Chapter 3
Crisp (Classical) Relations and Fuzzy Relations

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Fuzzy Relations

- Fuzzy relations map elements of one universe to elements of another universe through the Cartesian product.
- The strength of the relation between ordered pairs of the two universes is measured with a membership function ($\mu_{\underset{\sim}{R}}(x, y)$).
- The cardinality of fuzzy sets is infinity, the cardinality of a fuzzy relation between two or more universes is also infinity.

Fuzzy Relations: Operations

- Let R and S be fuzzy relations on the Cartesian space X×Y, then the following operations apply for the membership values:

Union $$\mu_{\underset{\sim}{R} \cup \underset{\sim}{S}}(x, y) = \max(\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{S}}(x, y)).$$

Intersection $$\mu_{\underset{\sim}{R} \cap \underset{\sim}{S}}(x, y) = \min(\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{S}}(x, y)).$$

Complement $$\mu_{\overline{\underset{\sim}{R}}}(x, y) = 1 - \mu_{\underset{\sim}{R}}(x, y).$$

Containment $$\underset{\sim}{R} \subset \underset{\sim}{S} \Rightarrow \mu_{\underset{\sim}{R}}(x, y) \leq \mu_{\underset{\sim}{S}}(x, y).$$

Fuzzy Relations: Properties

- As is the case in crisp relations, the properties of commutativity, associativity, distributivity, involution and idempotency are applicable for fuzzy relations.
- De Morgan's principles are applicable for fuzzy relations.
- Fuzzy relations are not constrained by the excluded middle axioms:

$$\underset{\sim}{R} \cup \overline{\underset{\sim}{R}} \neq \mathbf{E}.$$

$$\underset{\sim}{R} \cap \overline{\underset{\sim}{R}} \neq \mathbf{O}.$$

$$\mathbf{O} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{E} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Fuzzy Relations: Cartesian Product

- Suppose A and B are fuzzy sets on universes X and Y, respectively, then the Cartesian product is presented as follows:

$$\underset{\sim}{A} \times \underset{\sim}{B} = \underset{\sim}{R} \subset X \times Y,$$

- The fuzzy relation $\underset{\sim}{R}$ has the following membership function:

$$\mu_{\underset{\sim}{R}}(x, y) = \mu_{\underset{\sim}{A} \times \underset{\sim}{B}}(x, y) = \min(\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y)).$$

Example

- Suppose that

$$\underset{\sim}{A} = \frac{0.2}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3} \quad \text{and} \quad \underset{\sim}{B} = \frac{0.3}{y_1} + \frac{0.9}{y_2}.$$

- $\underset{\sim}{A} \times \underset{\sim}{B} = ?$

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.5 \\ 0.3 & 0.9 \end{bmatrix} \end{array}$$

Fuzzy Relations: Composition

- Suppose that $\underset{\sim}{R}$ and $\underset{\sim}{S}$ are fuzzy relations on the Cartesian space (X×Y) and (Y×Z), respectively, and $\underset{\sim}{T}$ is a fuzzy relation on (X×Z), then
- max–min composition can be defined as follows:

$$\underset{\sim}{T} = \underset{\sim}{R} \circ \underset{\sim}{S},$$

$$\mu_{\underset{\sim}{T}}(x, z) = \bigvee_{y \in Y} (\mu_{\underset{\sim}{R}}(x, y) \wedge \mu_{\underset{\sim}{S}}(y, z)),$$

Continue…

- max–product composition can be defined as follows:

$$\mu_{\underset{\sim}{T}}(x, z) = \bigvee_{y \in Y} (\mu_{\underset{\sim}{R}}(x, y) \bullet \mu_{\underset{\sim}{S}}(y, z)).$$

- Note that fuzzy composition is NOT commutative:

$$\underset{\sim}{R} \circ \underset{\sim}{S} \neq \underset{\sim}{S} \circ \underset{\sim}{R}.$$

Example

- Suppose that $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$, and $Z = \{z_1, z_2, z_3\}$
- The fuzzy relations are as follows:

$$\underset{\sim}{R} = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 0.7 & 0.5 \\ 0.8 & 0.4 \end{bmatrix} \end{array} \quad \text{and} \quad \underset{\sim}{S} = \begin{array}{c c} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} y_1 \\ y_2 \end{array} & \begin{bmatrix} 0.9 & 0.6 & 0.2 \\ 0.1 & 0.7 & 0.5 \end{bmatrix} \end{array}$$

- Using the max–min composition and max-product composition find the relation that relates the elements of universe X to the elements of universe Z.

Crisp Relations: Tolerance and Equivalence Relation

- A relation is considered as an equivalence relation if it has the following three properties:

Reflexivity	$(x_i, x_i) \in \mathrm{R}$ or $\chi_{\mathrm{R}}(x_i, x_i) = 1.$
Symmetry	$(x_i, x_j) \in \mathrm{R} \longrightarrow (x_j, x_i) \in \mathrm{R}$
	or $\chi_{\mathrm{R}}(x_i, x_j) = \chi_{\mathrm{R}}(x_j, x_i).$
Transitivity	$(x_i, x_j) \in \mathrm{R}$ and $(x_j, x_k) \in \mathrm{R} \longrightarrow (x_i, x_k) \in \mathrm{R}$
	or $\chi_{\mathrm{R}}(x_i, x_j)$ and $\chi_{\mathrm{R}}(x_j, x_k) = 1 \longrightarrow \chi_{\mathrm{R}}(x_i, x_k) = 1.$

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- A tolerance (proximity) relation: A relation that exhibits only the properties of reflexivity and symmetry.

