Chapter 2
Crisp (Classical) Sets and Fuzzy Sets

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Crisp Set: Definitions

- Universe of discourse (X): A collection of objects having the same characteristics.
- The elements (x) of a universe are either discrete or continuous.
- Cardinal number (n_x): The total number of elements in a universe.
- Discrete universes have a finite cardinal number, whereas continuous universes have an infinite cardinality.
- Set: Collections of elements within a **UNIVERSE**.
- Subset: Collections of elements within **SETS.**
- The whole set: collection of all elements in the set.

Crisp Set: Notations

- Suppose that A and B consist of collections of some elements in X, then

$x \in X$	$\rightarrow$	x belongs to X
$x \in A$	$\rightarrow$	x belongs to A
$x \notin A$	$\rightarrow$	x does not belong to A

Continue…

- Also

$A \subset B$	$\rightarrow$	A is fully contained in B (if $x \in A$, then $x \in B$)
$A \subseteq B$	$\rightarrow$	A is contained in or is equivalent to B
$(A \leftrightarrow B)$	$\rightarrow$	$A \subseteq B$ and $B \subseteq A$ (A is equivalent to B)

- Null set (Ø): The set that contains no elements.
- Power set (P(X)): All possible sets of X.

Continue…

- Example

- X= {1, 2, 3, 4}

- Cardinal number?
- Power set?
- Cardinal number of the power set?

Crisp Sets: Operations

- Suppose that A and B two sets on the universe X, then

Union	$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.
Intersection	$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.
Complement	$\overline{A} = \{x \mid x \notin A, x \in X\}$.
Difference	$A \mid B = \{x \mid x \in A \text{ and } x \notin B\}$.

- Such operations can be easily presented using Venn diagrams.

Crisp Sets: Properties

- Commutativity: $A \cup B = B \cup A$

  $A \cap B = B \cap A.$

- Associativity: $A \cup (B \cup C) = (A \cup B) \cup C$

  $A \cap (B \cap C) = (A \cap B) \cap C.$

- Distributivity: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$

Continue…

- Idempotency:

$$A \cup A = A$$
$$A \cap A = A.$$

- Identity:

$$A \cup \emptyset = A$$
$$A \cap X = A$$
$$A \cap \emptyset = \emptyset.$$
$$A \cup X = X.$$

Continue…

- Transitivity: If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
- Involution: $\overline{\overline{A}} = A$.
- Axiom of excluded middle: $A \cup \overline{A} = X$.
- Axiom of contradiction: $A \cap \overline{A} = \emptyset$.

Continue…

- De Morgan’s principles: The complement of a union or an intersection is equal to the intersection or union, respectively.

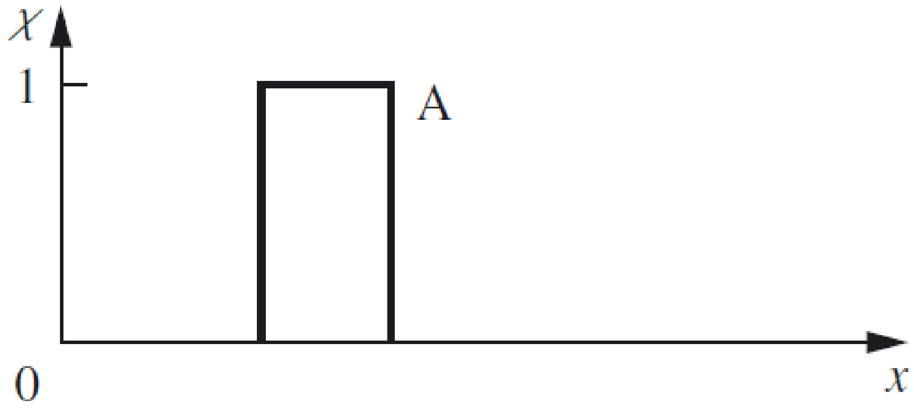
$$\overline{A \cap B} = \overline{A} \cup \overline{B}.$$

$$\overline{A \cup B} = \overline{A} \cap \overline{B}.$$

Mapping of Crisp Sets to Functions

- Relating set-theoretic forms to function-theoretic terms.
- Mapping elements (subsets) in an universe of discourse to elements (subsets) in another one.
- Membership function is a mapping for a crisp set:

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$



Continue…

- Suppose that A and B are two sets on the universe X, then the function-theoretic terms:

Union $\mathrm{A} \cup \mathrm{B} \longrightarrow \chi_{\mathrm{A}\cup\mathrm{B}}(x) = \chi_{\mathrm{A}}(x) \vee \chi_{\mathrm{B}}(x) = \max(\chi_{\mathrm{A}}(x), \chi_{\mathrm{B}}(x)).$

Intersection $\mathrm{A} \cap \mathrm{B} \longrightarrow \chi_{\mathrm{A}\cap\mathrm{B}}(x) = \chi_{\mathrm{A}}(x) \wedge \chi_{\mathrm{B}}(x) = \min(\chi_{\mathrm{A}}(x), \chi_{\mathrm{B}}(x)).$

Complement $\overline{\mathrm{A}} \longrightarrow \chi_{\overline{\mathrm{A}}}(x) = 1 - \chi_{\mathrm{A}}(x).$

Containment $\mathrm{A} \subseteq \mathrm{B} \longrightarrow \chi_{\mathrm{A}}(x) \leq \chi_{\mathrm{B}}(x).$

