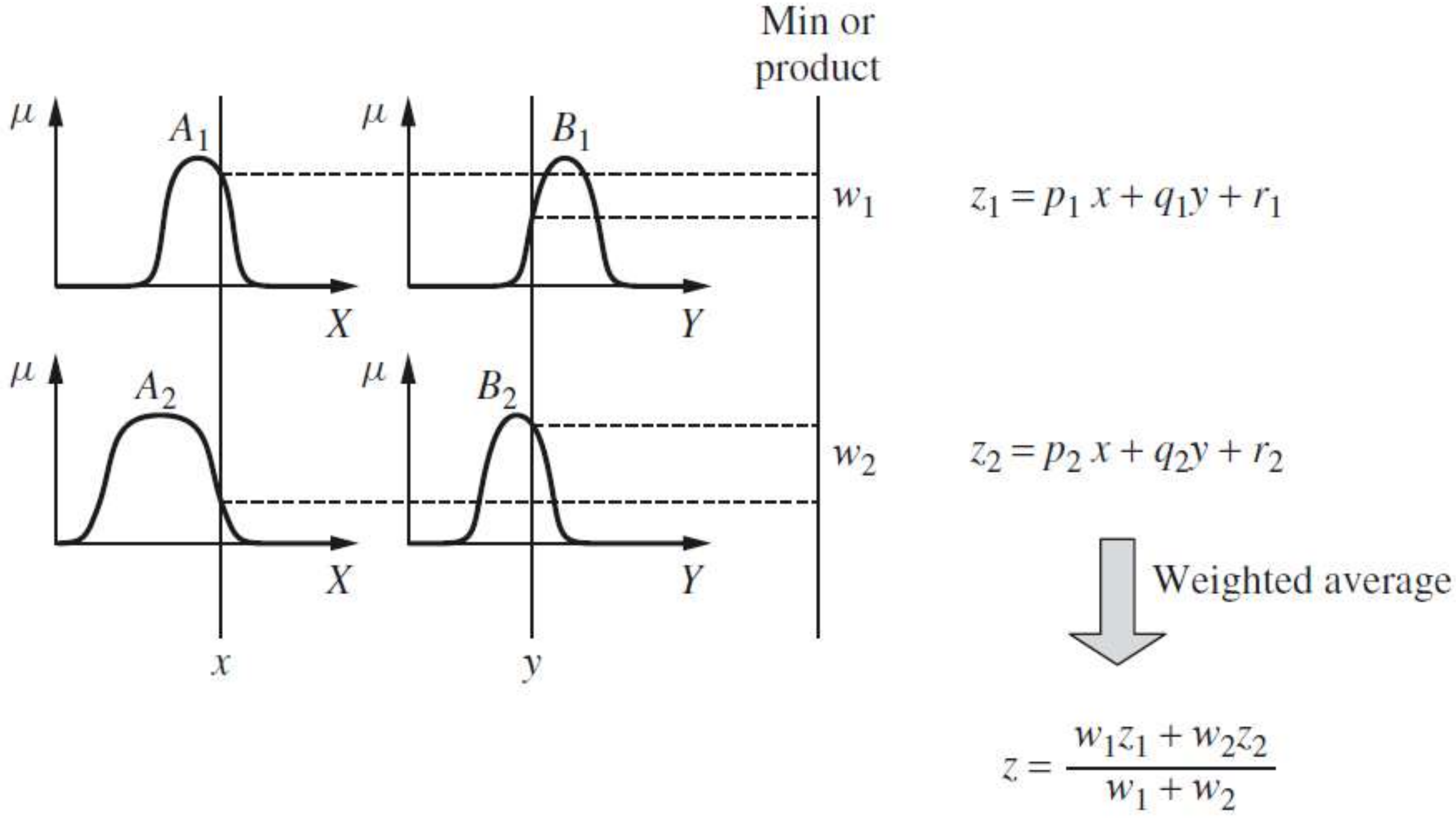


Note: Each rule has a crisp output, thus, the overall output is obtained via a weighted average defuzzification

Example

- A two-input, single-output Sugeno model with four rules is presented as follows (Jang *et al*.,1997):