- It can be reformed into an equivalence relation by AT MOST (CN−1) compositions with itself, as follows:

$$R_1^{n-1} = R_1 \circ R_1 \circ \cdots \circ R_1 = R$$

Fuzzy Relations: Tolerance and Equivalence Relation

- A fuzzy relation is considered as a fuzzy equivalence relation if it has the following properties:

Reflexivity $\mu_{\underset{\sim}{R}}(x_i, x_i) = 1.$

Symmetry $\mu_{\underset{\sim}{R}}(x_i, x_j) = \mu_{\underset{\sim}{R}}(x_j, x_i).$

Transitivity $\mu_{\underset{\sim}{R}}(x_i, x_j) = \lambda_1$ and $\mu_{\underset{\sim}{R}}(x_j, x_k) = \lambda_2 \longrightarrow \mu_{\underset{\sim}{R}}(x_i, x_k) = \lambda,$

where $\lambda \geq \min[\lambda_1, \lambda_2].$

Continue…

- A fuzzy tolerance (proximity) relation: A fuzzy relation that exhibits only the properties of reflexivity and symmetry.

- It can be reformed into an equivalence fuzzy relation by AT MOST (CN−1) compositions with itself, as follows:

$$\underset{\sim}{R}_1^{n-1} = \underset{\sim}{R}_1 \circ \underset{\sim}{R}_1 \circ \cdots \circ \underset{\sim}{R}_1 = \underset{\sim}{R}$$

Example (3.11)

- A fuzzy relation is as follows:

$$\underset{\sim}{R}_1 = \begin{bmatrix} 1 & 0.8 & 0 & 0.1 & 0.2 \\ 0.8 & 1 & 0.4 & 0 & 0.9 \\ 0 & 0.4 & 1 & 0 & 0 \\ 0.1 & 0 & 0 & 1 & 0.5 \\ 0.2 & 0.9 & 0 & 0.5 & 1 \end{bmatrix}$$

- Is it reflexive, symmetric, transitive?

Value Assignment Methods

- Cartesian product,
- Closed-form expression,
- Lookup table,
- Linguistic rules of knowledge,
- Classification,
- Automated methods from input/output data,
- Similarity methods in data manipulation.

Cosine Amplitude Method

- A similarity metric that uses a collection of data samples.
- It can be presented as follows:

$$r_{ij} = \frac{\left|\sum_{k=1}^{m} x_{ik} x_{jk}\right|}{\sqrt{\left(\sum_{k=1}^{m} x_{ik}^2\right)\left(\sum_{k=1}^{m} x_{jk}^2\right)}}, \quad \text{where } i, j = 1, 2, \ldots, n.$$

Max-Min Method

- A similarity metric that uses a collection of data samples. It is computationally simpler than the cosine amplitude method.
- It can be presented as follows:

$$r_{ij} = \frac{\sum_{k=1}^{m} \min(x_{ik}, x_{jk})}{\sum_{k=1}^{m} \max(x_{ik}, x_{jk})}, \quad \text{where } i, j = 1, 2, \ldots, n$$

Example

Five separate regions along the San Andreas fault in California have suffered damage from a recent earthquake. For purposes of assessing payouts from insurance companies to building owners, the five regions must be classified as to their damage levels. Expression of the damage in terms of relations will prove helpful. Surveys are conducted of the buildings in each region. All the buildings in each region are described as being in one of three damage states: no damage, medium damage, and serious damage. Each region has each of these three damage states expressed as a percentage (ratio) of the total number of buildings. The following table summarizes the findings of the survey team:

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Regions	x_1	x_2	x_3	x_4	x_5
x_{i1} – Ratio with no damage	0.3	0.2	0.1	0.7	0.4
x_{i2} – Ratio with medium damage	0.6	0.4	0.6	0.2	0.6
x_{i3} – Ratio with serious damage	0.1	0.4	0.3	0.1	0.0

By using the cosine amplitude and max-min methods, express these data as a fuzzy relation.

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Cosine amplitude method:

$$\underset{\sim}{R}_1 = \begin{bmatrix} 1 & & & & \\ 0.836 & 1 & & \text{sym} & \\ 0.914 & 0.934 & 1 & & \\ 0.682 & 0.6 & 0.441 & 1 & \\ 0.982 & 0.74 & 0.818 & 0.774 & 1 \end{bmatrix}$$

Max-min method:

$$\underset{\sim}{R}_1 = \begin{bmatrix} 1 & & & & \\ 0.538 & 1 & & \text{sym} & \\ 0.667 & 0.667 & 1 & & \\ 0.429 & 0.333 & 0.250 & 1 & \\ 0.818 & 0.429 & 0.538 & 0.429 & 1 \end{bmatrix}$$