Note that crisp sets are a special case of fuzzy sets

Fuzzy Sets

- Fuzzy sets contain elements that have varying degrees of membership.
- For discrete universe:

$$\underset{\sim}{\mathrm{A}} = \left\{ \frac{\mu_{\underset{\sim}{\mathrm{A}}}(x_1)}{x_1} + \frac{\mu_{\underset{\sim}{\mathrm{A}}}(x_2)}{x_2} + \cdots \right\} = \left\{ \sum_i \frac{\mu_{\underset{\sim}{\mathrm{A}}}(x_i)}{x_i} \right\}$$

- For continuous universe:

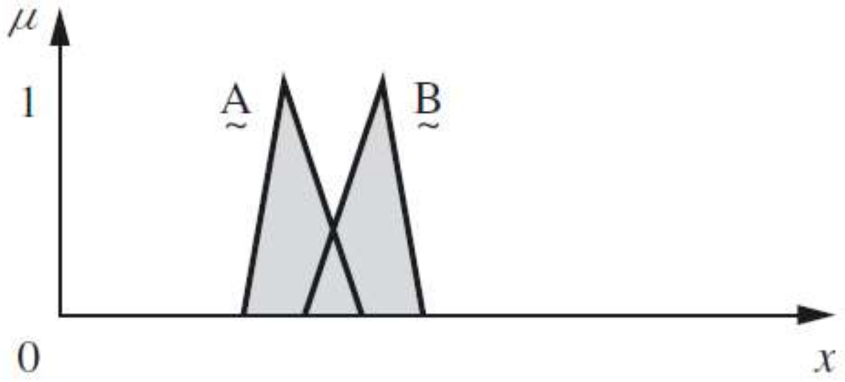
$$\underset{\sim}{\mathrm{A}} = \left\{ \int \frac{\mu_{\underset{\sim}{\mathrm{A}}}(x)}{x} \right\}$$

The summation and the integral signs are not algebraic symbols.

Fuzzy Sets: Operations

- Suppose that $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are two sets on the universe X, then the function-theoretic terms:

Union	$\mu_{\underset{\sim}{A} \cup \underset{\sim}{B}}(x) = \mu_{\underset{\sim}{A}}(x) \vee \mu_{\underset{\sim}{B}}(x).$
Intersection	$\mu_{\underset{\sim}{A} \cap \underset{\sim}{B}}(x) = \mu_{\underset{\sim}{A}}(x) \wedge \mu_{\underset{\sim}{B}}(x).$
Complement	$\mu_{\overline{\underset{\sim}{A}}}(x) = 1 - \mu_{\underset{\sim}{A}}(x).$



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- All the operations for classical sets are valid for fuzzy sets **EXCEPT for the excluded middle axioms.**
- The excluded middle axioms have been extended for fuzzy sets:

$$\underset{\sim}{A} \cup \overline{\underset{\sim}{A}} \neq X.$$

$$\underset{\sim}{A} \cap \overline{\underset{\sim}{A}} \neq \emptyset.$$

- De Morgan’s principles for crisp sets are valid for fuzzy sets.
- Fuzzy intersections and unions can be represented as t-norms and t-conorms, respectively.

Fuzzy Sets: Question

For a collection of fuzzy sets and subsets on a universe, what is:

- The fuzzy power set?
- The cardinal number of the fuzzy power set?

Fuzzy Sets: Example

- Suppose that we have two discrete fuzzy sets:

$$\underset{\sim}{A} = \left\{ \frac{1}{2} + \frac{0.5}{3} + \frac{0.3}{4} + \frac{0.2}{5} \right\} \quad \text{and} \quad \underset{\sim}{B} = \left\{ \frac{0.5}{2} + \frac{0.7}{3} + \frac{0.2}{4} + \frac{0.4}{5} \right\}$$

- Note that the membership function of 1 is Zero.
- Calculate: Complement, union, intersection and difference.

Continue…

Complement
$$\overline{\underset{\sim}{A}} = \left\{\frac{1}{1} + \frac{0}{2} + \frac{0.5}{3} + \frac{0.7}{4} + \frac{0.8}{5}\right\}.$$
$$\overline{\underset{\sim}{B}} = \left\{\frac{1}{1} + \frac{0.5}{2} + \frac{0.3}{3} + \frac{0.8}{4} + \frac{0.6}{5}\right\}.$$

Union
$$\underset{\sim}{A} \cup \underset{\sim}{B} = \left\{\frac{1}{2} + \frac{0.7}{3} + \frac{0.3}{4} + \frac{0.4}{5}\right\}.$$

Intersection
$$\underset{\sim}{A} \cap \underset{\sim}{B} = \left\{\frac{0.5}{2} + \frac{0.5}{3} + \frac{0.2}{4} + \frac{0.2}{5}\right\}.$$

Difference
$$\underset{\sim}{A}|\underset{\sim}{B} = \underset{\sim}{A} \cap \overline{\underset{\sim}{B}} = \left\{\frac{0.5}{2} + \frac{0.3}{3} + \frac{0.3}{4} + \frac{0.2}{5}\right\}.$$
$$\underset{\sim}{B}|\underset{\sim}{A} = \underset{\sim}{B} \cap \overline{\underset{\sim}{A}} = \left\{\frac{0}{2} + \frac{0.5}{3} + \frac{0.2}{4} + \frac{0.4}{5}\right\}.$$