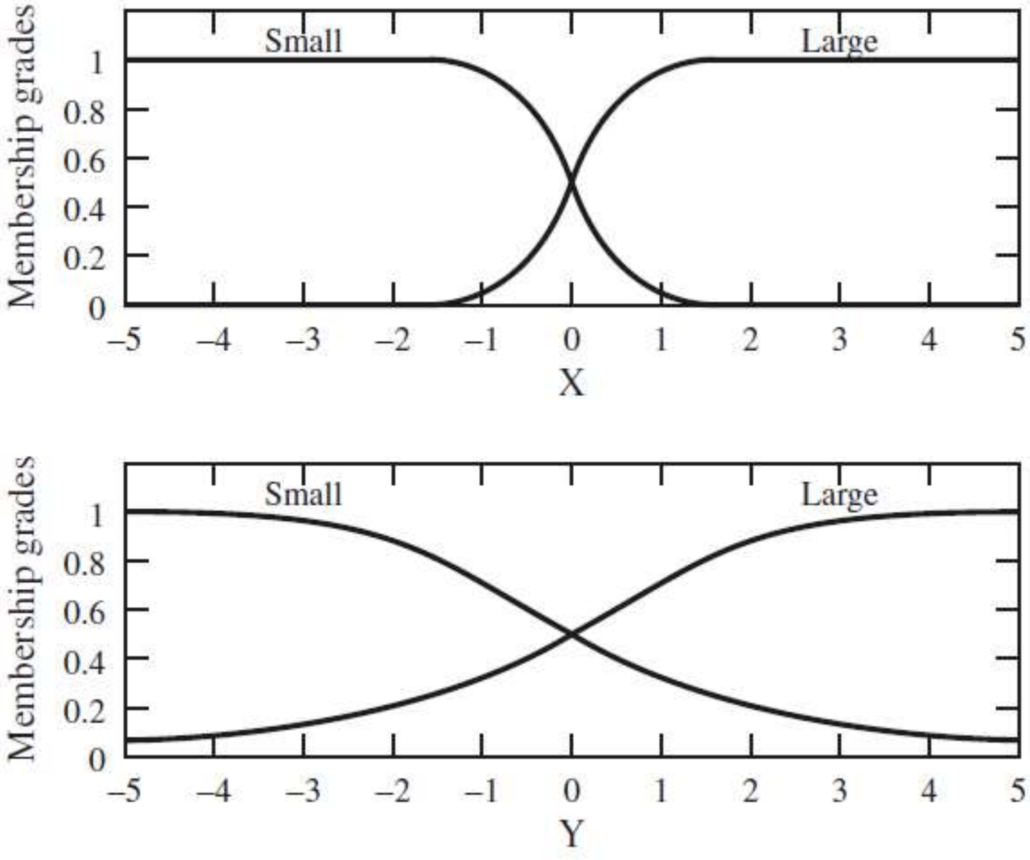
IF X is small and Y is small, THEN $z = -x + y + 1$.

IF X is small and Y is large, THEN $z = -y + 3$.

IF X is large and Y is small, THEN $z = -x + 3$.

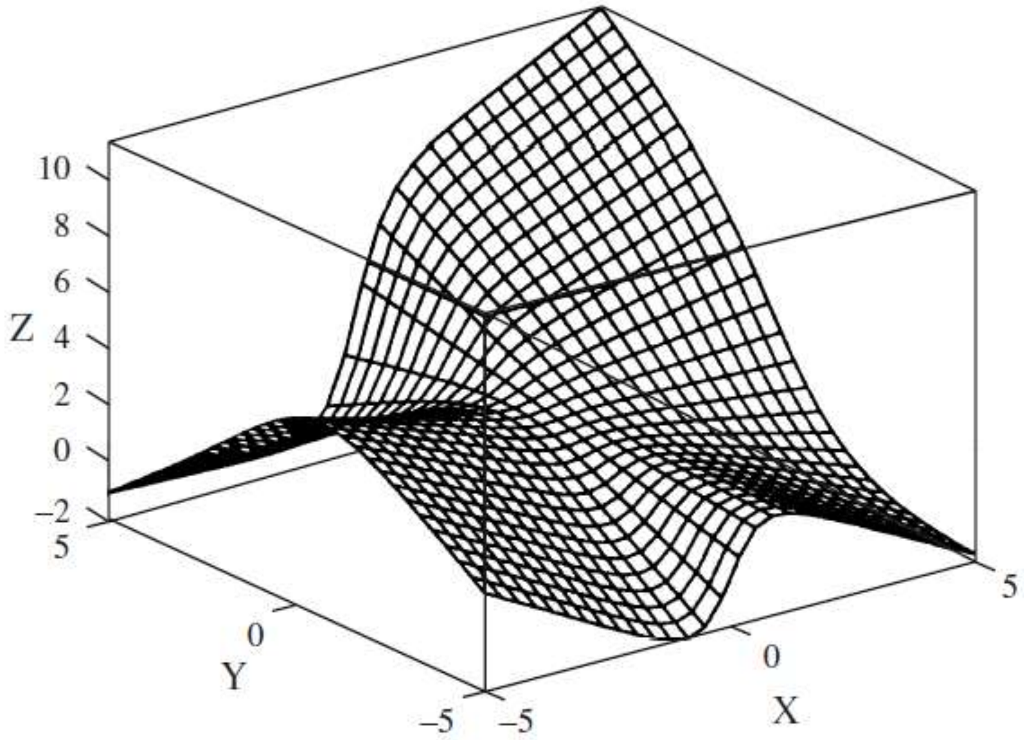
IF X is large and Y is large, THEN $z = x + y + 2$.

Continue…



Membership grades
Small
Large
1
0.8
0.6
0.4
0.2
0
–5
–4
–3
–2
–1
0
1
2
3
4
5
X
Membership grades
Small
Large
Y

(a)

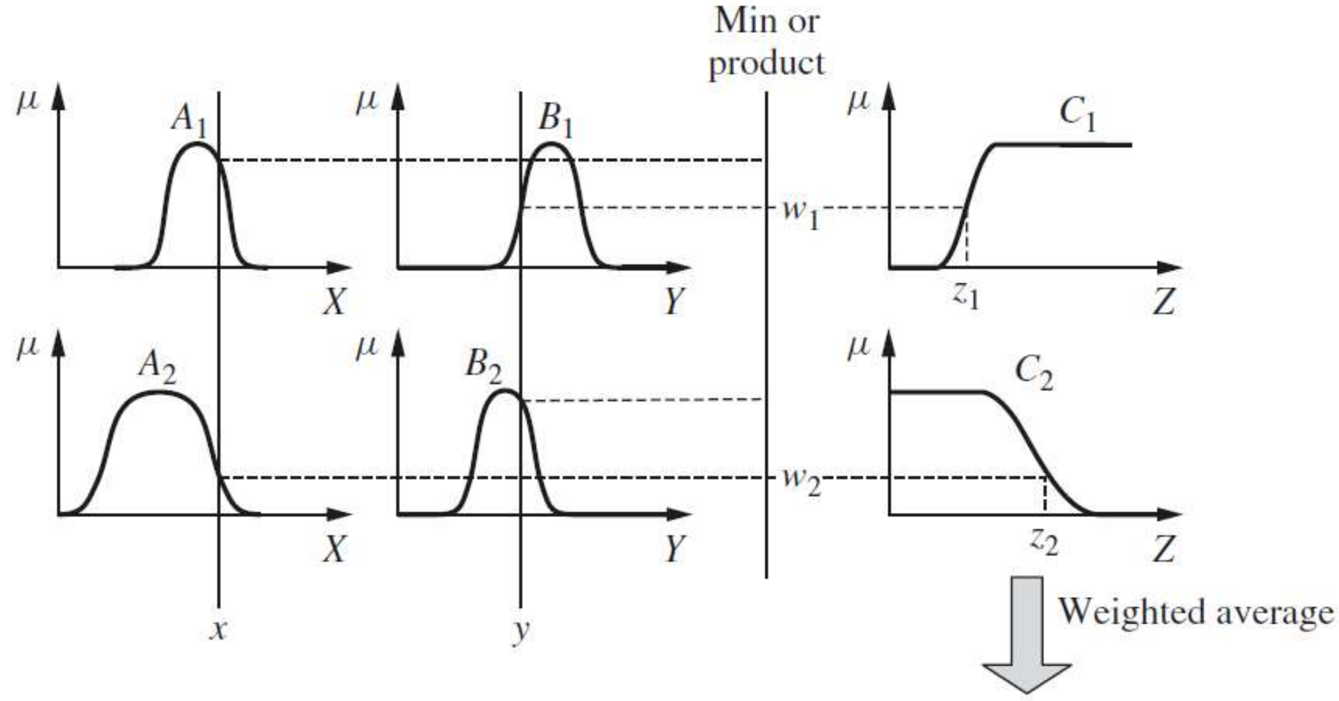


10
8
6
Z
4
2
0
–2
5
0
Y
–5
–5
0
X
5

(b)

Inference: Tsukamoto Systems

- The consequent of each fuzzy rule is represented by a fuzzy set with a monotonic (shoulder) membership function.



$$z = \frac{w_1 z_1 + w_2 z_2}{w_1 + w_2}$$

Example

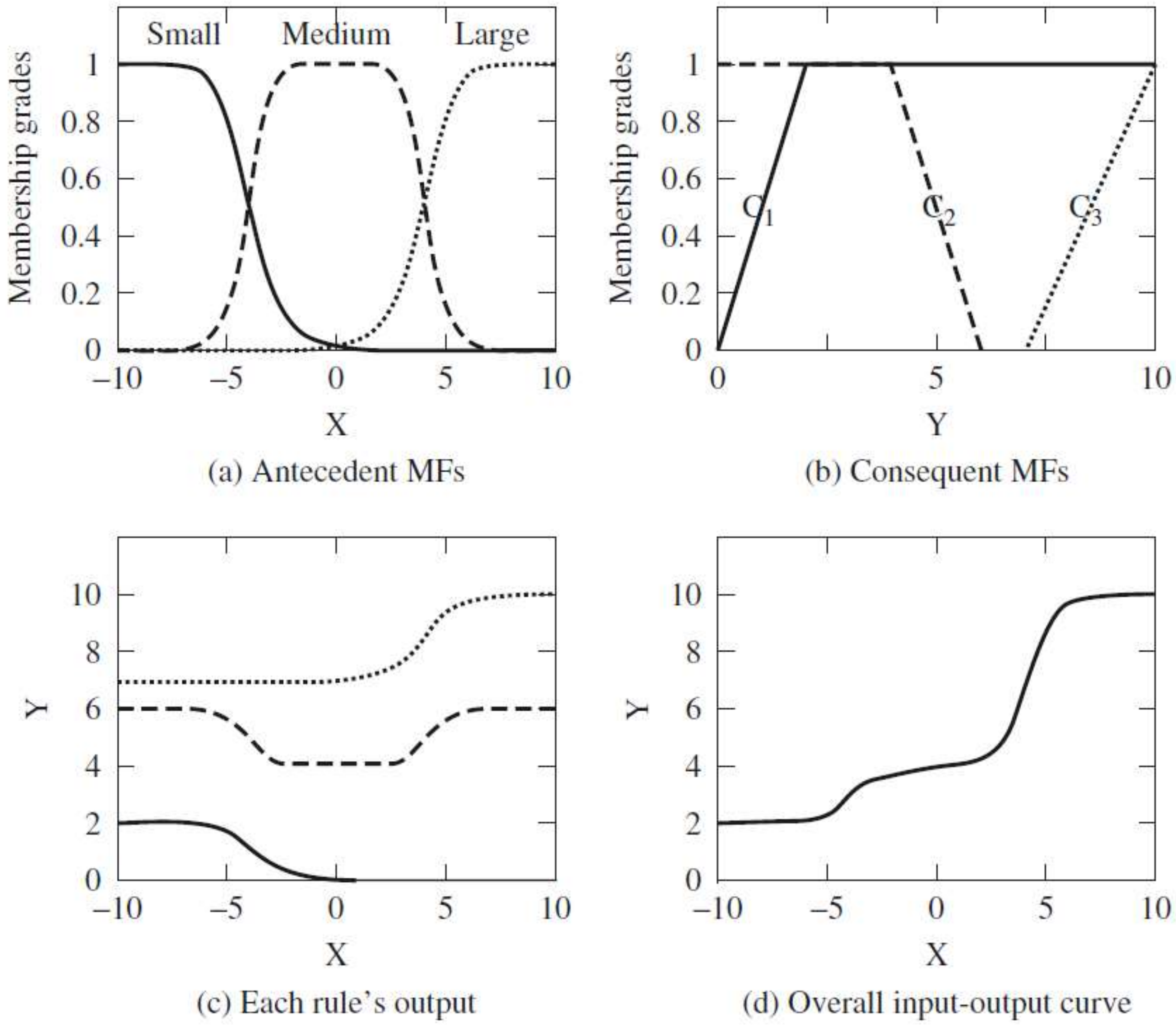
- A single-input, single-output Tsukamoto fuzzy model is given as follows:

IF X is small, THEN Y is C_1,

IF X is medium, THEN Y is C_2,

IF X is large, THEN Y is C_3,

Continue…



Small
Medium
Large
Membership grades
1
0.8
0.6
0.4
0.2
0
–10
–5
0
5
10
X
(a) Antecedent MFs
Membership grades
C1
C2
C3
Y
(b) Consequent MFs
Y
10
8
6
4
2
X
(c) Each rule's output
(d) Overall input-output curve